#A67 INTEGERS 20 (2020)

BERNOULLI-STIRLING NUMBERS

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Received: 5/29/19, Revised: 2/29/20, Accepted: 8/13/20, Published: 8/31/20

Abstract

Congruences modulo prime powers involving generalized Harmonic numbers are known. While looking for similar congruences, we have encountered a curious triangular array of numbers indexed with positive integers n, k, involving the Bernoulli and cycle Stirling numbers. These numbers are all integers and they vanish when $n-k$ is odd. This triangle has many similarities with the Stirling triangle. In particular, we show how it can be extended to negative indices and how this extension produces a *second kind* of such integers which may be considered as a new generalization of the Genocchi numbers and for which a generating function is easily obtained. But our knowledge of these integers remains limited, especially for those of the *first kind*.

1. Introduction

Let n and k be non-negative integers and let the generalized Harmonic numbers $H_n^{(k)}$, $G_n^{(k)}$ be defined as

$$H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k},$$

$$G_n^{(k)} := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \frac{1}{i_1 i_2 \cdots i_k},$$

with $H_n^{(1)} = G_n^{(1)} = \sum_{j=1}^{n} \frac{1}{j} = H_n = G_n$; $H_n^{(0)} = n$ and $G_n^{(0)} = 1$. We have [10]

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = n! G_n^{(k)},$$

$\begin{bmatrix} n \\ k \end{bmatrix}$ being the cycle Stirling number (or unsigned Stirling number of the first kind), so that the Harmonic and cycle Stirling numbers are inter-related by the convolution

$$k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = -\sum_{j=0}^{k-1} (-1)^{k-j} H_n^{(k-j)} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}, \tag{1.1}$$

which is obtained as a direct application of the well-known relation between elementary symmetric polynomials and power sums [9]. Extended congruences for the Harmonic numbers $H_{p-1}^{(k)}$, modulo any power of a prime p are known [16], [7]. Our initial motivation for the work reported in the present paper is to look for similar congruences modulo prime powers, involving $G_{p-1}^{(k)}$, or the cycle Stirling numbers $\left[{p \atop k+1}\right]$, instead of $H_{p-1}^{(k)}$. We will show that such similar congruences for $G_{p-1}^{(k)}$ do exist, but that they are just the particular prime instances of not very well-known but elementary identities for the cycle Stirling numbers. This will lead us to introduce a triangular array of integers, the *Bernoulli-Stirling numbers*, involving the Bernoulli and Stirling numbers which we believe is new.

2. Notation and Preliminaries

In addition to what was exposed in the previous introduction, further notation that we use throughout this paper is presented in this section, along with classical results which we will need. Most of these results can be found in textbooks like [6] and they are given hereafter without proof.

In the following, p denotes a prime number, x denotes the argument of a generating function, of a polynomial function or of a formal power series, $[[x^n]]f(x)$ the coefficient of $\frac{x^n}{n!}$ in $f(x)$ and $D^m f(x)$ is the m-order derivative of $f(x)$ with respect to x. We will use the Iverson bracket notation: $\left[\mathfrak{P}\right] = 1$ when proposition $\mathfrak{P}$ is true, and $\left[\mathfrak{P}\right] = 0$ otherwise. For q a rational number, we denote by $v_p(q)$ the p-adic order of q and we have $v_p(q_1 \cdot q_2) = v_p(q_1) + v_p(q_2)$ and $v_p(q_1 + q_2) \geq \min\left(v_p(q_1), v_p(q_2)\right)$. The binomial coefficients $\binom{n}{k}$ are defined by $\sum_k \binom{n}{k} x^k = (1+x)^n$, whatever the sign of integer n. They obviously vanish when $k < 0$. They are easily obtained by the basic recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. When $n > 0$, we have $\binom{-n}{k} = (-1)^k \binom{n+k-1}{n-1}$ and we have the well-known inversion formula

$$\sum_{k \geq 0} (-1)^{k-j} \binom{k}{j} \binom{n}{k} = [n = j]. \tag{2.1}$$

The cycle Stirling numbers $\left[{n \atop k}\right]$, $n \geq 0$ may be defined by the horizontal generating function

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{j=0}^{n-1} (x + j), \tag{2.2}$$

where an empty product is meant to be 1. Alternatively, they have the exponential generating function

$$\sum_{n \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{(-1)^k \big(\ln(1-x)\big)^k}{k!}. \tag{2.3}$$

They obviously vanish when $k < 0$ and $k > n$. They are easily obtained by the basic recurrence $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, valid for $n \geq 1$, with $\left[{0 \atop k}\right] = [k=0]$. They also obey the generalized recurrence relation

$$\left[{n+1 \atop m+1}\right] = \sum_{h \geq 0} \binom{h+m}{m} \left[{n \atop h+m}\right]. \tag{2.4}$$

We let $\left\{{n \atop k}\right\}$, $n \geq 0$, be the partition Stirling numbers (or Stirling numbers of the second kind). They also vanish when $k < 0$ and $k > n$. Their basic recurrence is $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$ for $n \geq 1$, with $\left\{{0 \atop k}\right\} = [k=0]$. They have the following exponential generating function

$$\sum_{n \geq 0} \left\{{n \atop k}\right\} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}. \tag{2.5}$$

We will make use of the following Lemma about the Stirling numbers, for which there is a proof in [6], p. 266-271.

