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GEOMETRIC PATTERNS IN THE DETERMINANT HOSOYA TRIANGLE

Matthew Blair
Department of Mathematical Sciences, The Citadel, Charleston, South Carolina
mblair1@citadel.edu

Rigoberto Flórez[1]
Department of Mathematical Sciences, The Citadel, Charleston, South Carolina
rigo.florez@citadel.edu

Antara Mukherjee[2]
Department of Mathematical Sciences, The Citadel, Charleston, South Carolina
antara.mukherjee@citadel.edu

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Abstract

The *determinant Hosoya triangle* is a triangular array of integers where each entry is the determinant of two-by-two matrices with Fibonacci-number entries. The underlying Fibonacci recursions imply similar recursions for the determinant Hosoya triangle. This paper uses a graphical approach to define magic cycles in the determinant Hosoya triangle; a linear algebra approach to identify identities and patterns in determinants of the triangle; and a geometric-combinatorial approach to highlight a multitude of patterns embedded within the triangle. The geometric approach allows seeing identities in determinants with Fibonacci-number entries.

1. Introduction

In 1976, Hosoya published an article exploring a triangular array, constructed using a double recursion [9] —its entries are products of Fibonacci numbers. Hosoya proved several interesting results; each of which connects a geometric or graphical pattern with an identity on the Fibonacci numbers which reflect the underlying Fibonacci recursion. Because of the geometric patterns involved, the results had elegant names such as magic diamond, amoeba, and crawling crab. Since 1976 several other papers studying triangles whose entries are recursively defined objects

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(numbers or polynomials) have appeared. Each of these papers continue connecting simply described geometric-graphical patterns with identities on the triangle's entries reflecting some fundamental underlying recursion. The names of these results reflect the underlying geometric patterns, for example, Star of David theorems.

The current paper continues this tradition by studying a Hosoya-like triangle with determinants of 2×2 matrices with Fibonacci numbers as its entries. The approach of connecting simply described geometric-graphical patterns with recursive identities is maintained. This paper proves the following results on this determinant triangle: Cassini-like identity, Catalan-like identity, magic cycle, hourglass, rhombus, hockey stick, and column generation properties.

The *Hosoya triangle* [9] is a triangular array where the entries are products of Fibonacci numbers. This triangle has many geometric properties providing a good tool to explore properties of Fibonacci numbers (see for example, [1, 4–7, 9, 10]).

In this paper, instead of using products of Fibonacci numbers, we utilize determinants of 2×2 matrices (with Fibonacci numbers as entries) to obtain a new triangle; we call this new triangle the *determinant Hosoya triangle* denoted by $\mathcal{H}$ (see Table 1). This triangle was originally discovered by Sloane ([11, A108038]). Its entries are formally described by,

$$H(r,k) = H(r-1,k)+H(r-2,k) \text{ and } H(r,k) = H(r-1,k-1)+H(r-2,k-2) \quad (1)$$

with initial conditions $H(1,1) = 0, H(2,1) = H(2,2) = 1$, and $H(3,2) = 3$ where $r \geq 1$ and $1 \leq k \leq r$. For brevity, we write $H_{r,k}$ instead of $H(r,k)$ for the rest of the paper. Throughout the paper, for simplicity and if there is no ambiguity, we say triangle to refer to the determinant Hosoya triangle and say points or Fibonacci matrices to refer to the entries of the triangle.

								0								
							1		1							
						1		3		1						
					2		4		4		2					
				3		7		5		7		3				
			5		11		9		9		11		5			
		8		18		14		16		14		18		8		
	13		29		23		25		25		23		29		13	
21		47		37		41		39		41		37		47		21

Table 1: Determinant Hosoya triangle $\mathcal{H}$.

A combinatorial interpretation of the determinant Hosoya triangle can be found in [2]. We take a linear algebra approach to this triangle to obtain a closed formula for the sum of any row within the triangle. The linear algebra approach allows discovery of a Cassini-like identity nested within the triangle. Using a geometric and a linear algebraic approach we show that the determinant of 2×2 matrices comprised of entries within specific rows, results in a multiple of a Fibonacci number. We call this the hourglass property because the specific entries of specific rows expand outward forming an hourglass shape. This hourglass property can be seen in Table 3.

We also consider a graph theoretic approach by considering the adjacent entries within the triangle to be vertices of a graph to obtain a directed graph and a specific identity associated with the graph. We call this identity the magic cycle, which can be seen in Figure 1.

Using the well-defined geometry of the triangle we show that the difference between any two points within the same row is a product of Fibonacci numbers. Using the geometric approach we also see that any two entries that are vertically aligned within the triangle will have an alternating sum or difference resulting in a Fibonacci number. For example, in Table 1, the sums of vertically aligned entries in rows 4 and 6 are

$$2 + 11 = 4 + 9 = 4 + 9 = 2 + 11 = 13 = F_7.$$

However, if we increase their vertical distance, then the differences between vertically aligned entries in rows 4 and 8 are

$$|2 - 23| = |4 - 25| = |4 - 25| = |2 - 23| = 21 = F_8.$$

The geometric approach shows that the sum of entries in the triangle displaying a hockey-stick pattern equals a nearby point. Note that the entries $H_{5,2}$, $H_{6,3}$, $H_{8,4}$, $H_{10,5}$ in Table 4 form the hockey stick within the triangle; their sum is $H_{11,6} = 105$, where $H_{r,k}$ is the entry in position k of the rth row. This pattern can be seen for everywhere within the triangle. In addition, we obtain two identities involving determinant of Fibonacci matrices (points of the triangle) from the hockey stick pattern. One of the identities is the following.

Let r, j, and t be positive integers where $r > 1$ and $1 \leq j \leq r + 2$, then

$$\begin{vmatrix} F_{r-j+4} & F_{r-j+3} \\ F_j & F_{j+1} \end{vmatrix} + \sum_{i=1}^{t} \begin{vmatrix} F_{r+i-j+3} & F_{r+i-j+2} \\ F_{i+j} & F_{i+j+1} \end{vmatrix} = \begin{cases} \begin{vmatrix} F_{r+t-j+4} & F_{r+t-j+3} \\ F_{t+j} & F_{t+j+1} \end{vmatrix}, & \text{if } t \text{ is even;} \\[2ex] \begin{vmatrix} F_{r+t-j+3} & F_{r+t-j+2} \\ F_{t+j+1} & F_{t+j+2} \end{vmatrix}, & \text{if } t \text{ is odd.} \end{cases}$$

We provide the details and proofs for the (hockey stick) identities in Section 4.3.

Finally a combinatorial approach using the underlying recursions of the triangle and the Fibonacci numbers yields a generating function.

We can see a plethora of patterns and determinant identities emerge from this triangle due to the recursive structure of the diagonals of the triangle. Therefore, we believe that these properties may hold in other recursively-defined geometric arrays and other recursively-generated sequences. It is also possible to derive graph theoretic results with matrices embedded in this triangle (see [3]).

2. The Determinant Hosoya Triangle

In this section we present a collection of identities connecting Fibonacci numbers and determinants of 2×2 Fibonacci matrices using a geometric approach. Proposition 1 tells us that the points of the triangle are determinants of matrices with Fibonacci numbers as entries. We show that summing these determinants gives a combination of Lucas and Fibonacci numbers. For example, we show that the sum of all points in the rth row of the triangle is $(rL_{r+2} - 4F_r)/5$.

Every diagonal in the triangle satisfies the underlying Fibonacci recursion albeit with different initial conditions. For instance, the fifth backslash diagonal in Table 1 is $3, 11, 14, 25, 39, \ldots$. This sequence corresponds with the generalized Fibonacci number $G_n^{(5)} = G_{n-1}^{(5)} + G_{n-2}^{(5)}$, where $G_1^{(5)} = 3$ and $G_2^{(5)} = 11$. In general, the entries of the mth diagonal of this triangle are given by the generalized Fibonacci number $G_n^{(m)} = G_{n-1}^{(m)} + G_{n-2}^{(m)}$, where $G_1^{(m)} = F_{m-1}$ and $G_2^{(m)} = L_m$ with F_{m-1} and L_m being the Fibonacci and Lucas numbers. Note that another triangle that also has determinants as entries is the Narayana numbers Triangle (see [11, A001263]).

