

Stochastic analysis of file-swarming systems

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Abstract

File swarming (or file sharing) is one of the most important applications in P2P networks. In this paper, we propose a stochastic framework to analyze a file-swarming system under realistic setting: constraints in upload/download capacity, collaboration among peers and incentive for chunk exchange. We first extend the results in the coupon system [L. Massoulié, M. Vojnovic, Coupon replication systems, in: Proc. ACM SIGMETRICS, Banff, Alberta, Canada, 2005] by providing a tighter performance bound. Then we generalize the coupon system by considering peers with limited upload and download capacity. We illustrate the *last-piece problem* and show the effectiveness of using forward error-correction (FEC) code and/or multiple requests to improve the performance. Lastly, we propose a framework to analyze an incentive-based file-swarming system. The stochastic framework we propose can serve as a basis for other researchers to analyze and design more advanced features of file-swarming systems.

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1. Introduction

In recent years, peer-to-peer (P2P) networks have emerged as a new paradigm for creating network applications. Recent network measurements have shown that P2P file-sharing applications constitute a large percentage of the network traffic. Also, P2P networks have a significant impact on the way new network services are designed. Unlike the traditional client–server computing paradigm, P2P networks allow the individual user (or peer) to play the roles of client and server at the same time. Therefore, peers in a P2P network can help other peers in file searching, file lookup, as well as file transfer.

File swarming (or file sharing) is one of the most important applications in P2P networks. In general, a file-swarming application has a good scalability property due to its collaborative mechanism, which can be intuitively explained as follows: a file is first partitioned into many disjoint *chunks*. Each peer can obtain these chunks either from a server, or from other peers holding those chunks that it does not already have. Each peer offers an upload service to other peers, and in return, each peer tries to obtain a missing chunk so as to maximize its ability to serve others;

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hence also the service it will receive. By coupling the service each peer can receive to its contribution to others, file-swarming applications successfully make each peer play the role of server and client at the same time. Therefore, as the number of peers increases, the service capacity of the whole system also increases. The file-swarming application is implemented in P2P file-sharing networks such as eDonkey, KaZaA, and it is the core functionality of the popular BitTorrent (BT) [25] protocol.

The work by the authors in [23] suggests that file-swarming systems (e.g. BT networks) are efficient in the sense that as the demand for the file increases, the service capacity also increases. However, which aspects of the system are critical to maintain the scalability property is not completely understood. The authors in [20] use a fluid model to represent the BT file-swarming protocol and derive a coarse approximation of the average file downloading time. Recently, a coupon model [15] was proposed to represent a generic file-swarming system. The authors analyze the system under the large population regime and show that the file-swarming system stabilizes around a finite equilibrium point and is indeed efficient. The results provide further support to the claim of [23], that the system performs well under the flash crowd scenario, even when the rarest first chunk selection policy is replaced by some random coupon selection policies. However, strong assumptions are made in [15]; in particular, the authors assume that peers have an infinite upload capacity (or relatively large as compared with the download capacity).

The aim of this paper is to provide a deeper understanding of file-swarming protocols and the efficiency of BitTorrent-like file-sharing systems. We propose a simple *density dependent jump Markov process* to model the dynamics of a file-swarming system, and we investigate the performance of the system under constraints on upload capacity, download capacity and peer selection policies (including random chunk selection and coordinated matching). The contributions of our work are as follows:

- We generalize some of the results in the coupon system [15] and provide a tighter bound for performance measures such as the average file downloading time.
- We consider the *last-piece problem* and analytically show the improvement in performance when a file-swarming system uses the forward error correction (FEC) [21] coding technique for file sharing.
- We relax the unlimited upload capacity assumption in [15], analyze the file-swarming system under a more realistic setting and provide asymptotic bounds on the average file downloading time.
- We propose a stochastic model for an incentive-based file-swarming system with coordinated matching, wherein chunk exchange is only allowed when both peers are deemed to be useful to each other.

Extensive simulations are carried out to validate our models and to illustrate some interesting design guidelines.

The balance of this paper is as follows. In Section 2, we present a generic model for a file-swarming system. In Section 3, we present an analytical model of an altruistic file-swarming system wherein each peer has an unlimited amount (or sufficiently large amount) of upload capacity and we derive performance measures such as the average file downloading time. In Section 4, we present the model of an altruistic file-swarming system with limited upload and download capacity. In Section 5, an analytical model of an incentive-based file-swarming system is presented and we derive various important performance measures. Extensive simulations and the related results are given in Section 6. Related work is given in Section 7 and Section 8 concludes.