Lemma 2.1. *Let n, k be non-negative integers. There exists a polynomial $Q_k \in \mathbb{Q}[X]$ of degree $2k$, such that $Q_k(n)$ coincides with $\left[{n \atop n-k}\right]$. If $k > 0$ then $0, 1, \cdots, k$ are roots of the polynomial Q_k. Moreover, we have $Q_k(-n) = \left\{{n+k \atop n}\right\}$ and in particular $Q_k(-1) = 1$.*

For $n \geq 0$, let B_n be the Bernoulli numbers. The first of them are $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots$ and for $h > 0$, we have $B_{2h+1} = 0$. They have the well-known exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!} \tag{2.6}$$

and they obey the recurrence

$$(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{2.7}$$

We will also make use of the Von Staudt-Clausen theorem which states that the denominator of B_n, in reduced form, is the product of all primes p such that $p-1$ divides n. In particular, any prime may divide the denominator of a Bernoulli number once at most.

3. Two Identities for the Cycle Stirling Numbers

In this section, we re-demonstrate two identities for the cycle Stirling numbers which, in spite of their similarity to Equation (2.4), do not seem to be very well-known.

Theorem 3.1. *Let m, n be non-negative integers. We have*

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = (-1)^{n-m} \sum_{h \geq 0} \binom{h+m}{m} \begin{bmatrix} n \\ h+m \end{bmatrix} (-n)^h. \tag{3.1}$$

Moreover, if $n > 0$,

$$\begin{bmatrix} n \\ m \end{bmatrix} = (-1)^{n-m} \sum_{h \geq 0} \binom{h+m-1}{m-1} \begin{bmatrix} n \\ h+m \end{bmatrix} (-n)^h. \tag{3.2}$$

In more compact and symmetric formulations, these two identies also read

$$(-1)^{n-m} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix} (-n)^m = \sum_h \binom{h}{m} \begin{bmatrix} n \\ h \end{bmatrix} (-n)^h \tag{3.3}$$

and, if $n > 0$,

$$(-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} (-n)^m = \sum_h \binom{h-1}{m-1} \begin{bmatrix} n \\ h \end{bmatrix} (-n)^h. \tag{3.4}$$

Remark. These identities are not in [6] where quite many finite sums, recurrences and convolutions involving Stirling numbers are reported. Our Equation (3.1) may be obtained as a particular case of Theorem 3 in [2]. An identity equivalent to our Equation (3.2), is obtained incidentally in [1], where it is not even labelled. Another identity, equivalent to our Equation (3.2), is the equation (18) from [15], where it is said to be new.

Proof of Theorem 3.1. Like in [1], our proof will highlight that Equation (3.1) and Equation (3.2) are actually closely related to the convolution Equation (1.1) between the Harmonic and cycle Stirling numbers. Let $f_n(x) := \prod_{h=0}^{n-1}(x-h)$. We are going to show that, for $m \geq 1$,

$$m \frac{D^m f_n(x)}{m!} = -\sum_{h=0}^{m-1} (-1)^{m-h} \frac{D^h f_n(x)}{h!} \sum_{j=0}^{n-1} \frac{1}{(x-j)^{m-h}}. \tag{3.5}$$

It is true for $m = 1$, since $D f_n(x) = \sum_{j=0}^{n-1} \prod_{h \neq j} (x-h) = f_n(x) \sum_{j=0}^{n-1} \frac{1}{x-j}$. We suppose (induction hypothesis) that it is true for some m, so that

$$(m+1) \frac{D^{m+1} f_n(x)}{(m+1)!} = D \frac{D^m f_n(x)}{m!} = \frac{1}{m} D \left(m \frac{D^m f_n(x)}{m!} \right)$$

$$\begin{aligned}
(m+1)\frac{D^{m+1}f_n(x)}{(m+1)!} &= \frac{1}{m}\left(-\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^{h+1}f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m-h}}\right)\\
&+\frac{1}{m}\left(\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^{h}f_n(x)}{h!}(m-h)\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)\\
&=\frac{1}{m}\left(\sum_{h=1}^{m}(-1)^{m-h}\frac{D^{h}f_n(x)}{(h-1)!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)\\
&+\frac{1}{m}\left(m\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^{h}f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)\\
&-\frac{1}{m}\left(\sum_{h=1}^{m-1}(-1)^{m-h}\frac{D^{h}f_n(x)}{(h-1)!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)\\
&=\frac{1}{m}\left(\frac{D^{m}f_n(x)}{(m-1)!}\sum_{j=0}^{n-1}\frac{1}{x-j}\right)\\
&+\frac{1}{m}\left(m\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^{h}f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\right)\\
&=\sum_{h=0}^{m}(-1)^{m-h}\frac{D^{h}f_n(x)}{h!}\sum_{j=0}^{n-1}\frac{1}{(x-j)^{m+1-h}}\ .
\end{aligned}$$

This establishes the validity of Equation (3.5). Now, when $x=n$, it reads

$$m\frac{D^m f_n(n)}{m!} = -\sum_{h=0}^{m-1}(-1)^{m-h}\frac{D^h f_n(n)}{h!}H_n^{(m-h)}. \tag{3.6}$$

This is the same recurrence as in the convolution Equation (1.1), with the same inital value, since by definition $f_n(n)=n!=\left[{n+1 \atop 1}\right]$. Then

$$\frac{D^m f_n(n)}{m!} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}.$$

On the other hand, from Equation (2.2), we have $f_n(x)=\sum_{h=0}^{n}\left[{n \atop h}\right](-1)^{n-h}x^h$, and hence

$$\begin{aligned}
D^m f_n(x) &= m!\sum_{h=0}^{n}\begin{bmatrix} n \\ h \end{bmatrix}\binom{h}{m}(-1)^{n-h}x^{h-m},\\
\frac{D^m f_n(n)}{m!} &= \sum_{h=0}^{n}\begin{bmatrix} n \\ h \end{bmatrix}\binom{h}{m}(-1)^{n-h}n^{h-m}.
\end{aligned}$$

