2

The classical groups

The *Linear, Unitary, Symplectic,* and *Orthogonal* groups have been collectively known as 'The classical groups' since the publication of Hermann Weyl's famous book of that name, which discussed them over the real and complex fields. Most of their theory has been generalized to the other Chevalley and twisted Chevalley groups. However, the classical definitions require little technical knowledge, lead readily to invariant treatments of the groups, and provide many techniques for easy calculations inside them. In this 𝔸𝕋𝕃𝔸𝕊 we take a severely classical viewpoint, for the most part. Later in this introduction, however, we shall quickly describe the larger class of groups, and the present section contains some forward references.

1. The groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$, and $PSL_n(q) = L_n(q)$

The *general linear group* $GL_n(q)$ consists of all the $n \times n$ matrices with entries in $\mathbb{F}_q$ that have non-zero determinant. Equivalently it is the group of all linear automorphisms of an n-dimensional vector space over $\mathbb{F}_q$. The *special linear group* $SL_n(q)$ is the subgroup of all matrices of determinant 1. The *projective general linear group* $PGL_n(q)$ and *projective special linear group* $PSL_n(q)$ are the groups obtained from $GL_n(q)$ and $SL_n(q)$ on factoring by the scalar matrices contained in those groups.

For $n \geqslant 2$ the group $PSL_n(q)$ is simple except for $PSL_2(2) = S_3$ and $PSL_2(3) = A_4$, and we therefore also call it $L_n(q)$, in conformity with Artin's convention in which single-letter names are used for groups that are 'generally' simple.

The orders of the above groups are given by the formulae

$$|GL_n(q)| = (q-1)N, \qquad |SL_n(q)| = |PGL_n(q)| = N,$$

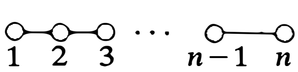
$$|PSL_n(q)| = |L_n(q)| = \frac{N}{d},$$

where

$$N = q^{\frac{1}{2}n(n-1)}(q^n - 1)(q^{n-1} - 1)\dots(q^2 - 1),$$

and $d = (q-1, n)$.

$L_{n+1}(q)$ is the adjoint Chevalley group $A_n(q)$, with Dynkin diagram



(n nodes).

The maximal parabolic subgroup correlated with the node labelled k in the diagram corresponds to the stabilizer of a k-dimensional vector subspace.

2. The groups $GU_n(q)$, $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q) = U_n(q)$

Let V be a vector space over $\mathbb{F}_{q^2}$. Then a function $f(x, y)$ which is defined for all x, y in V and takes values in $\mathbb{F}_{q^2}$ is called a *conjugate-symmetric sesquilinear form* if it satisfies

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y)$$

(linearity in x), and

$$f(y, x) = \overline{f(x, y)}$$

(conjugate-symmetry), where $x \to \bar{x} = x^q$ is the automorphism of $\mathbb{F}_{q^2}$ whose fixed field is $\mathbb{F}_q$. Such a form is necessarily *semilinear in* y, that is

$$f(x, \lambda_1 y_1 + \lambda_2 y_2) = \overline{\lambda_1} f(x, y_1) + \overline{\lambda_2} f(x, y_2).$$

It is called *singular* if there is some $x_0 \neq 0$ such that $f(x_0, y) = 0$ for all y. The *kernel* is the set of all such x_0. The *nullity* and *rank* are the dimension and codimension of the kernel.

A *Hermitian form* $F(x)$ is any function of the shape $f(x, x)$, where $f(x, y)$ is a conjugate-symmetric sesquilinear form. Since either of the forms F and f determines the other uniquely, it is customary to transfer the application of adjectives freely from one to the other. Thus $F(x) = f(x, x)$ is termed non-singular if and only if $f(x, y)$ is non-singular. Coordinates can always be chosen so that a given non-singular Hermitian form becomes

$$x_1\overline{x_1} + x_2\overline{x_2} + \dots + x_n\overline{x_n}.$$

The *general unitary group* $GU_n(q)$ is the subgroup of all elements of $GL(q^2)$ that fix a given non-singular Hermitian form, or, equivalently, that fix the corresponding non-singular conjugate-symmetric sesquilinear form. If the forms are chosen to be the canonical one above, then a matrix U belongs to $GU_n(q)$ (is *unitary*) just if $U^{-1} = \bar{U}^t$, the matrix obtained by replacing the entries of U^t by their qth powers.

The determinant of a unitary matrix is necessarily a $(q+1)$st root of unity. The *special unitary group* $SU_n(q)$ is the subgroup of unitary matrices of determinant 1. The *projective general unitary group* $PGU_n(q)$ and *projective special unitary group* $PSU_n(q)$ are the groups obtained from $GU_n(q)$ and $SU_n(q)$ on factoring these groups by the scalar matrices they contain.

For $n \geqslant 2$, the group $PSU_n(q)$ is simple with the exceptions

$$PSU_2(2) = S_3, \qquad PSU_2(3) = A_4, \qquad PSU_3(2) = 3^2 : Q_8,$$

and so we also give it the simpler name $U_n(q)$. We have $U_2(q) = L_2(q)$.

The orders of the above groups are given by

$$|GU_n(q)| = (q+1)N, \qquad |SU_n(q)| = |PGU_n(q)| = N,$$

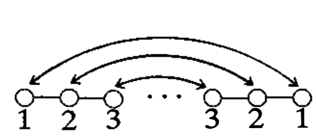
$$|PSU_n(q)| = |U_n(q)| = \frac{N}{d},$$

where

$$N = q^{\frac{1}{2}n(n-1)}(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})\dots(q^3 + 1)(q^2 - 1),$$

and $d = (q+1, n)$.

