#A9 INTEGERS 22 (2022)

SUMS OF CONSECUTIVE PRIME SQUARES

Janyarak Tongsomporn[1]
School of Science, Walailak University, Nakhon Si Thammarat, Thailand
tjanyarak@gmail.com

Saeree Wananiyakul
School of Science, Walailak University, Nakhon Si Thammarat, Thailand
s.wananiyakul@hotmail.com

Jörn Steuding
Department of Mathematics, Würzburg University, Würzburg, Germany
steuding@mathematik.uni-wuerzburg.de

Received: 4/16/21, Revised: 9/22/21, Accepted: 12/29/21, Published: 1/7/22

Abstract

We prove explicit bounds for the number of sums of consecutive prime squares below a given magnitude.

1. Motivation and the Main Result

Early last year the authors learned that 2020 can be represented as a sum of squares of consecutive prime numbers, namely

$$2020 = 17^2 + 19^2 + 23^2 + 29^2.$$

It is a natural question to ask what the next year with this property will be. We shall show that such a representation is a rare event.

Indeed, if $\mathrm{scp}(x)$ counts the number of sums of squares of consecutive primes below x, i.e.,

$$\mathrm{scp}(x) = \sharp \left\{ p_n^2 + p_{n+1}^2 + \ldots + p_{n-1+m}^2 \leq x \,:\, m \in \mathbb{N} \right\},$$

where p_j denotes the j-th prime number in ascending order, then $\lim_{x\to\infty} \mathrm{scp}(x)/x = 0$. The following theorem provides more precise bounds.

Theorem 1. *We have*

$$2\,\frac{x^{1/2}}{\log x} < \pi(\sqrt{x}) \leq \mathrm{scp}(x) < 28.4201\,\frac{x^{2/3}}{(\log x)^{4/3}}\,,$$

[1]Corresponding author

where the inequality on the far left is valid for $x \geq 289$ *and all those to the right for* $x \geq 377$.

Here, as usual, $\pi(N)$ is counting the number of primes $p \leq N$ and explicit bounds for this prime counting function are the main tool for proving the inequalities above; we have chosen a recent paper [1] by Pierre Dusart. The dear reader is invited to improve upon the bounds of our theorem; maybe it is even possible to prove an asymptotic formula with a main term of the form $C\,x^\alpha\,(\log x)^{-\beta}$ with constants α, β, C for the number of sums of consecutive prime squares below x; in this case one could expect the exponent α to be close to $2/3$ (because our lower bound is pretty rough). Note that we do not consider here the question whether or not an integer can have two or even more such representations or how many of these exist.

Using a computer algebra package one can verify that the next sum of squares of consecutive primes is given by the expected suspect, namely

$$2189 = 13^2 + 17^2 + 19^2 + 23^2 + 29^2.$$

A list with all integers below 5000 that can be written as a sum of consecutive prime squares can be found in the third and final section.

2. Proof of the Theorem

It is convenient to define, for fixed $m \in \mathbb{N}$, the counting function for sums of m consecutive prime squares, i.e.,

$$\mathrm{scp}_m(x) = \sharp\left\{p_n^2 + p_{n+1}^2 + \ldots + p_{n-1+m}^2 \leq x\right\}.$$

For the lower bound we first observe that the number of squares of prime numbers p^2 below or equal to x is given by $\pi(\sqrt{x})$.

In the sequel we shall use the explicit bounds

$$\frac{N}{\log N} < \pi(N) < 1.2551\,\frac{N}{\log N}, \tag{1}$$

where the left inequality is valid for $N \geq 17$ and the one on the right for $N > 1$ (see Corollary 5.2 in [1]); of course, the celebrated prime number theorem provides an asymptotic formula for $\pi(N)$ with main term $N/\log N$, however, for excluding the related error term for our analysis, we prefer the version above with the factor 1.2551. The corresponding range for these inequalities (resp. the range for x in our theorem) is also useful with respect to computer experiments.

It follows from Equation (1) that

$$\mathrm{scp}(x) \geq \mathrm{scp}_1(x) = \pi(\sqrt{x}) > \frac{\sqrt{x}}{\frac{1}{2}\log x},$$

which is valid for $x \geq 17^2 = 289$. This proves the lower bound.

The reasoning for the upper bound is a little more advanced. First we note that for $n = \mathrm{scp}(x)$ we have

$$mp_n^2 \leq p_n^2 + p_{n+1}^2 + \ldots + p_{n-1+m}^2 \leq x < p_n^2 + p_{n+1}^2 + \ldots + p_{n-1+m}^2 + p_{n+m}^2.$$

Hence, by the inequality in Equation (1),

$$\mathrm{scp}_m(x) = n \leq \pi(\sqrt{x/m}) < 1.2551\,\frac{\sqrt{x/m}}{\frac{1}{2}\log(x/m)}, \tag{2}$$

which is valid for $x > m$, which, obviously, is no severe restriction (since the largest integer $\leq x$ is a trivial upper bound for the length of a sum of consecutive primes squares $\leq x$).