Hence

$$\begin{aligned}\begin{bmatrix}n+1\\m+1\end{bmatrix} &= \sum_{h=0}^{n}\begin{bmatrix}n\\h\end{bmatrix}\binom{h}{m}(-1)^{n-h}n^{h-m}\\ &= \sum_{h=0}^{n-m}(-1)^{n-m}\begin{bmatrix}n\\h+m\end{bmatrix}\binom{h+m}{m}(-n)^{h}.\end{aligned}$$

This completes the proof of Equation (3.1). For the proof of Equation (3.2), we also use an induction argument, but on m and backward. Our induction hypothesis is

$$\begin{bmatrix}n\\m+1\end{bmatrix} = (-1)^{n-(m+1)}\sum_{h=0}^{n-(m+1)}\binom{h+m}{m}\begin{bmatrix}n\\h+m+1\end{bmatrix}(-n)^{h}.$$

Then

$$\begin{bmatrix}n\\m+1\end{bmatrix} = -(-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m}\begin{bmatrix}n\\h+m\end{bmatrix}(-n)^{h-1}.$$

Hence

$$n\begin{bmatrix}n\\m+1\end{bmatrix} = (-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m}\begin{bmatrix}n\\h+m\end{bmatrix}(-n)^{h}.$$

We subtract the latter equation from Equation (3.1), and we obtain

$$\begin{bmatrix}n+1\\m+1\end{bmatrix} - n\begin{bmatrix}n\\m+1\end{bmatrix} = (-1)^{n-m}\sum_{h=1}^{n-m}\left(\binom{h+m}{m}-\binom{h+m-1}{m}\right)\begin{bmatrix}n\\h+m\end{bmatrix}(-n)^{h}.$$

That is,

$$\begin{bmatrix}n\\m\end{bmatrix} = (-1)^{n-m}\sum_{h=1}^{n-m}\binom{h+m-1}{m-1}\begin{bmatrix}n\\h+m\end{bmatrix}(-n)^{h}.$$

To finish the proof, we just need that Equation (3.2) be true for $m=n$, which is obvious. □

4. Extended Congruences for the Harmonic Numbers $G_{p-1}^{(j+1)}$

Theorem 4.1. *Let $k\geq 0$ be an integer and p a prime number. We have*

$$G_{p-1}^{(k)} = (-1)^{k}\sum_{j\geq 0}(-1)^{j}\binom{j+k}{j}G_{p-1}^{(k+j)}p^{j}. \tag{4.1}$$

In particular, when $k=0$, we have

$$\sum_{j\geq 0}(-1)^{j}G_{p-1}^{(j+1)}p^{j} = 0. \tag{4.2}$$

Proof. Letting $n = p$ a prime number, and $m = k+1$ in Equation (3.2), and dividing throughout by $(p-1)!$ provides the desired result. □

Recall [7] that when $k \geq 1$, the generalized Harmonic numbers $H_{p-1}^{(k)}$ admit the following p-adically converging expansion:

$$H_{p-1}^{(k)} = (-1)^k \sum_{j\geq 0} \binom{j+k-1}{j} H_{p-1}^{(k+j)} p^j. \tag{4.3}$$

It is interesting to point out the similarity of Equation (4.3) and Equation (4.1), but also some differences. Contrary to Equation (4.3), the sum on the right-hand side of Equation (4.1) is finite: it is actually limited to $j = p-1-k$. We also notice that the sign alternates in Equation (4.1) and that there is a slight difference in the binomial coefficient.

It is also known [16] that, for odd prime p,

$$\sum_{j\geq 0} \binom{j+2k}{2k} B_j H_{p-1}^{(2k+j+1)} (-p)^j = 0, \tag{4.4}$$

the convergence of the series being understood p-adically. More precisely, when $p \geq 5$, the following identity was shown in [7]:

$$\sum_{j=0}^{2n+1} \binom{j+2k}{2k} B_j H_{p-1}^{(2k+j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{4.5}$$

Now, we look for an equation similar to Equation (4.4), but involving the cycle Stirling numbers. In the case where $k = 0$, we have, for $p \geq 5$

$$\sum_{j=0}^{2n+1} B_j H_{p-1}^{(j+1)} (-p)^j \equiv 0 \pmod{p^{2n+3}}. \tag{4.6}$$

For the first values of n, $n = 0, 1, 2...$, these congruences read

$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p^3}\ ,$$

$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} \equiv 0 \pmod{p^5}\ ,$$

$$H_{p-1} + \frac{p}{2} H_{p-1}^{(2)} + \frac{p^2}{6} H_{p-1}^{(3)} - \frac{p^4}{30} H_{p-1}^{(5)} \equiv 0 \pmod{p^7} \text{ ..., respectively.}$$

A clue for our search of an equation analogous to Equation (4.4) involving the Stirling cycle numbers is obtained by making use of Equation (1.1) in order to recursively compute $H_{p-1}^{(j+1)}$ as function of the $G_{p-1}^{(i+1)}$, with $i \leq j$. Then substituting

$H_{p-1}^{(j+1)}$ in the above congruences and finally reducing modulo p^{2n+3} as much as possible, by accounting for any previous congruence involving the $G_{p-1}^{(i+1)}$. In doing so, the following congruences are found, valid for $p \geq 5$:

$$G_{p-1} - pG_{p-1}^{(2)} \equiv 0 \pmod{p^3} ,$$

$$G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} \equiv 0 \pmod{p^5} ,$$

$$G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} - \frac{p^4}{6} G_{p-1}^{(5)} \equiv 0 \pmod{p^7} ,$$

$$G_{p-1} - pG_{p-1}^{(2)} + \frac{p^2}{2} G_{p-1}^{(3)} - \frac{p^4}{6} G_{p-1}^{(5)} + \frac{p^6}{6} G_{p-1}^{(7)} \equiv 0 \pmod{p^9} \text{ ... etc.}$$

The calculations become increasingly laborious as n increases, but we are able to guess that

$$\sum_{j=0}^{2n+1} (j+1) B_j G_{p-1}^{(j+1)} p^j \equiv 0 \pmod{p^{2n+3}}. \tag{4.7}$$

In an even broader generalization, we anticipate the following theorem which will be proved in the next section.