$U_{n+1}(q)$ is the twisted Chevalley group ${}^2A_n(q)$, with the Dynkin diagram and twisting automorphism indicated:



(n nodes).

The maximal parabolic subgroup correlated with the orbit of nodes labelled k in the diagram corresponds to the stabilizer of a k-dimensional totally isotropic subspace (i.e. a space on which $F(x)$ or equivalently $f(x, y)$ is identically zero).

3. The groups $Sp_n(q)$ and $PSp_n(q) = S_n(q)$

An *alternating bilinear form* (or *symplectic form*) on a vector space V over $\mathbb{F}_q$ is a function $f(x, y)$ defined for all x, y in V and taking values in $\mathbb{F}_q$, which satisfies

$$f(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 f(x_1, y) + \lambda_2 f(x_2, y)$$

(linearity in x), and also

$$f(y,x)=-f(x,y) \quad \text{and} \quad f(x,x)=0$$

(skew-symmetry and alternation). It is automatically linear in y also (and so *bilinear*).

The *kernel* of such a form is the subspace of x such that $f(x,y)=0$ for all y, and the *nullity* and *rank* of f are the dimension and codimension of its kernel. A form is called non-singular if its nullity is zero. The rank of a symplectic form is necessarily an even number, say $2m$, and coordinates can be chosen so that the form has the shape

$$x_1y_{m+1}+x_2y_{m+2}+\ldots+x_my_{2m}-x_{m+1}y_1-x_{m+2}y_2-\ldots-x_{2m}y_m.$$

For an even number $n=2m$, the *symplectic group* $Sp_n(q)$ is defined as the group of all elements of $GL_n(q)$ that preserve a given non-singular symplectic form $f(x,y)$. Any such matrix necessarily has determinant 1, so that the 'general' and 'special' symplectic groups coincide. The *projective symplectic group* $PSp_n(q)$ is obtained from $Sp_n(q)$ on factoring it by the subgroup of scalar matrices it contains (which has order at most 2). For $2m\geqslant 2$, $PSp_{2m}(q)$ is simple with the exceptions

$$PSp_2(2)=S_3, \qquad PSp_2(3)=A_4, \qquad PSp_4(2)=S_6$$

and so we also call it $S_{2m}(q)$. We have $S_2(q)=L_2(q)$.

If A, B, C, D are $m\times m$ matrices, then $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ belongs to the symplectic group for the canonical symplectic form above just if

$$A^tC-C^tA=0, \qquad A^tD-C^tB=I, \qquad B^tD-D^tB=0,$$

where M^t denotes a transposed matrix.

The orders of the above groups are given by

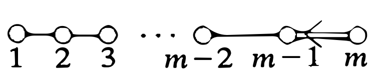
$$|Sp_{2m}(q)|=N, \qquad |PSp_{2m}(q)|=|S_{2m}(q)|=\frac{N}{d},$$

where

$$N=q^{m^2}(q^{2m}-1)(q^{2m-2}-1)\ldots(q^2-1)$$

and $d=(q-1,2)$.

$S_{2m}(q)$ is the adjoint Chevalley group $C_m(q)$ with Dynkin diagram



(m nodes).

The maximal parabolic group correlated with the node labelled k corresponds to the stabilizer of a k-dimensional totally isotropic subspace (that is, a space on which $f(x,y)$ is identically zero).

4. The groups $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, $PSO_n(q)$, and $O_n(q)$

A *symmetric bilinear form* on a space V over $\mathbb{F}_q$ is a function $f(x,y)$ defined for all x, y in V and taking values in $\mathbb{F}_q$ which satisfies

$$f(\lambda_1x_1+\lambda_2x_2,y)=\lambda_1f(x_1,y)+\lambda_2f(x_2,y)$$

(linearity in x), and also

$$f(y,x)=f(x,y)$$

(symmetry). It is then automatically linear in y. A *quadratic form* on V is a function $F(x)$ defined for x in V and taking values in $\mathbb{F}_q$, for which we have

$$F(\lambda x+\mu y)=\lambda^2F(x)+\lambda\mu f(x,y)+\mu^2F(y)$$

for some symmetric bilinear form $f(x,y)$.

The *kernel of f* is the subspace of all x such that $f(x,y)=0$ for all y, and the *kernel of F* is the set of all x in the kernel of f for which also $F(x)=0$.

When the characteristic is not 2, F and f uniquely determine each other, so that the two kernels coincide. The literature contains a bewildering variety of terminology adapted to describe the more complicated situations that can hold in characteristic 2. This can be greatly simplified by using only a few standard terms (rank, nullity, non-singular, isotropic), but always being careful to state to which of f and F they apply.

Thus we define the *nullity* and *rank* of either f or F to be the dimension and codimension of its kernel, and say that f or F is *non-singular* just when its nnllity is zero. A subspace is said to be (*totally*) *isotropic for f* if $f(x,y)$ vanishes for all x, y in that subspace, and (*totally*) *isotropic for F* if $F(x)$ vanishes for all x in the subspace. When the characteristic is not 2 our adjectives can be freely transferred between f and F.

The *Witt index* of a quadratic form F is the greatest dimension of any totally isotropic subspace for F. It turns out that if two non-singular quadratic forms on the same space over $\mathbb{F}_q$ have the same Witt index, then they are equivalent to scalar multiples of each other. The *Witt defect* is obtained by subtracting the Witt index from its largest possible value, $[\frac{1}{2}n]$. For a non-singular form over a finite field the Witt defect is 0 or 1.