2. Model description

Let us consider a P2P file-swarming system that distributes a given file \mathcal{F} to a number of peers. The file is divided into K equal size chunks, the i th chunk being denoted as \mathcal{C}_i , and $\mathcal{F} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_K$, with $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for $i \neq j$. To download the file \mathcal{F} , a peer needs to download all K chunks from other peers in this P2P file-swarming system. Let \mathcal{F}_A be the set of chunks that peer A possesses. Peer A maintains a *bitmap* to denote which chunks they possess. Whenever peer A finishes the downloading of a new chunk, it will update its bitmap. Peer A can upload chunk \mathcal{C}_k to others only after it has completely downloaded \mathcal{C}_k . New peers arrive to this system according to a Poisson process with an average rate λ . Using the BitTorrent's terminology, a peer that has at least one missing chunk of \mathcal{F} is called a *leecher*, while a peer that has all K unique chunks of \mathcal{F} is called a *seeder*. Note that, unlike the BitTorrent system, which has at least one seeder to start the file distribution and serve the leechers, we assume that every newly arrived peer will initially obtain one chunk from a server before entering this system.¹ This initial chunk is randomly chosen

¹ This assumption is similar to the one made in [15].

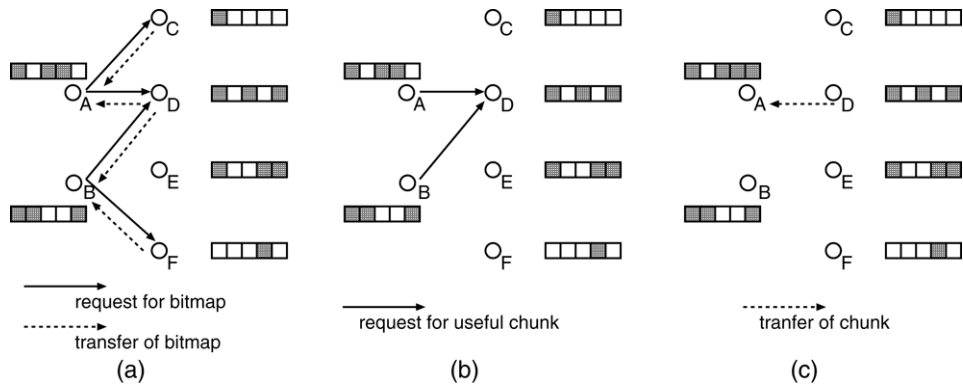


Fig. 1. A simple illustration of a transfer dynamic within one time slot with $\mathcal{F} = \mathcal{C}_1 \cup \mathcal{C}_2 \cdots \cup \mathcal{C}_5$.

by the server with equal probability $1/K$ for chunks $\mathcal{C}_1 \dots \mathcal{C}_K$. When a peer finishes downloading all K chunks, the peer will depart immediately.

Similarly to [15], we assume that this P2P file-swarming is slotted in the sense that uploading (or downloading) a single chunk takes one slot time. The file distribution process in each time slot can be described as follows. At the beginning of every time slot, a peer, say A , will select $m \geq 1$ other peers in the system and fetch their bitmaps. Note that the parameter m and the way it chooses these m peers will greatly affect the system performance, and we will investigate this further in later sections. Since the bitmap information can be greatly compressed, the transfer time of a bitmap is negligible compared to the transfer time of a chunk. Let peer B be one of these m peers. Upon receiving its bitmap, peer A can determine whether peer B is useful (i.e. peer B possesses at least one missing chunk of peer A , or $\mathcal{F}_B \setminus \mathcal{F}_A \neq \emptyset$). If no peer among these m selected peers is useful to peer A , then peer A will take no action but remain idle in the current time slot; otherwise, peer A will randomly select one of the useful peers to request a useful chunk for download. Assume the selected peer is B , then peer A will request one chunk which is uniformly chosen from the set of chunks possessed by peer B and is missing in peer A (i.e. a chunk $\mathcal{C}_k \subset \mathcal{F}_B \setminus \mathcal{F}_A$). Note that this can be viewed as a *blind chunk selection policy*, in contrast to the *rarest first policy* in the BitTorrent protocol by which peer A will select the chunk among $\mathcal{F}_B \setminus \mathcal{F}_A$ with the fewest number of copies among its neighbors [3]. As a result, peer B may receive *multiple* downloading requests. Based on the upload capacity constraint and service rule, peer B will choose one or more requests to satisfy (we will elaborate this in later sections). The transfer time of this chunk will take one time slot. At the end of a time slot, the process repeats.