Theorem 4.2. *If $i \geq 1$ is an integer and p a prime number, then we have*

$$\sum_{j \geq 0} B_j \binom{j+2i-1}{j} G_{p-1}^{(j+2i-1)} p^j = 0 \tag{4.8}$$

or equivalently,

$$\sum_{j \geq 0} B_j \binom{j+2i-1}{j} \begin{bmatrix} p \\ j+2i \end{bmatrix} p^j = 0. \tag{4.9}$$

Remark. Equation (4.8) looks very much like Equation (4.4) but the sums in Equation (4.8) and Equation (4.9) are actually finite.

5. The Aerated Triangular Array $\mathcal{A}_{n,k}$

For performing numerical verifications of Equation (4.9), we now introduce the number $\mathcal{A}_{n,k}$.

Definition. For non-negative integers n and k, we define the number $\mathcal{A}_{n,k}$ by

$$\mathcal{A}_{n,k} := \sum_{h \geq 0} B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h. \tag{5.1}$$

It is clear from this definition that $\mathcal{A}_{n,k}$ is zero when $k > n$ and that $\mathcal{A}_{n,n} = 1$. The first terms of the sequence $\mathcal{A}_{n,k}$ are computed numerically and displayed in the following table.

n	$\mathcal{A}_{n,0}$	$\mathcal{A}_{n,1}$	$\mathcal{A}_{n,2}$	$\mathcal{A}_{n,3}$	$\mathcal{A}_{n,4}$	$\mathcal{A}_{n,5}$	$\mathcal{A}_{n,6}$	$\mathcal{A}_{n,7}$	$\mathcal{A}_{n,8}$	$\mathcal{A}_{n,9}$	$\mathcal{A}_{n,10}$
0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0
3	0	-1	0	1	0	0	0	0	0	0	0
4	0	0	-5	0	1	0	0	0	0	0	0
5	0	24	0	-15	0	1	0	0	0	0	0
6	0	0	238	0	-35	0	1	0	0	0	0
7	0	-3396	0	1281	0	-70	0	1	0	0	0
8	0	0	-51508	0	4977	0	-126	0	1	0	0
9	0	1706112	0	-408700	0	15645	0	-210	0	1	0
10	0	0	35028576	0	-2267320	0	42273	0	-330	0	1

Table 1: The triangular array $\mathcal{A}_{n,k}$, for $0 \leq n, k \leq 10$.

It is striking that these numbers seem to be zero when $n - k$ is odd, which, if true, would imply the validity of Theorem 4.2. This will be demonstrated in the next theorem. It is also striking that they seem to be all integers.

Theorem 5.1. *Let n, k be non-negative integers. We have $\mathcal{A}_{n,k} = (-1)^{n-k}\mathcal{A}_{n,k}$. Equivalently, $\mathcal{A}_{n,k} = 0$ when $n - k$ is odd.*

Proof. When $n = 0$, this is obviously true. Supposing $n > 0$, we have

$$\begin{aligned}
\mathcal{A}_{n,k} &= \sum_h B_h \binom{k+h-1}{h} \begin{bmatrix} n \\ h+k \end{bmatrix} n^h \\
&= (-1)^n \sum_h \frac{B_h}{n^k} \binom{k+h-1}{h} (-1)^{n-(h+k)} \begin{bmatrix} n \\ h+k \end{bmatrix} (-n)^{h+k} \\
&= (-1)^n \sum_h \frac{B_h}{n^k} \binom{k+h-1}{h} \sum_g \binom{g-1}{h+k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^g \quad \text{by Equation (3.4).}
\end{aligned}$$

But, it is easy to see that $\binom{k+h-1}{h}\binom{g-1}{h+k-1} = \binom{g-k}{h}\binom{g-1}{k-1}$, so that

$$\begin{aligned}
\mathcal{A}_{n,k} &= (-1)^{n-k} \sum_g \sum_h B_h \binom{g-k}{h} \binom{g-1}{k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^{g-k} \\
&= (-1)^{n-k} \sum_g (-1)^{g-k} B_{g-k} \binom{g-1}{k-1} \begin{bmatrix} n \\ g \end{bmatrix} (-n)^{g-k} \quad \text{by Equation (2.7)} \\
&= (-1)^{n-k} \sum_g B_g \binom{k+g-1}{g} \begin{bmatrix} n \\ g+k \end{bmatrix} n^g = (-1)^{n-k}\mathcal{A}_{n,k}.
\end{aligned}$$

□

Theorem 5.2. *For non-negative integers n and k, we have $\mathcal{A}_{n,k} \in \mathbb{Z}$.*

The proof of this theorem will not be given until the next section. Before proceeding to this proof, we want to point out a similarity between the triangle $\mathcal{A}_{n,k}$ and the triangle of Stirling numbers of the first kind. We have the following theorem which is analogous to Lemma 2.1.