The *general orthogonal group* $GO_n(q,F)$ is the subgroup of all elements of $GL_n(q)$ that fix the particular non-singular quadratic form F. The determinant of such an element is necessarily ± 1, and the *special orthogonal group* $SO_n(q,F)$ is the subgroup of all elements with determinant 1. The *projective general orthogonal group* $PGO_n(q,F)$ and *projective special orthogonal group* $PSO_n(q,F)$ are the groups obtained from $GO_n(q,F)$ and $SO_n(q,F)$ on factoring them by the groups of scalar matrices they contain.

In general $PSO_n(q,F)$ is *not* simple. However, it has a certain subgroup, specified precisely later, that is simple with finitely many exceptions when $n\geqslant 5$. This subgroup, which is always of index at most 2 in $PSO_n(q,F)$, we call $O_n(q,F)$.

When $n=2m+1$ is odd, all non-singular quadratic forms on a space of dimension n over $\mathbb{F}_q$ have Witt index m and are equivalent up to scalar factors. When $n=2m$ is even, there are up to equivalence just two types of quadratic form, the *plus type*, with Witt index m, and the *minus type*, with Witt index $m-1$. (These statements make use of the finiteness of $\mathbb{F}_q$.) Accordingly, we obtain only the following distinct families of groups:

When n is odd $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, $PSO_n(q)$, $O_n(q)$, being the values of $GO_n(q,F)$ (etc.) for any non-singular F.

When n is even $GO_n^\varepsilon(q)$, $SO_n^\varepsilon(q)$, $PGO_n^\varepsilon(q)$, $PSO_n^\varepsilon(q)$, $O_n^\varepsilon(q)$ for either sign $\varepsilon=+$ or $-$, being the values of $GO_n(q,F)$ (etc.) for a form F of plus type or minus type respectively.

We now turn to the problem of determining the generally simple group $O_n(q)$ or $O_n^\varepsilon(q)$. This can be defined in terms of the invariant called the *spinor norm*, when q is odd, or in terms of the *quasideterminant*, when q is even. We define these below, supposing $n\geqslant 3$ (the groups are boring for $n\leqslant 2$).

A vector r in V for which $F(r)\neq 0$ gives rise to certain elements of $GO_n(q,F)$ called *reflections*, defined by the formula

$$x \longrightarrow x-\frac{f(x,r)}{F(r)}\cdot r.$$

We shall now define a group $\Omega_n^\varepsilon(q)$ of index 1 or 2 in $SO_n^\varepsilon(q)$. The image $P\Omega_n^\varepsilon(q)$ of this group in $PSO_n^\varepsilon(q)$ is the group we call $O_n^\varepsilon(q)$, which is usually simple. The Ω, $P\Omega$ notation was introduced by Dieudonné, who defined $\Omega_n^\varepsilon(q)$ to be the commutator subgroup of $SO_n^\varepsilon(q)$, but we have changed the definition so as to obtain the 'correct' groups (in the Chevalley sense) for small n. For $n\geqslant 5$ our groups agree with Dieudonné's.

When q is odd, $\Omega_n^\varepsilon(q)$ is defined to be the set of all those g in $SO_n^\varepsilon(q)$ for which, when g is expressed in any way as the product of reflections in vectors $r_1, r_2, \ldots, r_t$, we have

$$F(r_1)\,.\,F(r_2)\,.\,\ldots\,.\,F(r_t) \quad \text{a square in } \mathbb{F}_q.$$

The function just defined is called the *spinor norm* of g, and takes values in $\mathbb{F}_q^*/(\mathbb{F}_q^*)^2$, where $\mathbb{F}_q^*$ is the multiplicative group of $\mathbb{F}_q$. Then $\Omega_n^\varepsilon(q)$ has index 2 in $SO_n^\varepsilon(q)$. It contains the scalar matrix -1 just when $n=2m$ is even and $(4, q^m-\varepsilon)=4$.

When q is even, then if $n=2m+1$ is odd, $SO_n(q)=GO_n(q)$ is isomorphic to the (usually simple) symplectic group $Sp_{2m}(q)$, and we define $\Omega_n(q)=SO_n(q)$. To obtain the isomorphism, observe that the associated symmetric bilinear form f has a one-dimensional kernel, and yields a non-singular symplectic form on $V/ker(f)$.

For arbitrary q, and $n=2m$ even, we define the *quasideterminant* of an element to be $(-1)^k$, where k is the dimension of its fixed space. Then for q odd this homomorphism agrees with the determinant, and for q even we define Ω_n^ε to be its kernel. The quasideterminant can also be written as $(-1)^D$, where D is a polynomial invariant called the Dickson invariant, taking values in $\mathbb{F}_2$.

An alternative definition of the quasideterminant is available. When $\varepsilon=+$ there are two families of maximal isotropic subspaces for F, two spaces being in the same family just if the codimension of their intersection in either of them is even. Then the quasideterminant of an element is 1 or -1 according as it preserves each family or interchanges the two families. When $\varepsilon=-$, the maximal isotropic spaces defined over $\mathbb{F}_q$ have dimension $m-1$, but if we extend the field to $\mathbb{F}_{q^2}$, we obtain two families of m-dimensional isotropic spaces and can use the same definition.