Proposition 1 ([2] Proposition 2.1)**.** *If r, k are positive integers with $k \le r$ and $H_{r,k}$ as defined in* (1)*, then*

1. $H_{r,k} = F_{k+1}F_{r-k+2} - F_kF_{r-k+1}$. *Thus,*
$$H_{r,k} = \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix}.$$

2. $H_{r,k} = F_{k-1}F_{r-k+2} + F_kF_{r-k}$. *Thus,*
$$H_{r,k} = \begin{vmatrix} F_{r-k+2} & -F_{r-k} \\ F_k & F_{k-1} \end{vmatrix}.$$

Proposition 2. *If r and k are positive integers with $k \le r$, then the following hold*

$$\begin{aligned} \sum_{k=1}^{r} H_{r,k} = \sum_{k=1}^{r} \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix} &= F_{r+1} + \sum_{k=1}^{r-2} F_{k+1}F_{r-k+1} \\ &= [rL_{r+2} - 4F_r]/5 \\ &= ((7r-4)F_{r-1} + 4(r-1)F_{r-2})/5. \end{aligned}$$

Proof. From Proposition 1, we know that

$$\sum_{k=1}^{r} H_{r,k} = \sum_{k=1}^{r}(F_{k+1}F_{r-k+2} - F_{r-k+1}F_k) = \sum_{k=1}^{r}(F_kF_{r-k} + F_{k-1}F_{r-k+2}).$$

This and the identity $\sum_{i=0}^{n} F_iF_{n-i} = [nL_n - F_n]/5$ (see [11, A001629]), imply that

$$\begin{aligned}\sum_{k=1}^{r}[F_kF_{r-k} + F_{k-1}F_{r-k+2}] &= (rL_r - F_r)/5 + ((r+1)L_{r+1} - F_{r+1} - 5F_r)/5 \\ &= [rL_{r+2} - 4F_r]/5 \\ &= [(7r-4)F_{r-1} + 4(r-1)F_{r-2}]/5.\end{aligned}$$

This completes the proof. □

3. Identities Involving Determinants

In the first part of this section, we define a graph whose vertices are points in the determinant Hosoya triangle. We define a signed 4-cycle within the triangle. Using the signs of the edges of the cycle, we sum the vertices of the 4-cycle. The sum is a point that is the nearest neighbor to one of the four vertices. In the second part of this section, we obtain an identity for determinants. This identity is similar to the Cassini and Catalan identities. Therefore, we call this identity the Cassini identity for determinants.

3.1. Magic Cycle

Proposition 3 extends the Magic Diamond property in [9, Section 3] and [10], from the Hosoya triangle to the determinant Hosoya triangle. For the statement of Proposition 3, we borrow terminology from graph theory (see, for example, [14]). Recall, a *signed graph* is a graph in which each edge has a positive or negative sign as label. A cycle is called *balanced* if the product of the signs associated with the edges is positive (Figure 1). We define the entries of the triangle to be the vertices of a graph, where two consecutive vertices are adjacent if they are in the same diagonal of the triangle. For the following proposition, we add or subtract the entries located at the adjacent vertices, depending on the edge signs. A good feature of this proposition is that the location of signs on the graph does not matter; it can be consecutive or alternating and that the direction of the edges does not matter, but a direction is chosen at the the beginning and then followed in a cyclic pattern (see Figure 1 (b) and (c)). For example, given the 4-cycle on right-hand side of Figure 1, we see that given the balanced signed cycle $v_1v_2v_3v_4v_1$, where the edge signs are $\mathrm{sign}(v_1v_2) = -$; $\mathrm{sign}(v_2v_3) = +$; $\mathrm{sign}(v_3v_4) = -$; and $\mathrm{sign}(v_4v_1) = +$, the addition of the vertices of the cycle in a circular order is either $+v_1 - v_2 + v_3 - v_4$ or $-v_1 + v_4 - v_3 + v_2$ (if the direction of the arrows were flipped).

It is important to note that Proposition 3 extends into the negative entries of the determinant Hosoya triangle, however the goal of this paper is to explore the positive entries and further exploration is left to the reader. In addition, we note that if the cycle is unbalanced the conclusion of Proposition 3 does not hold.

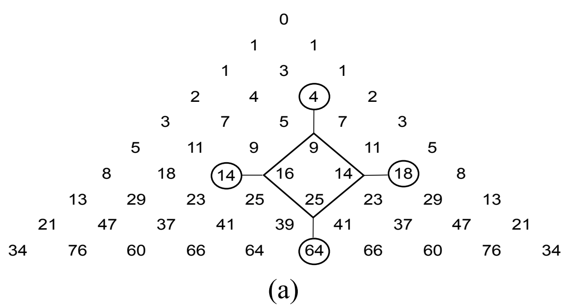

Figure 1: Magic cycle in the determinant Hosoya triangle.

We say that a vertex P is *external* to the 4-cycle C if it is located (in the neighborhood) in either the vertical line or the horizontal line crossing C. Figure 1 Part (a) shows four external vertices to C: $P_1 = 4$, $P_2 = 18$, $P_3 = 64$, and $P_4 = 14$.

Proposition 3. *Let C be a 4-cycle embedded in $\mathcal{H}$ and assume that none of its vertices lie on the border of $\mathcal{H}$. Let P_i be an external vertex of C, where $i \in \{1, 2, 3, 4\}$. The cycle C is balanced if and only if the addition of all vertices of C in a circular order is equal to $\pm P_i$ for some $i \in \{1, 2, 3, 4\}$.*

Proof. We prove this proposition by cases. We start by observing that there are 32 unique graphs which can be generated with any four adjacent entries. This is because a graph can be clockwise directional or counter-clockwise directional, and each of the four edges are either positive or negative. Combinatorially, this gives us 2^5 unique graphs. However, it is easy to see that the two graphs who share the same edge signs, but are read in opposite directions, are simply negative versions of each other. Therefore, we will only consider clockwise graphs.

We can show that the entries located between the external vertices located in either the vertical line or the horizontal line crossing the cycle can be represented as a linear combination of the vertices of the 4-cycle. Following the notation given in Figure 1 Parts (b) and (c), we let $v_1 = H_{r,k}, v_2 = H_{r+1,k+1}, v_3 = H_{r+2,k+1}$, and $v_4 = H_{r+1,k}$. Thus,

$$H_{r,k-1} = v_3 - v_4;\ H_{r-1,k-1} = v_2 - v_1;\ H_{r-1,k} = v_4 - v_1;\ H_{r,k+1} = v_3 - v_2; \quad (2)$$

$$H_{r+3,k+2} = v_4 + v_3;\ H_{r+3,k+1} = v_2 + v_3;\ H_{r+2,k} = v_1 + v_4;\ H_{r+2,k+2} = v_1 + v_2.$$

Due to the recursive definition of the triangle, we can generate each point P by combining two specific points listed above. In particular, we can see that each point can be generated in two different ways. Therefore, we show both ways and thus account for all 8 graphs which result in a point P. Note that to find $-P$, we simply flip every edge sign.

From (2), we use the recursive definition of the triangle to show that linear combinations of the vertices result in an adjacent external vertex.

Case 1. $H_{r+4,k+2} = H_{r+3,k+2} + H_{r+2,k+2} = v_1 + v_2 + v_3 + v_4$.

Case 2. $H_{r-2,k-1} = H_{r-1,k} - H_{r,k+1} = -v_1 + v_2 - v_3 + v_4$.

Case 3. $H_{r+1,k+2} = H_{r-1,k} + H_{r,k+1} = -v_1 - v_2 + v_3 + v_4$.

Case 4. $H_{r+1,k-1} = H_{r+3,k+1} - H_{r+2,k} = -v_1 + v_2 + v_3 - v_4$.

This shows that balanced graphs will always result in one of the four external vertices. Now we show how unbalanced graphs cannot produce an external vertex. For brevity, we only show four cases as the other four cases are the same except negative.