Theorem 5.3. *Let n, k be non-negative integers such that $0 \leq k \leq n$. There exists a polynomial $P_k \in \mathbb{Q}[X]$, of degree $2k$, such that $P_k(n)$ coincides with $\mathcal{A}_{n,n-k}$. Moreover, when $k > 0$, we have that $-1, 0, \cdots, k$ are $k+2$ roots of $P_k(x)$.*

Proof. By definition of $\mathcal{A}_{n,k}$, we have

$$\mathcal{A}_{n,n-k} = \sum_{h=0}^{k} B_h \binom{n-1-(k-h)}{h} \begin{bmatrix} n \\ n-(k-h) \end{bmatrix} n^h,$$

where the binomial coefficient is a polynomial in n from $\mathbb{Q}[X]$, of degree h, and after Lemma 2.1, $\left[\begin{smallmatrix} n \\ n-(k-h) \end{smallmatrix}\right]$ is a polynomial in n from $\mathbb{Q}[X]$, of degree $2(k-h)$. Therefore $\mathcal{A}_{n,n-k}$ is also a polynomial in n from $\mathbb{Q}[X]$, of degree $2k$. Let $k > 0$, and recall Q_j, the polynomial such that $Q_j(n) = \left[\begin{smallmatrix} n \\ n-j \end{smallmatrix}\right]$. We have

$$\begin{aligned} P_k(u) &= \sum_{h=0}^{k} B_h \binom{u+h-1-k}{h} Q_{k-h}(u) u^h \\ &= \sum_{h=0}^{k} B_h \frac{(u+h-1-k)\cdot\cdot(u-k)}{h!} Q_{k-h}(u) u^h. \end{aligned}$$

From Lemma 2.1, we know that if $k > h$ then $0, 1, \cdot\cdot\cdot, k-h$ are roots of the polynomial function $Q_{k-h}(x)$. Moreover when $k \geq h$, $Q_{k-h}(-1) = 1$.
Then, when $u = 0$, we have $P_k(0) = B_0\binom{-1-k}{0} Q_k(0) 0^0 = 0$, since $k > 0$. When $u > 0$, we have

$$\begin{aligned} P_k(u) &= \sum_{h=k-u+1}^{k} B_h \frac{(u-k)\cdot\cdot(u-k+h-1)}{h!} Q_{k-h}(u) u^h \\ &= \sum_{h=k-u+1}^{k} (-1)^h B_h \frac{(k-u)\cdot\cdot(k-u-h+1)}{h!} Q_{k-h}(u) u^h. \end{aligned}$$

If $0 < u \leq k$, for any h in the set $\{k-u+1, \cdot\cdot, k\}$ the product $(k-u)\cdot\cdot(k-u-h+1)$ must vanish because it has one factor which is zero, and then we also have $P_k(u) = 0$.

Finally, if $u = -1$,

$$\begin{aligned} P_k(-1) &= \sum_{h=0}^{k} B_h \frac{(h-2-k)\cdot\cdot(-1-k)}{h!}(-1)^h Q_{k-h}(-1) \\ &= \sum_{h=0}^{k} B_h \frac{(k+2-h)\cdot\cdot(k+1)}{h!} = \sum_{h=0}^{k} \binom{k+1}{h} B_h \\ &= \sum_{h=0}^{k+1} \binom{k+1}{h} B_h - B_{k+1} = 0 \quad \text{by Equation (2.7).} \end{aligned}$$

□

6. The Dual Triangle $\mathcal{B}_{n,k}$

We now introduce a dual triangle $\mathcal{B}_{n,k}$ which is similar to the triangle of Stirling numbers of the second kind. Coming back to the polynomial P_k, we may extend the definition of $\mathcal{A}_{n,k}$ to non-positive indices since, for non-negative n and k, it is natural to define $\mathcal{A}_{-n,-n-k} := P_k(-n)$. Then, we have

$$\begin{aligned} \mathcal{A}_{-n,-n-k} &= \sum_{h=0}^{k} B_h \binom{-n-1-(k-h)}{h} Q_{k-h}(-n)(-n)^h \\ &= \sum_{h=0}^{k} B_h \binom{n+k}{h} Q_{k-h}(-n) n^h. \end{aligned}$$

That is,

$$\mathcal{A}_{-n,-k} = \sum_{h=0}^{k-n} B_h \binom{k}{h} Q_{k-n-h}(-n) n^h$$

or, accounting for Lemma 2.1,

$$\mathcal{A}_{-n,-k} = \sum_{h=0}^{k-n} B_h \binom{k}{h} \left\{ {k-h \atop n} \right\} n^h.$$

Definition. Let n, k be positive integers. We define the number $\mathcal{B}_{n,k}$ by

$$\mathcal{B}_{n,k} := \sum_{h\geq 0} B_h \binom{n}{h} \left\{ {n-h \atop k} \right\} k^h. \tag{6.1}$$

It is then clear that for all integers n, k, positive or negative, we have the duality:

$$\mathcal{A}_{-n,-k} = \mathcal{B}_{k,n}. \tag{6.2}$$

This is similar to the duality $\left[{-n \atop -k}\right] = \left\{{k \atop n}\right\}$ which holds [6] for the usual Stirling numbers. It is also clear from this definition that $\mathcal{B}_{n,k}$ is zero when $k > n$ and that $\mathcal{B}_{n,n} = 1$. Also note that we have $\mathcal{B}_{x+n,x} = P_n(-x)$. The first $\mathcal{B}_{n,k}$ are computed numerically and displayed in the following table.

n	$\mathcal{B}_{n,1}$	$\mathcal{B}_{n,2}$	$\mathcal{B}_{n,3}$	$\mathcal{B}_{n,4}$	$\mathcal{B}_{n,5}$	$\mathcal{B}_{n,6}$	$\mathcal{B}_{n,7}$	$\mathcal{B}_{n,8}$	$\mathcal{B}_{n,9}$	$\mathcal{B}_{n,10}$	$\mathcal{B}_{n,11}$	$\mathcal{B}_{n,12}$
1	1	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0
4	0	-1	0	1	0	0	0	0	0	0	0	0
5	0	0	-5	0	1	0	0	0	0	0	0	0
6	0	3	0	-15	0	1	0	0	0	0	0	0
7	0	0	49	0	-35	0	1	0	0	0	0	0
8	0	-17	0	357	0	-70	0	1	0	0	0	0
9	0	0	-809	0	1701	0	-126	0	1	0	0	0
10	0	155	0	-13175	0	6195	0	-210	0	1	0	0
11	0	0	20317	0	-120395	0	18711	0	-330	0	1	0
12	0	-2073	0	706893	0	-760100	0	49203	0	-495	0	1

Table 2: The triangular array $\mathcal{B}_{n,k}$, for $1 \le n, k \le 12$.