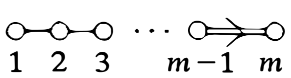
When $n=2m+1$ is odd, the groups have orders

$$|GO_n(q)|=dN, \qquad |SO_n(q)|=|PGO_n(q)|=|PSO_n(q)|=N,$$
$$|\Omega_n(q)|=|P\Omega_n(q)|=|O_n(q)|=N/d,$$

where

$$N=q^{m^2}(q^{2m}-1)(q^{2m-2}-1)\dots(q^2-1)$$

and $d=(2,q-1)$. $O_{2m+1}(q)$ is the adjoint Chevalley group $B_m(q)$, with Dynkin diagram



(m nodes).

The maximal parabolic subgroup correlated with the node labelled k corresponds to the stabilizer of an isotropic k-space for F.

When $n=2m$ is even, the groups have orders

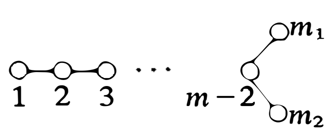
$$|GO_n^\varepsilon(q)|=2N, \qquad |SO_n^\varepsilon(q)|=|PGO_n^\varepsilon(q)|=2N/e,$$
$$|PSO_n^\varepsilon(q)|=2N/e^2, \qquad |\Omega_n^\varepsilon(q)|=N/e,$$
$$|P\Omega_n^\varepsilon(q)|=|O_n(q)|=N/d,$$

where

$$N=q^{m(m-1)}(q^m-\varepsilon)(q^{2m-2}-1)(q^{2m-4}-1)\dots(q^2-1)$$

and $d=(4,q^m-\varepsilon)$, $e=(2,q^m-\varepsilon)$.

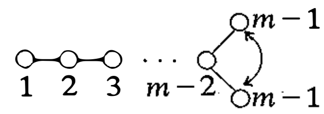
$O_{2m}^+(q)$ is the adjoint Chevalley group $D_m(q)$, with Dynkin diagram



(m nodes).

The maximal parabolic subgroup correlated with the node labelled $k\le m-2$ corresponds to the stabilizer of an isotropic k-space for F. Those correlated with the nodes labelled m_1 and m_2 correspond to stabilizers of members of the two families of isotropic m-spaces for F.

$O_{2m}^-(q)$ is the twisted Chevalley group ${}^2D_m(q)$, with the Dynkin diagram and twisting automorphism



(m nodes).

The maximal parabolic subgroup correlated with the orbit of nodes labelled $k\le m-1$ corresponds to the stabilizer of an isotropic k-space for F.

For $n\le 6$, the orthogonal groups are isomorphic to other classical groups, as follows:

$$O_3(q)=L_2(q), \quad O_4^+(q)=L_2(q)\times L_2(q), \quad O_4^-(q)=L_2(q^2),$$
$$O_5(q)=S_4(q), \quad O_6^+(q)=L_4(q), \quad O_6^-(q)=U_4(q).$$

The group $O_n(q)$ is simple for $n\ge 5$, with the single exception that $O_5(2)=S_4(2)$ is isomorphic to the symmetric group S_6.

5. Classification of points and hyperplanes in orthogonal spaces

Let V be a space equipped with a non-singular quadratic form F. Then for fields of odd characteristic many authors classify the vectors of V into three classes according as

$F(v)=0$,
$F(v)$ a non-zero square,
$F(v)$ a non-square,

since these correspond exactly to the three orbits of projective points under the orthogonal group of F.

In this 𝔸𝕋𝕃𝔸𝕊 we prefer a different way of making these distinctions, which is independent of the choice of any particular scalar multiple of the quadratic form F, and which does the correct thing in characteristic 2. We say that a subspace H of even dimension on which F is non-singular is of *plus type* or *minus type* according to the type of F when restricted to H, and if H is the hyperplane perpendicular to a vector v, we apply the same adjectives to v. Thus our version of the above classification is

v *isotropic* (or *null*),
v of *plus type*,
v of *minus type*.

Table 2. Structures of classical groups

d	$GL_n(q)$	$PGL_n(q)$	$SL_n(q)$	$PSL_n(q)=L_n(q)$
$(n,q-1)$	$d.\left(\frac{q-1}{d}\times G\right).d$	$G.d$	$d.G$	G

d	$GU_n(q)$	$PGU_n(q)$	$SU_n(q)$	$PSU_n(q)=U_n(q)$
$(n,q+1)$	$d.\left(\frac{q+1}{d}\times G\right).d$	$G.d$	$d.G$	G

d	$Sp_n(q)$	$PSp_n(q)=S_n(q)$
$(2,q-1)=2$	$2.G$	G
$(2,q-1)=1$	G	G

d	$GO_n^\varepsilon(q)$	$PGO_n^\varepsilon(q)$	$SO_n^\varepsilon(q)$	$PSO_n^\varepsilon(q)$	$\Omega_n^\varepsilon(q)$	$P\Omega_n^\varepsilon(q)=O_n^\varepsilon(q)$	
$(4,q^m-\varepsilon)=4$	$2.G.2^2$	$G.2^2$	$2.G.2$	$G.2$	$2.G$	G	$n=2m$
$(4,q^m-\varepsilon)=2$	$2\times G.2$	$G.2$	$2\times G$	G	G	G	$n=2m$
$(4,q^m-\varepsilon)=1$	$G.2$	$G.2$	$G.2$	$G.2$	G	G	$n=2m$
$(2,q-1)=2$	$2\times G.2$	$G.2$	$G.2$	$G.2$	G	G	$n=2m+1$
$(2,q-1)=1$	G	G	G	G	G	G	$n=2m+1$