Case 1. $v_1 + v_2 + v_3 - v_4 = H_{r+2,k+2} + H_{r,k-1}$.

Case 2. $v_1 + v_2 - v_3 + v_4 = H_{r+2,k+2} - H_{r,k-1}$.

Case 3. $v_1 - v_2 + v_3 + v_4 = -H_{r-1,k-1} + H_{r+3,k+2}$.

Case 4. $-v_1 + v_2 + v_3 + v_4 = H_{r-1,k-1} + H_{r+3,k+2}$.

This completes all 16 cases where 8 are all possible balanced graphs and 8 are all possible unbalanced graphs. Due to the definition in (1), we show that only the balanced graphs produce an external vertex, while the unbalanced graphs cannot. This completes the proof. □

We can view the above proposition visually in Figure 1. The entries 16, 9, 14, and 25 are the vertices and the edges connecting them will either be a negative or positive edge. If we have a balanced graph (even number of negative edges) embedded within the triangle, then it will produce one of the encircled entries. Once a vertex is fixed, let us say v_1 (in Figure 1), there are two closed paths $v_1v_2v_3v_4$ or $v_1v_4v_3v_2$, in both cases the absolute value of the results are the same. For example, from Figure 1 Parts (a) and (b) we can see that $v_1 = 9$, $v_2 = 14$, $v_3 = 25$, and $v_4 = 16$. Using Figure 1 Parts (b) we obtain that $+v_1-v_2+v_3-v_4 = H_{4,3}$. Thus, $9-14+25-16 = 4$. Using Figure 1 Parts (c) we obtain that $-v_1 + v_4 - v_3 + v_3 = -H_{4,3}$. Thus, $-9 + 16 - 25 + 14 = -4$. From Figure 1 Part (a), we have these five out of eight balanced cycles.

1.
$$\underbrace{H_{r-2,k-1}}_{P=4} = \underbrace{H_{r,k}}_{9} \underbrace{-}_{edge} \underbrace{H_{r+1,k+1}}_{14} \underbrace{+}_{edge} \underbrace{H_{r+2,k+1}}_{25} \underbrace{-}_{edge} \underbrace{H_{r+1,k}}_{16}.$$

2.
$$\underbrace{H_{r+1,k-1}}_{P=14} = \underbrace{-}_{edge} \underbrace{H_{r,k}}_{9} \underbrace{+}_{edge} \underbrace{H_{r+1,k+1}}_{14} \underbrace{+}_{edge} \underbrace{H_{r+2,k+1}}_{25} \underbrace{-}_{edge} \underbrace{H_{r+1,k}}_{16}.$$

3.
$$\underbrace{H_{r+4,k+2}}_{P=64} = \underbrace{H_{r,k}}_{9} \underbrace{+}_{edge} \underbrace{H_{r+1,k+1}}_{14} \underbrace{+}_{edge} \underbrace{H_{r+2,k+1}}_{25} \underbrace{+}_{edge} \underbrace{H_{r+1,k}}_{16}.$$

4.
$$\underbrace{H_{r+1,k+2}}_{P=18} = \underbrace{-}_{edge} \underbrace{H_{r,k}}_{9} \underbrace{-}_{edge} \underbrace{H_{r+1,k+1}}_{14} \underbrace{+}_{edge} \underbrace{H_{r+2,k+1}}_{25} \underbrace{+}_{edge} \underbrace{H_{r+1,k}}_{16}.$$

5.
$$-\underbrace{H_{r,k+2}}_{-P=-18} = \underbrace{H_{r,k}}_{9} \underbrace{+}_{edge} \underbrace{H_{r+1,k+1}}_{14} \underbrace{-}_{edge} \underbrace{H_{r+2,k+1}}_{25} \underbrace{-}_{edge} \underbrace{H_{r+1,k}}_{16}.$$

3.2. Cassini-like Identity

In this section we present a Cassini-like identity within the determinant Hosoya triangle. This identity is obtained by computing the determinants of 2×2 matrices represented as a rhombus embedded within the triangle similar to the one seen in Figure 1. For example, in the case of the rhombus seen in Figure 1, the determinant of the matrix is given by

$$\begin{vmatrix} 16 & 9 \\ 25 & 14 \end{vmatrix} = -1.$$

In addition, we observe that to obtain the Cassini-like identity, we compute the determinants of matrices which in turn have determinants as their entries. This follows from the fact that each entry of the triangle is represented as a determinant as shown in Proposition 2.

In the following proposition, we present a *Cassini-like* identity (on the variable r) in the determinant Hosoya triangle. For the proof of this proposition and Corollary 6 we use a 2×2 rhombus R with the notation given in Figure 2.

Proposition 4 (Cassini-like identity). *If $1 \le k \le r$ with $r, k \in \mathbb{Z}$, then*

$$H_{r,k}H_{r,k+1} - H_{r-1,k}H_{r+1,k+1} = (-1)^r.$$

Geometric Proof. Suppose that we have a rhombus R in a general position in the triangle (see Figure 2). Suppose that the corner points of R, in the triangle, are

$$v_1 = H_{r-1,k}, \quad v_2 = H_{r,k+1}, \quad v_3 = H_{r+1,k+1}, \quad \text{and} \quad v_4 = H_{r,k}. \tag{3}$$

Note that R can be associated to the determinant $v_4v_2 - v_1v_3$. Now we shift the rhombus through the diagonal D_1 containing v_1 and v_2 and the diagonal D_2 containing v_3 and v_4 to the left-hand side of the triangle.

For the simplicity of the discussion, when we shift R from a position A to another position B, we say that R in position B is a "new rhombus". This allows us to have an inductive proof, defining a sequence, $R_1, R_2, \dots, R_{k-1}$, of new rhombuses.

As a first step we shift R to a new rhombus, denoted by R_1, with corner points

$$v_1^{(1)} = H_{r-2,k-1}; \quad v_2^{(1)} = H_{r-1,k}; \quad v_3^{(1)} = H_{r,k}; \quad \text{and} \quad v_4^{(1)} = H_{r-1,k-1}.$$

The right-hand side of the second equation in (1) implies that the left-hand side of the equation given in the statement of this proposition is equal to

$$\begin{aligned}H_{r,k}(H_{r-1,k} + H_{r-2,k-1}) - H_{r-1,k}(H_{r,k} + H_{r-1,k-1}) = \\ H_{r,k}H_{r-2,k-1} - H_{r-1,k}H_{r-1,k-1}.\end{aligned}$$

This implies that R_1 gives that $v_1^{(1)}v_3^{(1)} - v_4^{(1)}v_2^{(1)}$. So, $(-1)R_1$ and R satisfy the same property. Similarly, we find a second new rhombus, denoted by R_2, satisfying

$$\begin{aligned}(H_{r-2,k-2} + H_{r-1,k-1})H_{r-2,k-2} - (H_{r-3,k-2} + H_{r-2,k-1})H_{r-1,k-1} = \\ H_{r-2,k-2}H_{r-2,k-1} - H_{r-3,k-2}H_{r-1,k-1}.\end{aligned}$$

This proves that R_2, with corner points

$$v_1^{(2)} = H_{r-3,k-2}; \quad v_2^{(2)} = H_{r-2,k-1}; \quad v_3^{(2)} = H_{r-1,k-1}; \quad \text{and} \quad v_4^{(2)} = H_{r-2,k-2},$$

satisfies the same property that R satisfies. Inductively, we shift R through the diagonals D_1 and D_2 to a k-th new rhombus, denoted by R_{k-1}. This gives that R_{k-1} satisfies

$$(-1)^k(v_4^{(k-1)}v_2^{(k-1)} - v_1^{(k-1)}v_3^{(k-1)}),$$

where

$$v_1^{(k-1)} = H_{r-k,1};\ v_2^{(k-1)} = H_{r-k+1,2};\ v_3^{(k-1)} = H_{r-k+2,2};\ \text{and}\ v_4^{(k-1)} = H_{r-k+1,1},$$

(note that the two corner points are on the border of the triangle). Therefore,

$$\begin{aligned}&(-1)^{k-1}\left(v_4^{(k-1)}v_2^{(k-1)} - v_1^{(k-1)}v_3^{(k-1)}\right)\\ &\qquad = (-1)^{k-1}\left(H_{r-k+1,1}H_{r-k+1,2} - H_{r-k,1}H_{r-k+2,2}\right)\\ &\qquad = (-1)^{k-1}(F_{r-k}(F_{r-k+1} + F_{r-k-1}) - F_{r-k-1}(F_{r-k+2} + F_{r-k})).\end{aligned}$$

After some simplifications, we obtain that this last equality is equivalent to

$$(-1)^{k-1}(F_{r-k}^2 - F_{r-k-1}F_{r-k+1}) = (-1)^r. \qquad \square$$

We now provide a second proof for this proposition.