Again, we see on Table 2 that the $\mathcal{B}_{n,k}$ seem to be all integers and to vanish when $n-k$ is odd: this will be the next theorem. We can also, as was done in [6] for the usual Stirling numbers, display $\mathcal{A}_{n,k}$ and $\mathcal{B}_{-k,-n}$ in tandem, for $n, k \in \mathbb{Z}$. This is the purpose of Table 3, where we have left void the zero entries for $k > n$ and for odd $n-k$. The numbers which appear now in the diagonal lines are the values of the polynomial function $P_{n-k}(x)$ for integer arguments.

$k \setminus n$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-8	1															
-7		1														
-6	-70		1													
-5		-35		1												
-4	357		-15		1											
-3		49		-5		1										
-2	-17		3		-1		1									
-1		0		0		0		1								
0	0		0		0		0		1							
1		0		0		0		0		1						
2	0		0		0		0		0		1					
3		0		0		0		0		-1		1				
4	0		0		0		0		0		-5		1			
5		0		0		0		0		24		-15		1		
6	0		0		0		0		0		238		-35		1	
7		0		0		0		0		-3396		1281		-70		1

Table 3: $\mathcal{A}_{n,k}$ and $\mathcal{B}_{-k,-n}$ in tandem, for $-8 \le n, k \le 7$.

We now proceed to the proof of the integrality of these numbers.

Theorem 6.1. *If n and k are non-negative integers, then $\mathcal{B}_{n,k}$ is a triangular array of integers such that $\mathcal{B}_{n,k} = 0$ when $n-k$ is odd. Moreover, we have the inter-relations*

$$\mathcal{B}_{x,x-n} = \sum_{u\geq 0} \binom{n+x}{n-u}\binom{n-x}{n+u}\mathcal{A}_{n+u,u} \tag{6.3}$$

$$\mathcal{A}_{x,x-n} = \sum_{u\geq 0} \binom{n+x}{n-u}\binom{n-x}{n+u}\mathcal{B}_{n+u,u}. \tag{6.4}$$

Remark. Theorem 5.2 easily follows from Theorem 6.1.
Remark. Equation (6.3) and Equation (6.4) are formally the same as

$$\left\{ {x \atop x-n} \right\} = \sum_{u\geq 0} \binom{n+x}{n-u}\binom{n-x}{n+u}\left[{u+n \atop u} \right]$$
$$\left[{x \atop x-n} \right] = \sum_{u\geq 0} \binom{n+x}{n-u}\binom{n-x}{n+u}\left\{ {u+n \atop u} \right\},$$

respectively, which hold [6] for the usual Stirling numbers.

Proof of Theorem 6.1. For proving that $\mathcal{B}_{n,k}$ is integer, we are going to show that for any prime p, we have $v_p(\mathcal{B}_{n,k}) \geq 0$.

Firstly, we consider the case where p divides k. For all h such that $h \geq 1$, we have $v_p\left(B_h\binom{n}{h}\left\{ {n-h \atop k} \right\}k^h\right) \geq 0$, since $v_p(B_h) \geq -1$ by the Von Staudt-Clausen theorem and $v_p(k^h) \geq 1$. Moreover, $v_p\left(B_0\binom{n}{0}\left\{ {n-0 \atop k} \right\}k^0\right) = v_p\left(\left\{ {n \atop k} \right\}\right) \geq 0$, obviously. Then $v_p(\mathcal{B}_{n,k}) = v_p\left(\left\{ {n \atop k} \right\} + \sum_{h\geq 1} B_h\binom{n}{h}\left\{ {n-h \atop k} \right\}k^h\right) \geq 0$.

Then, we consider the case where p does not divide k. We may write

$$B_{n,k} = \sum_{h\geq 0} B_h\binom{n}{h}\left\{ {n-h \atop k} \right\}(k^h-1) + \sum_{h\geq 0} B_h\binom{n}{h}\left\{ {n-h \atop k} \right\}.$$

By the rule of multiplication of exponential generating functions [17] and given the exponential generating functions Equation (2.3) and Equation (2.6), we have

$$\begin{aligned}\sum_{h\geq 0} B_h\binom{n}{h}\left\{ {n-h \atop k} \right\} &= [[x^n]]\left(\frac{x}{e^x-1}\frac{(e^x-1)^k}{k!}\right)\\ &= \frac{1}{k}[[x^{n-1}]]\left(\frac{(e^x-1)^{k-1}}{(k-1)!}\right)\\ &= \frac{1}{k}\left\{ {n-1 \atop k-1} \right\}.\end{aligned}$$

But $v_p\left(\frac{1}{k}\left\{ {n-1 \atop k-1} \right\}\right) \geq 0$, since p does not divide k, and then it suffices to show that for all values of h, $v_p\left(B_h(k^h-1)\right) \geq 0$. This is true because either $p-1$ does not divide h, and then by the Von Staudt-Clausen theorem, $v_p(B_h) \geq 0$, or $p-1$ divides h and then $v_p(B_h) = -1$. But in this case, where p does not divide k and $p-1$ divides h, Fermat's little theorem holds and $v_p\left(k^h-1\right) \geq 1$.