6. The Clifford algebra and the spin group

The *Clifford algebra* of F is the associative algebra generated by the vectors of V with the relations $x^2 = F(x)$, which imply $xy + yx = f(x, y)$. If V has basis $e_1, \ldots, e_n$, then the Clifford algebra is 2^n-dimensional, with basis consisting of the formal products

$$e_{i_1} e_{i_2} \ldots e_{i_k} \quad (i_1 < i_2 < i_3 < \ldots < i_k, k \leqslant n).$$

The vectors r with $F(r) \neq 0$ generate a subgroup of the Clifford algebra which is a central extension of the orthogonal group, the vector r in the Clifford algebra mapping to the negative of the reflection in r. When the ground field has characteristic $\neq 2$, this remark can be used to construct a proper double cover of the orthogonal group, called the *spin group.*

7. Structure tables for the classical groups

Table 2 describes the structure of all the groups mentioned, in terms of the usually simple group G, which is the appropriate one of $L_n(q)$, $U_n(q)$, $S_n(q)$, $O_n(q)$.

8. Other notations for the simple groups

There are many minor variations such as $L_n(\mathbb{F}_q)$ or $L(n, q)$ for $L_n(q)$ which should give little trouble. However, the reader should be aware that although the 'smallest field' convention which we employ in this ATLAS is rapidly gaining ground amongst group theorists, there are still many people who write $U_n(q^2)$ or $U(n, q^2)$ for what we call $U_n(q)$. Artin's 'single letter for simple group' convention is not universally adopted, so that many authors would use $U_n(q)$ and $O_n(q)$ for what we call $GU_n(q)$ and $GO_n(q)$. The notations $E_2(q)$ and $E_4(q)$ have sometimes been used for $G_2(q)$ and $F_4(q)$.

Dickson's work has had a profound influence on group theory, and his notations still have some currency, but are rapidly becoming obsolete. Here is a brief dictionary:

(Linear fractional) $LF(n, q) = L_n(q)$

(Hyperorthogonal) $HO(n, q^2) = U_n(q)$

(Abelian linear) $A(2m, q) = S_{2m}(q)$

(First orthogonal) $\begin{cases} FO(2m+1, q) = O_{2m+1}(q) \\ FO(2m, q) = O^{\varepsilon}_{2m}(q) \end{cases}$

(Second orthogonal) $SO(2m, q) = O^{-\varepsilon}_{2m}(q)$

where $\varepsilon = \pm 1$, $q^m \equiv \varepsilon$ modulo 4.

(First hypoabelian) $FH(2m, q) = O^{+}_{2m}(q)$

(Second hypoabelian) $SH(2m, q) = O^{-}_{2m}(q)$

Dickson's first orthogonal group is that associated with the quadratic form $x_1^2 + x_2^2 + \ldots + x_n^2$. His orthogonal groups are defined only for odd q, and his hypoabelian groups only for even q. He uses the notation $GLH(n, q)$ (General Linear Homogeneous group) for $GL_n(q)$.

3

The Chevalley and twisted Chevalley groups

1. The untwisted groups

The Chevalley and twisted Chevalley groups include and neatly generalize the classical families of linear, unitary, symplectic, and orthogonal groups.

In the entries for individual groups in the ATLAS, we have preferred to avoid the Chevalley notation, since it requires considerable technical knowledge, and since most of the groups we discuss also have classical definitions. The classical descriptions, when available, permit easy calculation, and lead readily to the desired facts about subgroups, etc.

However, for a full understanding of the entire set of finite simple groups, the Chevalley theory is unsurpassed. In particular the isomorphisms such as $L_4(q) = O_6^+(q)$ between different classical groups of the same characteristic become evident. The full Chevalley theory is beyond the scope of this ATLAS, but in the next few pages we give a brief description for those already acquainted with some of the terminology of Lie groups and Lie algebras. We reject any reproach for the incompleteness of this treatment. It is intended merely to get us to the point where we can list all the groups, and the isomorphisms among them, and also ~~to~~ specify their Schur multipliers and outer automorphism groups.

In 1955 Chevalley discovered a uniform way to define bases for the complex simple Lie algebras in which all their structure constants were rational integers. It follows that analogues of these Lie algebras and the corresponding Lie groups can be defined over arbitrary fields. The resulting groups are now known as the *adjoint Chevalley groups*. Over finite fields, these groups are finite groups which are simple in almost all cases. The definition also yields certain covering groups, which are termed the *universal Chevalley groups*. If a given finite simple group can be expressed as an adjoint Chevalley group, then in all but finitely many cases its abstract universal cover is the corresponding universal Chevalley group.