Algebraic Proof of Proposition 4. From the definition given in (1), we can see that the k-th column of Table 5 is a generalized Fibonacci number of the form $G_{n,k} = G_{n-1,k} + G_{n-2,k}$ where $G_{1,k} = F_{k-1}$ and $G_{2,k} = L_k$ (the first and the second entries of the column k) for $n \geq 1$ and $k \geq 1$. From this we have that $G_{n,k} = L_k F_{n-1} + F_{k-1} F_{n-2}$. Thus,

$$H_{n,k} = L_k F_{n-k} + F_{k-1} F_{-k+n-1}. \tag{4}$$

Using this part we have that

$$\begin{aligned} H_{r,k} H_{r,k+1} - H_{r-1,k} H_{r+1,k+1} &= (F_k L_k - F_{k-1} L_{k+1})(F_{r-k-2} F_{r-k} - F^2_{r-k-1}) \\ &= (F_k F_{k+1} - F_{k-1} F_{k+2})(F_{r-k-2} F_{r-k} - F^2_{r-k-1}) \\ &= (-1)^{k+1}(-1)^{r-k+1} = (-1)^r. \end{aligned}$$

This completes the proof. □

We observe that the identity in the statement of Proposition 4 has a similarity with the Cassini identity when we look at only the indices involving r (in the proof of the proposition, the connection is clear). The Cassini-like identity using the determinant notation can be written as

$$\begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix} \cdot \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_{k+1} & F_{k+2} \end{vmatrix} - \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_k & F_{k+1} \end{vmatrix} \cdot \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_{k+1} & F_{k+2} \end{vmatrix} = (-1)^r.$$

If we focus solely on the r indices —setting $H(r) := H_{r,i}$ for every i, thus, ignoring k—, we have $H(r)^2 - H(r-1) \cdot H(r+1) = (-1)^r$. This expression has the same functional form as the Cassini identity and highlights a strong connection between the determinant Hosoya triangle and well-known Fibonacci identities. We can extend this identity further as we increase the size of our rhombus structure. It is a *Catalan-like* identity.

Proposition 5 (Catalan-like identity)**.** *If $1 \leq k \leq r$ and $i \leq r$, then*

$$\begin{vmatrix} H_{r,k} & H_{r-i-1,k} \\ H_{r+i+1,k+i+1} & H_{r,k+i+1} \end{vmatrix} = (-1)^{r-i} F^2_{i+1}.$$

Proof. This proposition can be proved geometrically in the same manner as in Proposition 4. However, for brevity, we give an algebraic proof.

Using (4) we have that

$$\begin{aligned} &H_{r,k} H_{r,k+1+i} - H_{r-1-i,k} H_{r+1+i,k+1+i} = \\ &\qquad (L_k F_{i+k} - F_{k-1} L_{i+k+1})(F_{r-k} F_{r-k-i-2} - F_{r-k-1} F_{r-k-i-1}). \end{aligned}$$

It is easy to verify that $L_kF_{i+k} - F_{k-1}L_{i+k+1} = F_{k+1}F_{i+k} - F_{k-1}F_{i+k+2}$. From [13, Identity 20a], we have that

$$\begin{aligned} F_{r-k}F_{r-k-i-2} - F_{r-k-1}F_{r-k-i-1} &= (-1)^{r-k+i+1}F_{i+1} \quad \text{and} \\ F_{k+1}F_{i+k} - F_{k-1}F_{i+k+2} &= (-1)^{k+1}F_{i+1}. \end{aligned}$$

Therefore, $H_{r,k}H_{r,k+1+i} - H_{r-1-i,k}H_{r+1+i,k+1+i} = (-1)^{r-i}F_{i+1}^2$. Note that the left-hand side of this equation is equivalent to

$$\begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix} \cdot \begin{vmatrix} F_{r-k+1-i} & F_{r-k-i} \\ F_{k+1+i} & F_{k+2+i} \end{vmatrix} - \begin{vmatrix} F_{r-k+1-i} & F_{r-k-1-i} \\ F_k & F_{k+1} \end{vmatrix} \cdot \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_{k+1+i} & F_{k+2+i} \end{vmatrix}.$$

This completes the proof. □

3.3. Identities from a Rhombus

The determinant Hosoya triangle has strong geometric properties. We, in particular, would like to utilize linear algebra techniques to highlight these properties. In Figure 2, we can see the triangle with specific points highlighted. These points can be viewed as a matrix, but in this paper we view these entries as a rhombus to show the geometry. We would like to note that these entries are specifically grouped. We consider 4×4 rhombuses in the triangle, similar to the one seen in Figure 2. Corollary 6 evaluates determinants from this type of 4×4 rhombus. The three parts of Corollary 3.4 are illustrated by the following examples: First we form a 2×2 matrix with all corner points of the outer 4×4 rhombus, the second matrix is comprised of the points in the inner 2×2 rhombus and finally we have two 2×2 matrices using the non-corner points in the parallel side of the rhombus. Note that the points within the ellipses are located in the diagonal and the anti-diagonal of the matrices.

$$\begin{vmatrix} 14 & 4 \\ 64 & 18 \end{vmatrix} = -4; \qquad \begin{vmatrix} 16 & 9 \\ 25 & 14 \end{vmatrix} = -1; \qquad \begin{vmatrix} 9 & 41 \\ 23 & 5 \end{vmatrix} + \begin{vmatrix} 25 & 7 \\ 11 & 39 \end{vmatrix} = 0.$$

Corollary 6. *If $r, k \in \mathbb{Z}_{>0}$ with $k \le r$, then these hold*

1. *The determinant of the corner points of any 4×4 rhombus (see encircled entries in Figure 2) is given by,*

$$\begin{aligned} \begin{vmatrix} H_{r,k} & H_{r-3,k} \\ H_{r+3,k+3} & H_{r,k+3} \end{vmatrix} &= \begin{vmatrix} \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_k & F_{k+1} \end{vmatrix} & \begin{vmatrix} F_{r-k-1} & F_{r-k-2} \\ F_k & F_{k+1} \end{vmatrix} \\ \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_{k+3} & F_{k+4} \end{vmatrix} & \begin{vmatrix} F_{r-k-1} & F_{r-k-2} \\ F_{k+3} & F_{k+4} \end{vmatrix} \end{vmatrix} \\ &= (-1)^r F_3^2 = (-1)^r 4. \end{aligned}$$

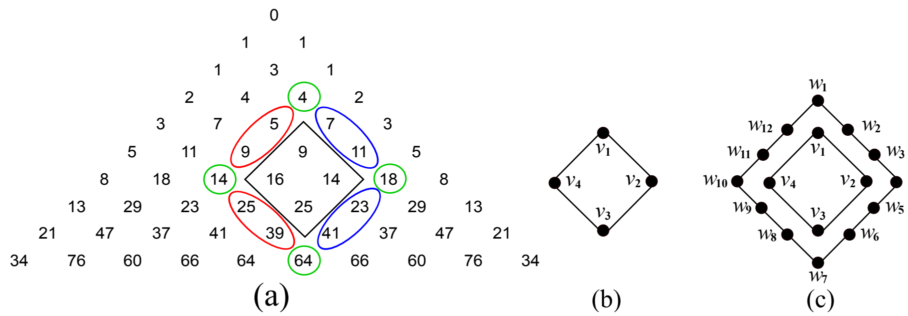

Figure 2: Geometry in the determinant Hosoya triangle.