Now, we turn to the proof of the inter-relations Equation (6.3) and Equation (6.4). Let p_n be a polynomial of degree n from $\mathbb{Q}[X]$. The set of binomial coefficents $\left\{\binom{x}{k}; 0 \leq k \leq n\right\}$ forms a basis for the vector space of the polynomials from $\mathbb{Q}[X]$ of degree less than $k+1$ and therefore there exists $a_{n,k}$ such that $p_n(x) = \sum_{k=0}^{n} a_{n,k}\binom{x}{k}$. By the inversion formula Equation (2.1), it is easy to verify that $a_{n,k} = \sum_u (-1)^{k-u}\binom{k}{u}p_n(u)$. We have seen that $\mathcal{A}_{x,x-n} = P_n(x)$ where P_n is a polynomial of degree $2n$ from $\mathbb{Q}[X]$, so we can apply the above general inversion scheme to $P_n(-x) = \mathcal{B}_{x+n,x}$ and we obtain

$$\begin{aligned}
\mathcal{B}_{x+n,x} &= \sum_{k=0}^{2n}\sum_{u=0}^{k}(-1)^{k-u}\binom{k}{u}\mathcal{A}_{u,u-n}\binom{-x}{k} \\
&= \sum_{k=n}^{2n}\sum_{u=n}^{k}(-1)^{u}\binom{k}{u}\binom{x+k-1}{k}\mathcal{A}_{u,u-n}
\end{aligned}$$

which, given Theorem 5.1, clearly shows that $B_{n,k} = 0$ when $n-k$ is odd. Now, since $\binom{k}{u}\binom{x+k-1}{k} = \binom{x+k-1}{k-u}\binom{x+u-1}{u}$, we have

$$\begin{aligned}
\mathcal{B}_{x+n,x} &= \sum_{k=n}^{2n}\sum_{u=n}^{k}(-1)^{u}\binom{x+k-1}{k-u}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}(-1)^{u}\sum_{k=u}^{2n}\binom{x+k-1}{x+u-1}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}(-1)^{u}\binom{2n+x}{2n-u}\binom{x+u-1}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=n}^{2n}\binom{2n+x}{2n-u}\binom{-x}{u}\mathcal{A}_{u,u-n}.
\end{aligned}$$

Then, replacing x by $x-n$ we have

$$\begin{aligned}
\mathcal{B}_{x,x-n} &= \sum_{u=n}^{2n}\binom{n+x}{2n-u}\binom{n-x}{u}\mathcal{A}_{u,u-n} \\
&= \sum_{u=0}^{n}\binom{n+x}{n-u}\binom{n-x}{n+u}\mathcal{A}_{u+n,u}.
\end{aligned}$$

Similarly, let $R_n(x) = \mathcal{B}_{x,x-n}$, which is a polynomial of degree $2n$ from $\mathbb{Q}[X]$. We apply the inversion to $R_n(-x) = \mathcal{A}_{x+n,x}$ which gives the similar identity where $\mathcal{A}$ and $\mathcal{B}$ are exchanged, and this completes the proof of the theorem. □

For the usual Stirling numbers, there exist polynomials $\sigma_n(x)$ of degree $n-1$ from $\mathbb{Q}[X]$ (also known as Stirling polynomials in [6], [11]), such that

$$Q_n(x) = \begin{bmatrix} x \\ x-n \end{bmatrix} = x(x-1)\cdots(x-n)\sigma_n(x) \tag{6.5}$$

or equivalently

$$Q_n(-x) = \begin{Bmatrix} x+n \\ x \end{Bmatrix} = (-1)^{n+1}x(x+1)\cdots(x+n)\sigma_n(-x). \tag{6.6}$$

Similarly, accounting for Theorem 5.3, we can define the polynomial $\mathcal{S}_n(x)$ of degree $n-2$, such that

$$P_n(x) = \mathcal{A}_{x,x-n} = (x+1)x(x-1)\cdots(x-n)\mathcal{S}_n(x) \tag{6.7}$$

or equivalently

$$P_n(-x) = \mathcal{B}_{x+n,x} = (-1)^{n+2}(x-1)x(x+1)\cdots(x+n)\mathcal{S}_n(-x). \tag{6.8}$$

In the following table, we give the first instances of $\mathcal{S}_n(x)$, together with the Stirling polynomial $\sigma_n(x)$, given in [6].

n	1	2	3	4
$\sigma_n(x)$	$\frac{1}{2}$	$\frac{1}{24}(3x-1)$	$\frac{1}{48}(x^2-x)$	$\frac{1}{5760}(15x^3-30x^2+5x+2)$
$\mathcal{S}_n(x)$	0	$-\frac{1}{24}$	0	$\frac{1}{5760}(7x^2+3x+2)$

Table 4: $\sigma_n(x)$ and $\mathcal{S}_n(x)$ for n in the range 1 to 4.

The second and third columns of Table 2 are known to the OEIS [14]. Up to the sign and discarding the zeros, we find in these columns the even index *Genocchi numbers* G_{2n} and *Glaisher's G-numbers*, A001469 and A002111 at the OEIS, respectively, for which exponential generating functions are known. More generally, we have the following exponential generating function for $\mathcal{B}_{n,k}$.