In the standard notation, the complex Lie algebras are

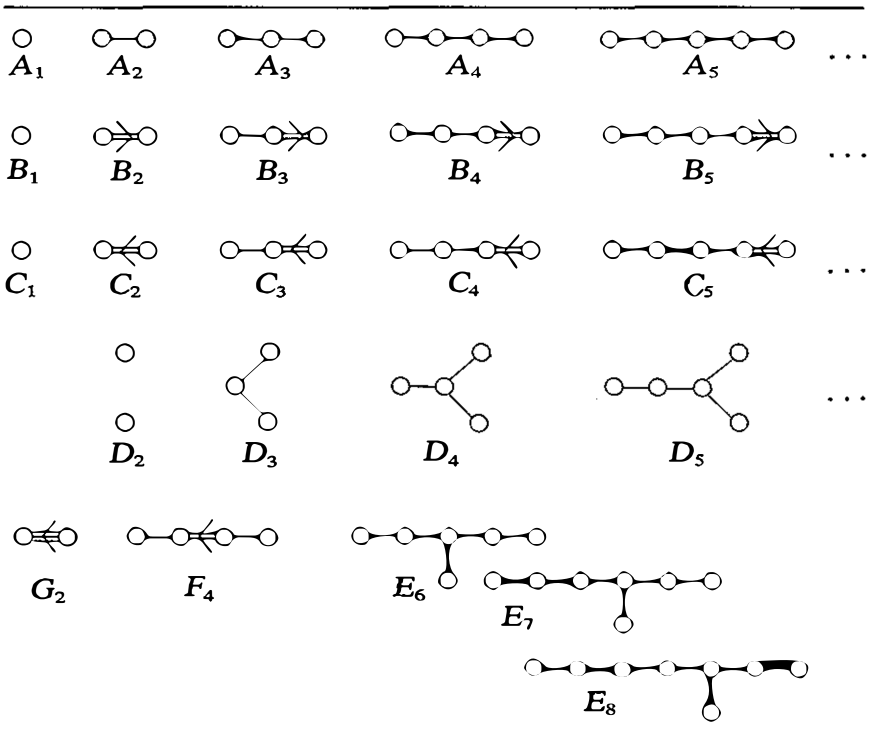
$$A_n \quad B_n \quad C_n \quad D_n \quad G_2 \quad F_4 \quad E_6 \quad E_7 \quad E_8$$

where to avoid repetitions we may demand that $n \geqslant 1, 2, 3, 4$ for A_n, B_n, C_n, D_n respectively. The corresponding adjoint Chevalley groups are denoted by

$$A_n(q), B_n(q), C_n(q), D_n(q), G_2(q), F_4(q), E_6(q), E_7(q), E_8(q).$$

The corresponding *Dynkin diagrams*, which specify the structure of the fundamental roots, appear in Table 3. In the cases when a p-fold branch appears ($p = 2$ or 3), the arrowhead points from long roots to short ones, the ratio of lengths being $\sqrt{p}$. We have included the non-simple case $D_2 = A_1 \oplus A_1$, and the repetitions $A_1 = B_1 = C_1$, $B_2 = C_2$, $A_3 = D_3$, since these help in the understanding of the relations between various classical groups.

Table 3. Dynkin diagrams



2. The twisted groups

Steinberg showed that a modification of Chevalley's procedure could be made to yield still more finite groups, and in particular, the unitary groups.

Any symmetry of the Dynkin diagram (preserving the direction of the arrowhead, if any) yields an automorphism of the Lie group or its Chevalley analogues, called an *ordinary graph automorphism*. Let us suppose that α is such an automorphism, of order t, and call it the *twisting automorphism*. We now define the *twisted Chevalley group* ${}^tX_n(q, q^t)$ to be set of elements of $X_n(q^t)$ that are fixed by the quotient of the twisting automorphism and the field automorphism induced by the Frobenius map $x \to x^q$ of F_{q^t}.

The particular cases are

$$\begin{aligned} &{}^2A_n(q, q^2) = U_{n+1}(q), \quad (n \geqslant 2) \\ &{}^2D_n(q, q^2) = O_{2n}^-(q), \quad (n \geqslant 3) \\ &{}^3D_4(q, q^3), \\ &{}^2E_6(q, q^2), \end{aligned}$$

the last two families being discovered by Steinberg. We usually abbreviate ${}^tX_n(q, q^t)$ to ${}^tX_n(q)$.

A further modification yields the infinite families of simple groups discovered by Suzuki and Ree. If the Dynkin diagram of X_n has a p-fold edge ($p = 2, 3$), then over fields of characteristic p the Chevalley group is independent of the direction of the arrowhead on that edge. In other words, there is an isomorphism between $X_n(p^f)$ and $Y_n(p^f)$, where Y_n is the diagram obtained from X_n by reversing the direction of the arrowhead. Thus for example, $B_n(2^f) = C_n(2^f)$, or in classical notation $O_{2n+1}(2^f) = S_{2n}(2^f)$, as we have already seen.

In the three cases $B_2 = C_2$, G_2, F_4, the diagram has an automorphism reversing the direction of the p-fold edge, and so over fields of the appropriate characteristic p, the resulting Chevalley groups have a new type of graph automorphism, which we call an *extraordinary graph automorphism*, whose square is the field automorphism induced by the Frobenius map $x \to x^p$.

Now when $q = p^{2m+1}$ is an odd power of this p, the field automorphism induced by $x \to x^{p^{m+1}}$ has the same square, and the elements of $X_n(p^{2m+1})$ fixed by the quotient of these two automorphisms form a new type of twisted Chevalley group, called ${}^2X_n(*, p^{2m+1})$, and usually abbreviated to ${}^2X_n(p^{2m+1})$. The particular cases are

$${}^2B_2(*, 2^{2m+1}) = {}^2C_2(*, 2^{2m+1}),$$ a *Suzuki group*,

$${}^2G_2(*, 3^{2m+1}),$$ a *Ree group* of characteristic 3,

$${}^2F_4(*, 2^{2m+1}),$$ a *Ree group* of characteristic 2.

These groups are often written

$$Sz(q), R_1(q), R_2(q),$$

where $q = p^{2m+1}$, but the subscripts 1 and 2 for the two types of Ree group can be omitted without risk of confusion.