2. *The determinant of the corner points of any 2×2 rhombus (see the rhombus in Figure 2 Part (b) or the internal rhombus in Part (a)) is given by*

$$\begin{aligned}\begin{vmatrix} H_{r,k+1} & H_{r-1,k+1} \\ H_{r+1,k+2} & H_{r,k+2} \end{vmatrix} &= \begin{vmatrix} \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_{k+1} & F_{k+2} \end{vmatrix} & \begin{vmatrix} F_{r-k} & F_{r-k+1} \\ F_{k+1} & F_{k+2} \end{vmatrix} \\ \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_{k+2} & F_{k+3} \end{vmatrix} & \begin{vmatrix} F_{r-k} & F_{r-k-1} \\ F_{k+2} & F_{k+3} \end{vmatrix} \end{vmatrix} \\ &= (-1)^r F_1^2 = (-1)^r.\end{aligned}$$

3. *Consider the non-corner point of any 4×4 rhombus (see w_2, w_3, w_5, w_6, w_8, w_9, w_{11} and w_{12} in Figure 2 Part (c) or the ellipses in Part (a)). Then*

$$\begin{vmatrix} w_{11} & w_6 \\ w_5 & w_{12} \end{vmatrix} + \begin{vmatrix} w_9 & w_2 \\ w_3 & w_8 \end{vmatrix} = \begin{vmatrix} H_{r-1,k} & H_{r+2,k+3} \\ H_{r+1,k+3} & H_{r-2,k} \end{vmatrix} + \begin{vmatrix} H_{r+1,k+1} & H_{r-2,k+1} \\ H_{r-1,k+2} & H_{r+2,k+2} \end{vmatrix} = 0.$$

That is,

$$\begin{vmatrix} \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_{k} & F_{k+1} \end{vmatrix} & \begin{vmatrix} F_{r-k+1} & F_{r-k} \\ F_{k+3} & F_{k+4} \end{vmatrix} \\ \begin{vmatrix} F_{r-k} & F_{r-k-1} \\ F_{k+3} & F_{k+4} \end{vmatrix} & \begin{vmatrix} F_{r-k} & F_{r-k-1} \\ F_{k} & F_{k+1} \end{vmatrix} \end{vmatrix} + \begin{vmatrix} \begin{vmatrix} F_{r-k+2} & F_{r-k+1} \\ F_{k+1} & F_{k+2} \end{vmatrix} & \begin{vmatrix} F_{r-k-1} & F_{r-k-2} \\ F_{k+1} & F_{k+2} \end{vmatrix} \\ \begin{vmatrix} F_{r-k-1} & F_{r-k-2} \\ F_{k+2} & F_{k+3} \end{vmatrix} & \begin{vmatrix} F_{r-k-2} & F_{r-k-3} \\ F_{k+2} & F_{k+3} \end{vmatrix} \end{vmatrix} = 0.$$

Proof. The proofs of Part (1) and Part (2) follows from Proposition 5. The first part holds using $i = 2$ and the second part holds using $i = 0$ and replacing k by $k + 1$.

Proof of Part (3). The points w_2, w_3, w_5, w_6, w_8, w_9, w_{11}, and w_{12} can be expressed in terms of the points that are within the 2×2 rhombus as given in Figure 2 and the notation given in (3). Thus, using the recursive definition given in (1) we have that

$$\begin{array}{lll} H_{r-2,k} = v_2 - v_1, & H_{r-1,k} = v_3 - v_4, & H_{r-2,k+1} = v_4 - v_1, \\ H_{r-1,k+2} = v_3 - v_2, & H_{r+2,k+3} = v_4 + v_3, & H_{r+1,k+3} = v_1 + v_2, \\ H_{r+1,k+1} = v_1 + v_4, & H_{r+2,k+2} = v_2 + v_3. & \end{array}$$

These imply that

$$\begin{aligned} &\begin{vmatrix} H_{r-1,k} & H_{r+2,k+3} \\ H_{r+1,k+3} & H_{r-2,k} \end{vmatrix} + \begin{vmatrix} H_{r+1,k+1} & H_{r-2,k+1} \\ H_{r-1,k+2} & H_{r+2,k+2} \end{vmatrix} = \\ &\qquad\qquad \begin{vmatrix} v_3 - v_4 & v_4 + v_3 \\ v_1 + v_2 & v_2 - v_1 \end{vmatrix} + \begin{vmatrix} v_1 + v_4 & v_4 - v_1 \\ v_3 - v_2 & v_2 + v_3 \end{vmatrix} = 0. \end{aligned}$$

This completes the proof. □

4. Geometric Properties of the Triangle

In this section, we continue describing recursive properties of geometric figures within the triangle. We prove theorems related to hourglass, hockey stick, and ladder.

4.1. Ladder Property

We look at entries which are aligned vertically within the determinant Hosoya triangle for the ladder property [5]. When we look at two entries in the triangle which are vertically aligned, either their sum or difference results in a Fibonacci multiple of a Fibonacci number. We alternate between taking the sum or difference between two points based on their vertical distance. We give an abstract representation of the ladder configuration in Table 2. Note that this property does not depend on the entries in the triangle, the ladder property only relies on the geometry.

As we mentioned above, when two points are in a ladder as seen in Table 2, their sum is a Fibonacci number. If we increase the distance by one, then their difference is also a Fibonacci number. If we continue this pattern by taking an alternating sum or difference then this produces a sequence of Fibonacci multiples of Fibonacci numbers. This phenomenon is shown in Proposition 7 below.

Proposition 7 (Ladder Property)**.** *For positive integers r, k, and non-negative integer j, the following holds:*

$$(-1)^{j-1} H_{r,k} + H_{r+2j,k+j} = F_j F_{r+j+2}.$$

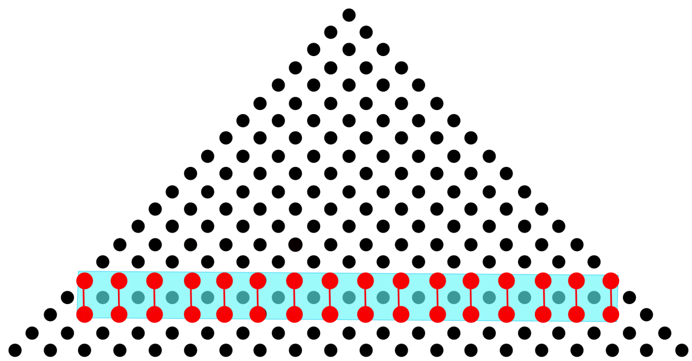

Table 2: Ladder property in the determinant Hosoya triangle.

Proof. Let L and L' be horizontal lines in the determinant Hosoya triangle with L at a distance i above L' . Let A_1, A_2 be points in L and let A_1', A_2' be points in L', where A_1 and A_1' are vertically aligned and A_2 and A_2' are vertically aligned.

Let $S(i)$ be the statement:

$$A_1 + (-1)^i A_1' = A_2 + (-1)^i A_2'.$$

We prove this statement using mathematical induction on i. The basis step is $i = 0$. Let us take two consecutive rows forming a square (see Table 2). Observe that each diagonal (slash and backslash) of this square has three points —two corner points and one inner point. Those two diagonals intersect at an inner point p. From the recursive definition given in (1) (of the entries of the determinant Hosoya triangle) and the point p, it is easy to see that the difference of the two corner points of one diagonal of the square is equal to the difference of the corner points of the other diagonal of the square. This implies that the addition of any two vertically aligned entries are equal across a row. This completes the basis step.

Suppose that the statement $S(i)$ is true for i equal to some $t > 0$. Let the points of the first line be denoted by P_j and the points of the line at the distance t from the first line be denoted by Q_s. Suppose that P_{j_1} and Q_{s_1} are vertically aligned points and suppose that P_{j_2} and Q_{s_2} are also vertically aligned points, where P_{j_1} and P_{j_2} are consecutive points and Q_{s_1} and Q_{s_2} are consecutive points. From the inductive hypothesis, we have that

$$Q_{s_1} + (-1)^t P_{j_1} = Q_{s_2} + (-1)^t P_{j_2}.$$

Now we consider the points of a third line at the distance $t + 1$ from line one. Suppose that these points are denoted by M_l. Again suppose that M_{l_1} and M_{l_2} are consecutive and that P_{j_1}, Q_{s_1}, and M_{l_2}, are vertically aligned and that P_{j_2}, Q_{s_2}, and M_{l_2} are vertically aligned.