Theorem 6.2. *Let n, k be non-negative integers. We have*

$$\sum_{n\geq 0} \mathcal{B}_{n,k}\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}\frac{kx}{e^{kx}-1}. \tag{6.9}$$

Proof. The proof is straightforward: we use the rule of multiplication of exponential generating functions [17] and we have

$$\begin{aligned}
\frac{(e^x-1)^k}{k!}\frac{kx}{e^{kx}-1} &= \left(\sum_{j\geq 0}\left\{ j \atop k \right\}\frac{x^j}{j!}\right)\left(\sum_{j\geq 0}B_j\frac{(kx)^j}{j!}\right)\\
&=\sum_{u\geq 0}\left(\sum_{j+h=u}B_jk^j\binom{u}{h}\left\{ h \atop k \right\}\right)\frac{x^u}{u!}\\
&=\sum_{u\geq 0}\left(\sum_{j\geq 0}B_j\binom{u}{j}\left\{ u-j \atop k \right\}k^j\right)\frac{x^u}{u!}.
\end{aligned}$$

□

Unfortunately, the derivation of a similar generating function for $\mathcal{A}_{n,k}$ seems much more difficult.

We finish this section by pointing out a notable difference with the usual Stirling numbers. Whereas it is well-known that the usual Stirling matrices of both kinds are inverses of each other, this is not the case for the $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$. Let $\mathcal{A}'_{n,k}$ be the inverse of the matrix $\mathcal{A}_{n,k}$. The first entries of $\mathcal{A}'_{n,k}$ are displayed in the following table.

n	$\mathcal{A}'_{n,1}$	$\mathcal{A}'_{n,2}$	$\mathcal{A}'_{n,3}$	$\mathcal{A}'_{n,4}$	$\mathcal{A}'_{n,5}$	$\mathcal{A}'_{n,6}$	$\mathcal{A}'_{n,7}$	$\mathcal{A}'_{n,8}$	$\mathcal{A}'_{n,9}$	$\mathcal{A}'_{n,10}$
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	1	0	1	0	0	0	0	0	0	0
4	0	5	0	1	0	0	0	0	0	0
5	−9	0	15	0	1	0	0	0	0	0
6	0	−63	0	35	0	1	0	0	0	0
7	1485	0	−231	0	70	0	1	0	0	0
8	0	18685	0	−567	0	126	0	1	0	0
9	−844757	0	125515	0	−945	0	210	0	1	0
10	0	−14862727	0	600655	0	−693	0	330	0	1

Table 5: The triangular array $\mathcal{A}'_{n,k}$, for $1 \leq n, k \leq 10$.

We do not see any evident link between the matrix $\mathcal{A}'_{n,k}$ and the matrix $\mathcal{B}_{n,k}$. Moreover, whereas the Stirling matrices are convolution matrices in the sense of [11], this is not the case for the matrices $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$. This is easily checked on their first entries, as indicated in [11]. If $\mathcal{A}_{n,k}$ and $\mathcal{B}_{n,k}$ were convolution matrices, inverse to each other, there would exist a function $f(x)$ such that $\sum_n \mathcal{B}_{n,k}\frac{x^n}{n!} = \frac{f(x)^k}{k!}$ and the generating function for $\mathcal{A}_{n,k}$ would read $\sum_n \mathcal{A}_{n,k}\frac{x^n}{n!} = \frac{g(x)^k}{k!}$, where g is the compositional inverse of f. The triangular array $\frac{n!}{k!}\mathcal{B}_{n,k}$ is not even a Riordan array for which (see for instance [13]) the ordinary generating function reads $d(x) \cdot h(x)^k$ for some power series $d(x)$ and $h(x)$, with $h(0) = 0$ and $Dh(0) \neq 0$. Knowing the generating function Equation (6.9) for $\mathcal{B}_{n,k}$, the difficulty to find an analogous generating function for $\mathcal{A}_{n,k}$ has to do with these observations.

7. Discussion and Questions

Since the second column of the triangle $\mathcal{B}_{n,k}$ corresponds to the Genocchi numbers, we might consider the other columns as some sort of generalized Genocchi numbers. However, these numbers are not the same as the already known generalized Genocchi numbers from [4], nor as those from [12]. For the classical Genocchi numbers, there exists a recursion so that for $n \geq 1$, it is possible to compute $\mathcal{B}_{2n,2}$ recursively:

$$\mathcal{B}_{2n,2} = n - \frac{1}{2}\sum_{j=1}^{n-1}\binom{2n}{2j}\mathcal{B}_{2j,2}.$$

Moreover, a combinatorial interpretation has been given to the Genocchi numbers [5], but to the author's knowledge, this is not the case for Glaisher's G-numbers, $|\mathcal{B}_{2n+1,3}|$. Here we raise the more general questions: for a given $k > 2$, find a recursion for the *Bernoulli-Stirling numbers of the second kind* $\mathcal{B}_{n,k}$ and find combinatorial objects that they enumerate.

As for the *Bernoulli-Stirling numbers of the first kind* $\mathcal{A}_{n,k}$, we have even more questions. Apart from their appearance in the above investigation of congruences modulo prime powers for the cycle Stirling numbers, we don't know their mathematical interest. Any recurrence that would allow to compute an entry in this triangular array from entries of previous lines would be insightful, and might lead to a direct proof of Theorem 5.2. Moreover $\mathcal{A}_{n,k}$ cries for a generating function, of any kind, or at least a functional equation involving such a function. There also remains the problem of the combinatorial interpretation of $\mathcal{A}_{n,k}$.

By comparison to these quite complicated combinatorics questions, the study of the arthmetic properties of the Bernoulli-Stirling numbers would seem more easy, as it could be made use of their explicit expression in terms of Bernoulli and Stirling numbers and then take advantage the existing knowledge on the arithmetic properties of the latter. In particular, we might expect that the Bernoulli-Stirling numbers satisfy some sort of Kummer congruence, as do the regular Bernoulli numbers.

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