Fig. 1 illustrates the P2P file-sharing model with $m = 2$. We have six peers: A, B, C, D, E and F . The file has five chunks and the shaded boxes represent the chunks that peers possess. For example, peer A has $\mathcal{C}_1, \mathcal{C}_3$ and \mathcal{C}_4 . In Fig. 1(a), peer A (peer B) requests bitmaps from peer C and D (peer D and F) and these peers reply with their respective bitmaps. Peer A determines that peer C is not useful while peer D is useful. Peer B , on the other hand, determines that both peer D and F are useful. Both peers select one peer for a chunk transfer and Fig. 1(b) shows that both peer A and B choose D for the chunk transfer. Peer D receives two transfer requests, it randomly picks one peer to serve in this example, and it chooses peer A . Fig. 1(c) shows that peer D transfers \mathcal{C}_5 which is requested by A . At the end of a time slot, peer A obtains \mathcal{C}_5 while peer B wastes one time slot.

The above model is in fact, quite general. For example, when one considers the case that $m = 1$ (or each peer just randomly chooses one peer to fetch the bitmap), and that there is no constraint on peers' upload capacity, then this becomes the model studied in the coupon replication system [15]. In this work, we generalize their model and study the performance of the system when $m \geq 1$, which means that each peer can first fetch multiple bitmaps from different peers but can choose at most one peer to request chunk transfer. Surprisingly, such a simple modification can improve the performance of the system to achieve a near-optimal average file downloading time. Furthermore, we also relax the assumption of large or infinite upload capacity in [15]. This is in fact a very important step because for the current Internet, the bottleneck is usually not at the network core but rather at the network edge, and the upload capacity of an end host is indeed limited (e.g. ADSL system, cable system). Therefore, this capacity constraint model is in fact a more realistic representation for file-swarming systems. In this uplink/downlink constrained system, we study two different uploading policies.

1. *Altruistic uploading service*: Under this policy, a peer will provide an upload service to other peers regardless of whether or not these peers have provided an upload service to other peers. In other word, this is a perfect collaborative system and it is similar to the “optimistic unchoking” feature in the BitTorrent protocol.
2. *Incentive uploading service*: Under this policy, a peer follows a given incentive mechanism similar to the “tit-for-tat” feature used in the BitTorrent protocol to decide on uploading.

Although our system model is a simple representation of some realistic P2P file-swarming system (e.g. BitTorrent), it has already captured many essential features such as the *collaborative* upload and download, as well as incentive-based chunk exchange in P2P file-swarming systems. In later sections, we will derive the performance of such a system, and show why and how it can achieve good performance.

3. Altruistic file-swarming system with constraint in download capacity

In this section, we consider the file-swarming system where each peer has a constraint in the download capacity and we place *no upper-bound restriction* on the upload capacity. So at every time slot, each peer will first contact $m \geq 1$ other peers randomly in the system to acquire their bitmaps. If more than one peer is useful, it will randomly choose one to request a useful chunk. It is possible that a peer may acquire many downloading requests. Since we assume that there is no restriction on uploading bandwidth, all requests will be satisfied. Also, due to the abundance of uploading bandwidth, there is no need to enforce an incentive mechanism for data transfer. Lastly, it is important to note that when $m = 1$, this corresponds to the model described in the coupon replication system [15].