Since the line with points Q_s and the line with points M_l have distance zero, we have that $Q_{s_1} + M_{l_1} = Q_{s_2} + M_{l_2}$. Subtracting $Q_{s_1} + M_{l_1} = Q_{s_2} + M_{l_2}$ from $Q_{s_1} + (-1)^t P_{j_1} = Q_{s_2} + (-1)^t P_{j_2}$ we have $M_{l_1} + (-1)^{t+1} P_{j_1} = M_{l_2} + (-1)^{t+1} P_{j_2}$.

We now prove the right-hand side of the identity in the statement of Proposition 7. Using (4) we have that

$$\begin{aligned} H_{2i+r+2,i+2} + (-1)^i H_{r,1} &= L_{i+2} F_{i+r} + (-1)^i F_{r-1} + F_{i+1} F_{i+r-1} \\ &= F_{i+1} L_{i+r+1} + F_{i+1} F_{i+r-1} \\ &= F_{i+1} F_{r+i+3}. \end{aligned}$$

This completes the proof. □

Corollary 8 is a direct application of the ladder property. Let us consider a vertical line in the triangle with even number of entries. Then the sum of the two midpoints of this line is the average of the two entries right above and below the central entries. Specifically, these two points are equidistant from the midpoints. In addition, the sum of the last two points in the same vertical line is a Fibonacci multiple of the sum of the midpoints.

For example and for brevity, suppose our vertical line passes through the first 6 entries on the median of the triangle (see Table 3). These entries are: $0, 3, 5, 16, 39, 105$. Our midpoints are entries 5 and 16. So,

$$2(5 + 16) = 39 + 3 \text{ and } 5(5 + 16) = 105 + 0.$$

This pattern will continue anywhere within the triangle and with any sized line.

Corollary 8. *If r, k, and i are positive integers with $k \leq r$, then the following holds:*

$$H_{r-2i,k-1} + H_{r+2(i+1),k+2i} = F_{2i+1}(H_{r,k} + H_{r+2,k+1}) = F_{2i+1} F_{r+3}.$$

Proof. The proof of this corollary is a direct application of Proposition 7. It follows from replacing r by $r - 2i$, k by $k - 1$ and j by $2i + 1$ in Proposition 7. □

Similar to Proposition 7, we can see that the difference between two horizontally aligned entries (entries on the same row) within the determinant Hosoya triangle result in a Fibonacci product. The *distance* between two aligned points is the number of entries between them. We can define *width* to be the difference between two points which are in the same row. We also define *height* to be the difference between two vertically aligned entries within the triangle. It is important to note that the width is twice the product of two Fibonacci numbers and height is the product of two Fibonacci numbers. The width and the height may be negative depending on the order in which the points are subtracted. This result is shown in Corollary 9.

When the horizontal and vertical distance are the same, we obtain a square within the triangle. This gives rise to the hourglass property or Proposition 10 where we observe the result of taking a determinant of the four entries which are located at the corner points of our square.

Corollary 9. *If r, k, j and i are positive integers with $k \leq r$, then this holds*

$$|H_{r,k+i} - H_{r,k}| = |H_{r+2j,k+j} - H_{r+2j,k+j+i}| = |2F_{r-2k-i+1}F_i|.$$

Proof. From Proposition 7, we know that

$$(-1)^{j-1}H_{r,k} + H_{r+2j,k+j} = (-1)^{j-1}H_{r,k+i} + H_{r+2j,k+i+j}.$$

Thus, $H_{r+2j,k+j} - H_{r+2j,k+j+i} = (-1)^{j-1}(H_{r,k+i} - H_{r,k})$. This previous identity proves that the difference between two entries on the same row is equivalent as we go upward or downward through the triangle. This proves the first equality.

Using the first equality with $j = 1 - k$, we are able to shift our entries upward until the left-hand entry is on the boarder of the triangle. This previous fact implies

$$|(-1)^k(H_{r,k+i} - H_{r,k})| = |H_{r+2(1-k),1} - H_{r+2(1-k),i+1}| = |F_{r-2k+1} - H_{r-2k+2,i+1}|.$$

This previous identity, Proposition 1, part 2, and $F_iF_{r-2k-i+3} + F_{i-1}F_{r-2k-i+2} = F_{r-2k+2}$ (see [13]) imply that $|F_{r-2k+1} - H_{r-2k+2,i+1}|$ is equal to

$$\begin{aligned} & |F_{r-2k+1} - (F_iF_{r-2k-i+3} + F_{i+1}F_{r-2k-i+1})| \\ = \; & |F_{r-2k+1} - (F_{r-2k+2} - F_{i-1}F_{r-2k-i+2} + F_{i+1}F_{r-2k-i+1})| \\ = \; & |2F_iF_{r-2k-i+1}|. \end{aligned}$$

The above identity shows that the middle and the right-hand side are equal, thus completing the proof. □

4.2. Hourglass Property

The hourglass property (see Table 3) shows how the difference between two entries on a row is mirrored by two other entries which is reflected across a center row. If we increase the distance between two entries and decrement the row by the same amount, then the difference remains invariant for any two rows with the same distance on opposite sides of the center line. We can see an identical case reflected across the midpoint as we increment the row. Continuation of this process recursively generates an hourglass shape, hence the name of this property. We also observe that the determinant of the 2×2 matrix formed by taking the entries located at the endpoints of the hourglass presents a pattern. Let

$$a_i := H_{r-i+1,k-i+1};\ b_i := H_{r-i+1,k+1};\ c_i := H_{r+i+1,k+1};\ d_i := H_{r+i+1,k+i+1}, \quad (5)$$

where i, r, k are positive integers and $i < k$. This point forms an hourglass configuration within the triangle. For example, from the hourglass in Table 3 and using Proposition 7 and Corollary 9, we have that

$$\begin{vmatrix} 23 & 25 \\ 66 & 64 \end{vmatrix} = |2F_2F_{11}F_1^2|;$$

$$\begin{aligned} \begin{vmatrix} 18 & 16 \\ 107 & 105 \end{vmatrix} &= |\underbrace{(18-16)}_{Width}\underbrace{(18-107)}_{Height}| = \underbrace{|2F_2F_2|| - F_{11}F_2|}_{Area} \\ &= | - 2F_2F_{11}F_2^2|; \end{aligned} \tag{6}$$

$$\begin{vmatrix} 5 & 9 \\ 173 & 169 \end{vmatrix} = |2F_2\, F_{11}F_3^2|.$$

Note that in this example, the second determinant (6) corresponds to the area of the square formed by these points: top width $= \{18, 14, 16\}$, bottom width $= \{107, 103, 105\}$, left-hand side height $= \{18, 37, 107\}$, and right-hand side height $= \{16, 39, 105\}$.

In addition, the determinants mentioned above also generate an infinite family of recursively-defined sequences by considering the values generated by the hourglass property. This is summarized in the following proposition (which can also be considered as a corollary of Proposition 7).

0											
1	1										
1	3	1									
2	4	4	2								
3	7	5	7	3							
5	11	9	9	11	5						
8	18	14	16	14	18	8					
13	29	23	25	25	23	29	13				
21	47	37	41	39	41	37	47	21			
34	76	60	66	64	64	66	60	76	34		
55	123	97	107	103	105	103	107	97	123	55	
89	199	157	173	167	169	169	167	173	157	199	89

Table 3: Hourglass property in the determinant Hosoya triangle.

In this paper we put the restrictions $r, k, i \in \mathbb{Z}_{>0}$, because we want a geometric interpretation of the identity in the following theorem. However, the identity can be generalized to negative integers, with some caution to the signs. In this theorem we give a formal statement of the above discussion.