To continue we shall next bound the length m of possible sums of consecutive prime squares below x. For this purpose we shall use an old result due to Barkley Rosser [3] which has been improved several times, in particular by Dusart [1], however, we prefer the more simple inequality

$$p_n > n \log n,$$

valid for all $n \in \mathbb{N}$; this lower bound is trivial for $n = 1$. We shall use this so-called Rosser theorem for the sum of the squares of the *first* primes:

$$p_1^2 + p_2^2 + \ldots + p_M^2 > \sum_{2 \leq n \leq M} (n \log n)^2.$$

If we can show that the right hand side is larger than x, then the least sum of M consecutive prime squares already exceeds the given magnitude. Assuming that this M is the least positive integer with this property, this leads to a bound for M depending on x. This estimate in combination with the previous one allows us to derive the upper bound of the theorem. Alternatively, one could also use partial summation here together with the prime number theorem, however, it is our intention to circumvent error terms.

Obviously, for $M \geq 6$,

$$\begin{aligned} \sum_{2 \leq n \leq M} (n \log n)^2 &\geq \sum_{\sqrt{M} \leq n \leq M} n^2 (\log n)^2 \\ &\geq (\tfrac{1}{2} \log M)^2 \sum_{\sqrt{M} \leq n \leq M} n^2 \geq \tfrac{1}{12} M^3 (\log M)^2, \end{aligned}$$

where we have used in the final step the well-known formula

$$1 + 2^2 + 3^2 + \ldots + M^2 = \tfrac{1}{6}\, M(M+1)(2M+1)$$

and some pen and paper. It thus follows that every sum of consecutive prime squares below x has less than roughly $x^{1/3}$ summands. For a more precise bound we observe that substituting

$$M = \lfloor 5\, x^{1/3} (\log x)^{-2/3} \rfloor \tag{3}$$

into the lower bound above yields a quantity slightly larger than x; here $\lfloor z \rfloor$ denotes the largest integer less than or equal to z. Note that one could replace the constant 5 by $124^{1/3}$, however, for the sake of readability we have chosen the rough bound. Since $M \geq 6$, this implies that $x \geq 377$.

To use this for an upper bound we first observe that we have $m < M$ in Equation (2) and

$$\log(x/m) \geq \log x - \log M \geq \log x - \log\left(5x^{1/3}\right) = \frac{2}{3} \log x - \log 5 \geq 0.395 \log x,$$

valid for $x \geq 377$; restricting x further would lead to a smaller upper bound, however, the factor 1/3 above can by this reasoning not be replaced by anything larger than 2/3. Now Equation (2) implies

$$\mathrm{scp}(x) \leq \sum_{1 \leq m \leq M} \mathrm{scp}_m(x) < 6.355 \, \frac{x^{1/2}}{\log x} \sum_{1 \leq m \leq M} m^{-1/2}.$$

In general, we have, for $\alpha \in (0, 1)$,

$$\sum_{1 \leq m \leq M} m^{-\alpha} < 1 + \sum_{2 \leq m \leq M} \int_{m-1}^{m} u^{-\alpha} \mathrm{d}u = 1 + \int_{1}^{M} u^{-\alpha} \mathrm{d}u = \frac{M^{1-\alpha} - \alpha}{1 - \alpha}.$$

This in combination with Equation (3) leads to

$$\mathrm{scp}(x) < 12.7099 \, \frac{(xM)^{1/2}}{\log x} \leq 12.7099 \cdot 5^{1/2} \, \frac{x^{2/3}}{(\log x)^{4/3}}.$$

This proves the upper bound of the theorem.

3. Explicit Sums of Consecutive Prime Squares

We conclude with a list of all integers below 5000 that can be written as a sum of consecutive prime squares:

4	9	13	25	34	38	49
74	83	87	121	169	170	195
204	208	289	290	339	361	364
373	377	458	529	579	628	650
653	662	666	819	841	890	940
961	989	1014	1023	1027	1179	1348
1369	1370	1469	1518	1543	1552	1556
1681	1731	1802	1849	2020	2189	2209
2310	2330	2331	2359	2384	2393	2397
2692	2809	2981	3050	3150	3171	3271
3320	3345	3354	3358	3481	3530	3700
3721	4011	4058	4061	4350	4489	4519
4640	4689	4714	4723	4727	4852	4899

This sequence of sums of consecutive prime squares appears as A340771 in the On-Line Encyclopedia of Integer Sequences [2] (founded by Neal Sloane in 1964). Related sequences, listed earlier in this encyclopedia, are A069484 consisting of the integers that are sums of two squares and A034707 consisting of sums of squares of any number of consecuted primes (but not squared).

Acknowledgements. The authors would like to thank National Research Council of Thailand (NRCT) and Walailak University for facilities and financial supports (grant number NRCT5-RSA63019-06 (2563NRCT322800)). Moreover, the authors are grateful to the anonymous referee for her or his valuable comments and corrections.

References

[1] P. Dusart, Explicit estimates of some functions over primes, *Ramanujan J.* **45** (2018), 227–251.

[2] The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.

[3] B. Rosser, The n-th prime is greater than $n \log n$, *Proc. Lond. math. Soc.* (2) **45** (1938), 21-44.