It turns out that the first group in each of these families is not simple, but all the later ones are simple. The cases are

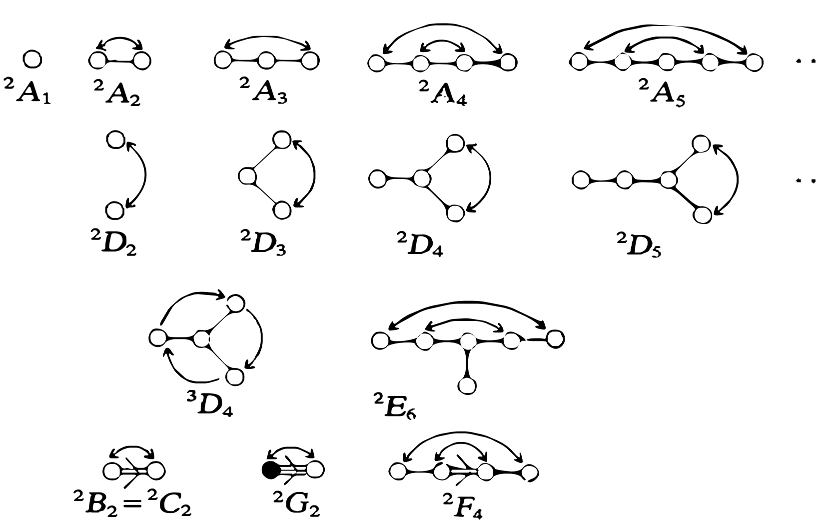
${}^2B_2(2) = 5:4$, the Frobenius group of order 20,

${}^2G_2(3) = L_2(8):3$, the extension of the simple group $L_2(8)$ of order 504 by its field automorphism,

${}^2F_4(2) = T.2$, where T is a simple group not appearing elsewhere in the classification of simple groups, called the *Tits group*.

Table 4 gives the Dynkin diagrams and twisting automorphisms for all the twisted groups. We remark that what we have called simply the twisted Chevalley groups are more fully called

Table 4. Diagrams and twisting automorphisms for twisted groups



the *adjoint* twisted Chevalley groups, and that, like the untwisted groups, they have certain multiple covers called the *universal twisted Chevalley groups*. We have included the untwisted group ${}^2A_1 = A_1$ in the table, and also the case ${}^2D_2$ obtained by twisting a disconnected diagram, and the repetition ${}^2A_3 = {}^2D_3$. The reason is again that these special cases illuminate relations between some classical groups.

3. Multipliers and automorphisms of Chevalley groups

The Schur multiplier has order de, and the outer automorphism group has order dfg, where the order of the base field is $q = p^f$ (p prime), and the numbers d, f, g are tabulated in Table 5. (An entry '2 if ...' means 1 if *not*.)

The Schur multiplier is the direct product of groups of orders d (the *diagonal multiplier*) and e (the *exceptional multiplier*). The diagonal multiplier extends the adjoint group to the corresponding universal Chevalley group. The exceptional multiplier is always a p-group (for the above p), and is trivial except in finitely many cases.

The outer automorphism group is a semidirect product (in this order) of groups of orders d (*diagonal automorphisms*), f (*field automorphisms*), and g (*graph automorphisms modulo field automorphisms*), except that for

$$B_2(2^f),\ G_2(3^f),\ F_4(2^f)$$

the (extraordinary) graph automorphism squares to the generating field automorphism. The groups of orders d, e, f, g are cyclic except that 3! indicates the symmetric group of degree 3, and orders written as powers indicate the corresponding direct powers of cyclic groups.

4. Orders of the Chevalley groups

In Table 6, the parameters have been chosen so as to avoid the 'generic' isomorphisms.
N is the order of the universal Chevalley group.
N/d is the order of the adjoint Chevalley group.

5. The simple groups enumerated

The exact list of finite simple groups is obtained from the union of

- the list of Chevalley groups (Table 5)
- the alternating groups A_n, for $n \geq 5$
- the cyclic groups of prime order
- the 26 sporadic groups, and finally
- the Tits simple group $T = {}^2F_4(2)'$

by taking into account the exceptional isomorphisms below. Each of these isomorphisms is either between two of the above groups, or between one such group and a non-simple group. For the reader's convenience, we give both the Chevalley and classical notations.

The exceptional isomorphisms:

$A_1(2) = L_2(2) \cong S_3$	$B_2(2) = S_4(2) \cong S_6$
$A_1(3) = L_2(3) \cong A_4$	$G_2(2) \cong {}^2A_2(3).2 = U_3(3).2$
$A_1(4) = L_2(4) \cong A_5$	${}^2A_2(2) = U_3(2) \cong 3^2.Q_8$
$A_1(5) = L_2(5) \cong A_5$	${}^2A_3(2) = U_4(2) \cong B_2(3) = S_4(3)$
$A_1(7) = L_2(7) \cong A_2(2) = L_3(2)$	${}^2B_2(2) = Sz(2) \cong 5:4$
$A_1(9) = L_2(9) \cong A_6$	${}^2G_2(3) = R(3) \cong A_1(8).3 = L_2(8).3$
$A_3(2) = L_4(2) \cong A_8$	${}^2F_4(2) = R(2) \cong T.2.$

6. Parabolic subgroups

With every proper subset of the nodes of the Dynkin diagram there is associated a *parabolic subgroup*, which is in structure a p-group extended by the Chevalley group determined by the subdiagram on those nodes. (Here p, as always, denotes the characteristic of the ground field $\mathbb{F}_q$.) The *maximal parabolic subgroups* are those associated to the sets containing all but one of the nodes—we shall say that such a subgroup is *correlated* to the remaining node. According to the *Borel–Tits theorem* the maximal p-local subgroups of a Chevalley group are to be found among its maximal parabolic subgroups. These statements hold true for the twisted groups, provided we replace 'node' by 'orbit of nodes under the twisting automorphism'.