Proposition 10. *If $r, k,$ and $i < k$ are positive integers, then the following hold*

1. *Hourglass identity*

$$H_{r-i+1,k-i+1}H_{r+i+1,k+i+1} - H_{r-i+1,k+1}H_{r+i+1,k+1} = (-1)^{r+k+(r+i-1 \bmod 2)}2F_{r-2k}F_{r+3}F_i^2.$$

2. *The hourglass identity gives rise to the recurrence relation*

$$S_{r,k}(n) = (-1)^{n+1}(2 \cdot |S_{r,k}(n-1)| + 2 \cdot |S_{r,k}(n-2)| - |S_{r,k}(n-3)|),$$

with $S_{r,k}(0) = 0$ and $S_{r,k}(1) = -S_{r,k}(2) = 2F_{r-2}F_{r+3}$.

Proof. To prove Part (1), we observe from (5) that $a_i, b_i, c_i,$ and d_i are vertices of a rectangle, where $\overline{a_ic_i}$ and $\overline{b_id_i}$ are vertical and parallel, $\overline{a_ib_i}$ and $\overline{c_id_i}$ are horizontal and parallel. This and Proposition 7 imply that $h := d_i+(-1)^{i-1}b_i = c_i+(-1)^{i-1}a_i$, where $h = (-1)^{i-1}H_{r-i+1,k-i+1} + H_{r+i+1,k+1} = F_iF_{r+3}$ (from Proposition 7). Thus,

$$\begin{vmatrix} H_{r-i+1,k-i+1} & H_{r-i+1,k+1} \\ H_{r+i+1,k+1} & H_{r+i+1,k+i+1} \end{vmatrix} = \begin{vmatrix} a_i & b_i \\ h-a_i & h-b_i \end{vmatrix} = -h(b_i - a_i).$$

From Corollary 9, we have $b_i - a_i = (-1)^{i-1}2F_{r-2k}F_i$ and from Proposition 7, we have $h = F_iF_{r+3}$. It is easy to see that $-h(b_i - a_i) = (-1)^i 2F_{r-2k}F_{r+3}F_i^2$. This completes the proof of Part (1).

To prove Part (2), we start by observing that $2F_{r-2k}F_{r+3}$ is a fixed constant for each unique sequence. Therefore, we can define a closed formula to find the nth term of our sequence as $S_{r,k}(n) = (-1)^{n+1}2F_{r-2k}F_{r+3}F_n^2$. From Part (1), it is easy to see that

$$(-1)^{n+1}(2 \cdot |S_{r,k}(n-1)| + 2 \cdot |S_{r,k}(n-2)| - |S_{r,k}(n-3)|)$$

equals

$$(-1)^{n+1}2F_{r-2k}F_{r+3}(2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2) = (-1)^{n+1}2F_{r-2k}F_{r+3}F_n^2.$$

This completes the proof of Part (2). □

As shown in (6), when the distance between the end points in a component of the hourglass is odd, then it can be interpreted as an area.

Corollary 11. *If i is even, then the closed formula of the hourglass can be expressed geometrically by finding the area of the hourglass which is | width | $\cdot$| height |. Thus,*

$$\begin{aligned} |H_{r-i+1,k-i+1} - H_{r-i+1,k+1}||H_{r-i+1,k-i+1} - H_{r+i+1,k+1}| &= \\ |H_{r-i+1,k-i+1}H_{r+i+1,k+i+1} - H_{r-i+1,k+1}H_{r+i+1,k+1}| &= |2F_{r-2k}F_{r+3}F_i^2|. \end{aligned}$$

4.3. Hockey Stick Property

The entries of the Pascal triangle exhibit a property called the hockey stick property (see [12]). The entries in the shaft of the hockey stick (the long length of the stick between the end and the blade) add up to give the entry in the blade. The shaft can be of any length and the hockey stick can be located in any part of the triangle (it must include a border point). The Hosoya triangle exhibits a similar property with certain restrictions (see [5]). In the determinant Hosoya triangle, we see a property that is similar to the hockey stick property, but the position of the shaft and the blade are reversed. Note that the hockey stick can be of any length, located in any part of the triangle and the blade can face either the left or the right, as seen in Table 4. For example, we can see in the Table 4 that for the longer hockey stick on the left side of the median,

$$H_{5,2} + H_{6,3} + H_{8,4} + H_{10,5} = 7 + 9 + 25 + 64 = 105 = H_{11,6}.$$

Note that the entry $H_{5,2}$ forms the blade, while the points $H_{6,3}, H_{8,4}, H_{10,5}$ form the shaft of the hockey stick. In addition, we note that there are odd number of entries (in particular three) in the shaft and only one entry in the blade. On the other hand, if the shaft has an even number of entries, then the sum is on the left-hand side of the shaft (instead of the right). We also note that the position of the entry in the blade of the hockey stick is in the $(r+2)$-th row due to the shape of the hockey stick in general. So, for the shorter hockey stick on the left of the median, the entry in the blade is in the seventh row which implies that $r = 5$. Therefore, the sum of the entries here is

$$H_{7,1} + H_{8,2} + H_{10,3} = 8 + 29 + 60 = 97 = H_{11,3}.$$

The last example in Table 4, the hockey stick on the right of the median has its shaft facing the right. The sum of the entries in this case is

$$H_{5,5} + H_{6,5} + H_{8,6} + H_{10,7} = 3 + 11 + 23 + 66 = 103 = H_{11,7}.$$

Proposition 12. *Let t, r, and j be positive integers, where $r > 1$, $1 \le j \le r+2$, and t is the number of entries in the shaft of the hockey stick. Then the following hold.*

(a) If the blade of the hockey stick faces the left, then

$$H_{r+2,j} + \sum_{i=1}^{t} H_{r+2i+1,i+j} = \begin{cases} H_{r+2t+2,t+j}, & \text{if } t \text{ is even;} \\ H_{r+2t+2,t+j+1}, & \text{if } t \text{ is odd.} \end{cases}$$

0										
1	1									
1	3	1								
2	4	4	2							
3	7	5	7	3						
5	11	9	9	11	5					
8	18	14	16	14	18	8				
13	29	23	25	25	23	29	13			
21	47	37	41	39	41	37	47	21		
34	76	60	66	64	64	66	60	76	34	
55	123	97	107	103	105	103	107	97	123	55

Table 4: Hockey stick in the determinant Hosoya triangle.

(b) If the blade of the hockey stick faces the right, then

$$H_{r+2,j} + \sum_{i=0}^{t-1} H_{r+2i+3,i+j} = \begin{cases} H_{r+2t+2,t+j}, & \textit{if } t \textit{ is even;} \\ H_{r+2t+2,t+j-1}, & \textit{if } t \textit{ is odd.} \end{cases}$$

Proof. Part (a). We prove this part by the method of induction on t, the number of entries in the shaft of the hockey stick.

Let $S(k)$ be the statement:

$$H_{r+2,j} + \sum_{i=1}^{k} H_{r+2i+1,i+j} = \begin{cases} H_{r+2t+2,t+j}, & \text{if } k \text{ is even;} \\ H_{r+2k+2,k+j+1} & \text{if } k \text{ is odd.} \end{cases}$$

As the basis step we prove $S(1)$ and $S(2)$. From the definition given in (1), we have $H_{r+2,j} + H_{r+3,j+1} = H_{r+4,j+2}$ proving $S(1)$. The second case follows using the definition given in (1) again. Thus,

$$\begin{aligned} H_{r+2,j} + \sum_{i=1}^{2} H_{r+2i+1,i+j} &= H_{r+2,j} + (H_{r+3,j+1} + H_{r+5,j+2}) \\ &= (H_{r+2,j} + H_{r+3,j+1}) + H_{r+5,j+2}. \qquad (7) \end{aligned}$$

Applying the recursive relation given in (1), we see that the expression in (7) equals $H_{r+4,j+2} + H_{r+5,j+2}$ which in turn equals $H_{r+6,j+2}$. This proves $S(2)$.

For the inductive step we assume that $S(t)$ is true for $t \geq 2$ and prove that

$S(t+1)$ is true. Therefore, we assume that

$$H_{r+2,j} + \sum_{i=1}^{t} H_{r+2i+1,i+j} = \begin{cases} H_{r+2t+2,t+j}, & \text{if } t \text{ is even;} \\ H_{r+2t+2,t+j+1}, & \text{if } t \text{ is odd,} \end{cases}$$

and prove that S(t+1) is also true. To prove $S(t+1)$ we consider two cases.

`Case` $t+1$ `is even`. We first observe that t is odd. Therefore, using the inductive hypothesis for the odd case and the recursive relation given in (1), we have that

$$\begin{aligned} H_{r+2,j} + \sum_{i=1}^{t+1} H_{r+2i+1,i+j} &= \left(H_{r+2,j} + \sum_{i=1}^{t} H_{r+2i+1,i+j} \right) + H_{r+2(t+1)+1,(t+1)+j} \\ &= H_{r+2t+2,t+j+1} + H_{r+2t+3,t+j+1} \\ &= H_{r+2t+4,t+j+1} = H_{r+2(t+1)+2,(t+1)+j}. \end{aligned}$$

`Case` $t+1$ `is odd`. We observe that t is even in this case. Therefore, using the inductive hypothesis for the odd case and the recursive relation given in (1), we have that

$$\begin{aligned} H_{r+2,j} + \sum_{i=1}^{t+1} H_{r+2i+1,i+j} &= \left(H_{r+2,j} + \sum_{i=1}^{t} H_{r+2i+1,i+j} \right) + H_{r+2(t+1)+1,(t+1)+j} \\ &= H_{r+2t+2,t+j} + H_{r+2t+3,t+j+1} \\ &= H_{r+2t+4,t+j+2} = H_{r+2(t+1)+2,(t+1)+j+1}. \end{aligned}$$

This completes the proof of Part (a).

The proof of Part (b) follows similarly using the method of induction on t. Let $S(k)$ be the statement:

$$H_{r+2,j} + \sum_{i=0}^{k-1} H_{r+2i+3,i+j} = \begin{cases} H_{r+2k+2,k+j}, & \text{if } k \text{ is even;} \\ H_{r+2k+2,k+j-1}, & \text{if } k \text{ is odd.} \end{cases}$$

As the basis step we prove $S(1)$ and $S(2)$. From (1), we have $H_{r+2,j} + H_{r+3,j} = H_{r+4,j}$ proving $S(1)$. The proof of $S(2)$ also follows in the same way as in the proof of Part (a).

For the inductive step we assume that $S(t)$ is true for $t \geq 2$ and prove that $S(t+1)$ is true. Therefore, we assume that

$$H_{r+2,j} + \sum_{i=0}^{t-1} H_{r+2i+3,i+j} = \begin{cases} H_{r+2t+2,t+j}, & \text{if } t \text{ is even;} \\ H_{r+2t+2,t+j-1}, & \text{if } t \text{ is odd;} \end{cases}$$

and prove that S(t+1) is also true. To prove $S(t+1)$ we consider two cases. We prove the case when $(t+1)$ is even. The proof for $(t+1)$ is odd is similar to the same case in the proof of Part (a), so we omit it.

`Case` $t+1$ `is even`. Since t is odd, using the inductive hypothesis for the odd case and recursive relation given in (1), for the left-hand side of $S(t+1)$, we have that

$$\begin{aligned} H_{r+2,j}+\sum_{i=0}^{t} H_{r+2i+3,i+j} &= \left(H_{r+2,j}+\sum_{i=0}^{t-1} H_{r+2i+3,i+j}\right)+H_{r+2t+3,t+j} \\ &= H_{r+2t+2,t+j-1}+H_{r+2t+3,t+j} \\ &= H_{r+2t+4,t+j+1}=H_{r+2(t+1)+2,(t+1)+j}. \end{aligned}$$

This completes the proof of Part (b). $\square$

Rewriting the statements of both parts of Proposition 12 using the definition of the entries of the determinant Hosoya triangle in Proposition 1, we obtain the following identity.

If r, j, and t are positive integers where $r>1$ and $1 \leq j \leq r+2$, then:

(i)

$$\begin{vmatrix} F_{r-j+4} & F_{r-j+3} \\ F_{j} & F_{j+1} \end{vmatrix} + \sum_{i=1}^{t} \begin{vmatrix} F_{r+i-j+3} & F_{r+i-j+2} \\ F_{i+j} & F_{i+j+1} \end{vmatrix} = \begin{cases} \begin{vmatrix} F_{r+t-j+4} & F_{r+t-j+3} \\ F_{t+j} & F_{t+j+1} \end{vmatrix}, & t \text{ is even}, \\ \begin{vmatrix} F_{r+t-j+3} & F_{r+t-j+2} \\ F_{t+j+1} & F_{t+j+2} \end{vmatrix}, & t \text{ is odd}; \end{cases}$$

(ii)

$$\begin{vmatrix} F_{r-j+4} & F_{r-j+3} \\ F_{j} & F_{j+1} \end{vmatrix} + \sum_{i=0}^{t-1} \begin{vmatrix} F_{r+i-j+5} & F_{r+i-j+4} \\ F_{i+j} & F_{i+j+1} \end{vmatrix} = \begin{cases} \begin{vmatrix} F_{r+t-j+4} & F_{r+t-j+3} \\ F_{t+j} & F_{t+j+1} \end{vmatrix}, & t \text{ is even}, \\ \begin{vmatrix} F_{r+t-j+5} & F_{r+t-j+4} \\ F_{t+j-1} & F_{t+j} \end{vmatrix}, & t \text{ is odd}. \end{cases}$$

We believe that the identities can be further generalized to extend beyond the entries of the triangle. However, we are only interested in the entries obtained from the triangle, so we leave the discussion about the identities here.

Finally, we note that if the number of entries in the blade is more than one, then the hockey stick property does not hold. For example, from Table 4, we have that, if $H_{3,1}, H_{4,2}, H_{5,3}$ are in the blade while $H_{7,4}, H_{9,5}$ are in the shaft of the stick, then the sum of these entries in not equal to the entry $H_{10,5}$. In fact,

$$H_{3,1} + H_{4,2} + H_{5,3} + H_{7,4} + H_{9,5} = 1 + 4 + 5 + 16 + 39 \neq 64.$$

5. Column Generators

Hoggatt [8] found the generating functions for the diagonals of the Pascal triangle. Here, we use a similar technique to find the generating functions of the diagonals of the determinant Hosoya triangle. Since the first entry in the first column of Table 5 is zero, we conclude that this triangle is not a Riordan array.

0								
1	1							
1	3	1						
2	4	4	2					
3	7	5	7	3				
5	11	9	9	11	5			
8	18	14	16	14	18	8		
13	29	23	25	25	23	29		
21	47	37	41	39	41	37	$\ddots$	
34	76	60	66	64	64	66	$\cdots$	F_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\frac{x}{1-x-x^2}$	$\frac{2x+1}{1-x-x^2}$	$\frac{3x+1}{1-x-x^2}$	$\frac{5x+2}{1-x-x^2}$	$\frac{8x+3}{1-x-x^2}$	$\frac{13x+5}{1-x-x^2}$	$\frac{21x+8}{1-x-x^2}$	$\cdots$	$\frac{F_{k+1}x+F_{k-1}}{1-x-x^2}$
1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}	7^{th}	$\cdots$	k^{th}

Table 5: Generating functions of $\mathcal{H}$.

From the recurrence of $G_{n,k}$ given in (4), it is easy to see that the generating function $g(x)$ of $G_{n,k}$ for a fixed k is given by

$$g(x) = \frac{G_{1,k} + (G_{2,k} - G_{1,k})x}{1 - x - x^2} = \frac{F_{k-1} + F_{k+1}x}{1 - x - x^2}.$$

The entries of the triangle (see Table 5) are also given by the generating function $(x + y + xy)/((1 - x - x^2)(1 - y - y^2))$ (see [11, A108038]). Another generating function to generate the triangle given in Table 5 can be found in [2].

From Proposition 2, we know that

$$\sum_{k=1}^{n-2} F_{k+1}F_{n-k+1} + F_{n+1} = [nL_{n+2} - 4F_n]/5.$$

This implies that the generating function for the row sums is given by $\dfrac{x^2(x+2)}{(x^2+x-1)^2}$. Table 5 summarizes the above discussion.

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