7. The fundamental representations

The ordinary representation theory of Chevalley groups, as recently developed by Deligne and Lusztig, is very complex, and the irreducible modular representations are not yet completely described. But certain important representations (not always irreducible) can be obtained from the representation theory of Lie groups.

The irreducible representations of a Lie group are completely classified, and can be written as 'polynomials' in certain *fundamental representations*, one for each node of the Dynkin diagram. The degrees of the representations for the various Lie groups X_n are given in Table 7. By change of field, we obtain the so-called fundamental representations of the universal Chevalley group $X_n(q)$, which are representations over the field $\mathbb{F}_q$ having the given degrees. In the classical cases these are fairly easily described geometrically—for example in $A_n(q) = L_{n+1}(q)$ the fundamental representation corresponding to the rth node is that of $SL_{n+1}(q)$ on the rth exterior power of the original vector space V.

Table 5. Automorphisms and multipliers of the Chevalley groups

Condition	Group	Definitions of d	f	g	Cases when $e \neq 1$
	$A_1(q)$	$(2, q-1)$	$q=p^f$	1	$A_1(4) \to 2$, $A_1(9) \to 3$
$n \geqslant 2$	$A_n(q)$	$(n+1, q-1)$	$q=p^f$	2	$A_2(2) \to 2$, $A_2(4) \to 4^2$, $A_3(2) \to 2$
$n \geqslant 2$	${}^2A_n(q)$	$(n+1, q+1)$	$q^2=p^f$	1	${}^2A_3(2) \to 2$, ${}^2A_3(3) \to 3^2$, ${}^2A_5(2) \to 2^2$
	$B_2(q)$	$(2, q-1)$	$q=p^f$	2 if $p=2$	$B_2(2) \to 2$
f odd	${}^2B_2(q)$	1	$q=2^f$	1	${}^2B_2(8) \to 2^2$
$n \geqslant 3$	$B_n(q)$	$(2, q-1)$	$q=p^f$	1	$B_3(2) \to 2$, $B_3(3) \to 3$
$n \geqslant 3$	$C_n(q)$	$(2, q-1)$	$q=p^f$	1	$C_3(2) \to 2$
	$D_4(q)$	$(2, q-1)^2$	$q=p^f$	3!	$D_4(2) \to 2^2$
	${}^3D_4(q)$	1	$q^3=p^f$	1	none
$n>4$, even	$D_n(q)$	$(2, q-1)^2$	$q=p^f$	2	none
$n>4$, odd	$D_n(q)$	$(4, q^n-1)$	$q=p^f$	2	none
$n \geqslant 4$	${}^2D_n(q)$	$(4, q^n+1)$	$q^2=p^f$	1	none
	$G_2(q)$	1	$q=p^f$	2 if $p=3$	$G_2(3) \to 3$, $G_2(4) \to 2$
f odd	${}^2G_2(q)$	1	$q=3^f$	1	none
	$F_4(q)$	1	$q=p^f$	2 if $p=2$	$F_4(2) \to 2$
f odd	${}^2F_4(q)$	1	$q=2^f$	1	none
	$E_6(q)$	$(3, q-1)$	$q=p^f$	2	none
	${}^2E_6(q)$	$(3, q+1)$	$q^2=p^f$	1	${}^2E_6(2) \to 2^2$
	$E_7(q)$	$(2, q-1)$	$q=p^f$	1	none
	$E_8(q)$	1	$q=p^f$	1	none

Table 6. Orders of Chevalley groups

G	N	d
$A_n(q), n \geqslant 1$	$q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1}-1)$	$(n+1, q-1)$
$B_n(q), n \geqslant 2$	$q^{n^2} \prod_{i=1}^{n} (q^{2i}-1)$	$(2, q-1)$
$C_n(q), n \geqslant 3$	$q^{n^2} \prod_{i=1}^{n} (q^{2i}-1)$	$(2, q-1)$
$D_n(q), n \geqslant 4$	$q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1} (q^{2i}-1)$	$(4, q^n-1)$
$G_2(q)$	$q^6(q^6-1)(q^2-1)$	1
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	1
$E_6(q)$	$q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)$	$(3, q-1)$
$E_7(q)$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)$	$(2, q-1)$
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	1
${}^2A_n(q), n \geqslant 2$	$q^{n(n+1)/2} \prod_{i=1}^{n} (q^{i+1}-(-1)^{i+1})$	$(n+1, q+1)$
${}^2B_2(q), q=2^{2m+1}$	$q^2(q^2+1)(q-1)$	1
${}^2D_n(q), n \geqslant 4$	$q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1} (q^{2i}-1)$	$(4, q^n+1)$
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	1
${}^2G_2(q), q=3^{2m+1}$	$q^3(q^3+1)(q-1)$	1
${}^2F_4(q), q=2^{2m+1}$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	1
${}^2E_6(q)$	$q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$	$(3, q+1)$

Table 7. Degrees of the fundamental representations of Lie and Chevalley groups. (The degrees are the numbers below or to the right of the nodes.)