3.1. Model formulation

First we assume that all types of chunk in the system are uniformly distributed. This assumption can be guaranteed by the random chunk selection policy (as described in Section 2). We classify peers into different types according to the number of chunks it possesses. A peer holding i chunks is called a type i peer, for $i = 1, 2, \dots, K - 1$ ($i \neq K$ because a peer will immediately depart from the system when it finishes downloading all K chunks). After receiving a new chunk, a type i peer will become a type $(i + 1)$ peer. Let $p_{i,j}$ denote the probability that a type j peer B is useful to a type i peer A . When $i < j$, it is clear that $p_{i,j} = 1$; When $i \geq j$, we have $p_{i,j} = 1 - \text{Prob}\{\mathcal{F}_B \subseteq \mathcal{F}_A\}$. Thus

$$p_{i,j} = \begin{cases} 1 & 1 \leq i < j \leq K - 1, \\ 1 - \frac{C_i^j}{C_K^j} & 1 \leq j \leq i \leq K - 1 \end{cases} \quad (C_x^y \text{ is the binomial coefficient}). \tag{1}$$

Let $y_i(t)$ denote the number of type i peers in the system at time t . The total number of peers in the system at time t is $y(t) = \sum_{i=1}^{K-1} y_i(t)$. When a type i peer randomly picks another peer and requests its bitmap, the probability that this selected peer is useful is $q_i(t) = \sum_{j=1}^{K-1} p_{i,j} y_j(t) / y(t)$, $i = 1, 2, \dots, K - 1$.

Given the system state $\mathbf{Y}(t) = \{y_i(t)\}_{i \in \{1, \dots, K-1\}}$, it is easy to verify that $(\mathbf{Y}(t))_{t \geq 0}$ is a Markov process taking its values in \mathcal{Z}_+^{K-1} (\mathcal{Z}_+^{K-1} is a $K - 1$ dimensions vector with non-negative integer entities). Denoting by e_i the unit vector of \mathcal{Z}_+^{K-1} whose i -coordinate equals 1, and with all other coordinates equal to zero, the non-zero transition rates of this Markov process are, for all $i \in \{1, \dots, K - 1\}$,

$$\begin{aligned} \mathbf{Y} &\longrightarrow \mathbf{Y} + e_1 && \text{with rate } \lambda, \\ \mathbf{Y} &\longrightarrow \mathbf{Y} - e_i + e_{i+1} && \text{with rate } y_i (1 - (1 - q_i)^m), i \in \{1, \dots, K - 2\} \\ \mathbf{Y} &\longrightarrow \mathbf{Y} - e_{K-1} && \text{with rate } y_{K-1} (1 - (1 - q_{K-1})^m). \end{aligned}$$

We analyze the system under a large population asymptotic regime. Note that this is a *density dependent jump Markov process* [13]. It converges to the solution of the differential equations

$$\frac{dy_i(t)}{dt} = \begin{cases} \lambda - y_1(t) [1 - (1 - q_1(t))^m] & i = 1, \\ y_{i-1}(t) [1 - (1 - q_{i-1}(t))^m] - y_i(t) [1 - (1 - q_i(t))^m] & i = 2, \dots, K - 1 \end{cases} \tag{2}$$

for some initial condition $\mathbf{Y}(0)$.

3.2. Steady state analysis

In this section, we derive the average file downloading time for the above P2P file-swarming system. We also extend our analysis to a file-swarming system that provides an FEC service.

3.2.1. Altruistic file-swarming without FEC

In this section we focus on the steady state performance and its *equilibrium point*. An equilibrium point is the point $\hat{Y} = (y_1, y_2, \dots, y_{K-1})$ such that if $Y(t) = \hat{Y}$, then $Y(t') = \hat{Y}$ for all $t' \geq t$. The necessary and sufficient condition for \hat{Y} to be an equilibrium point is $\frac{dy_i(t)}{dt} = 0$, for $1 \leq i \leq K - 1$. Applying these conditions to Eq. (2), we have the following equations at the equilibrium point \hat{Y} : $\lambda = y_i (1 - (1 - q_i)^m)$, $i = 1, 2, \dots, K - 1$.

Let T_i be the average sojourn time for type i peers, that is, the average time for a type i peer to receive a new chunk and become type $(i + 1)$. One can derive this measure from the equilibrium point $\hat{Y} = (y_1, \dots, y_{K-1})$ by using Little’s theorem [12]: $\lambda T_i = y_i$. Define $T = \sum_{j=1}^{K-1} T_j$ as the average file downloading time in the P2P file-swarming system, we have $y_i/y = T_i/T$. Finally, one can obtain the following equations at the equilibrium point \hat{Y} :

$$T_i = \frac{1}{1 - (1 - q_i)^m} \quad \text{and} \quad q_i = \sum_{j=1}^{K-1} \frac{T_j}{T} p_{i,j}, \quad \text{for } i = 1, 2, \dots, K - 1. \tag{3}$$

One can observe that T_i of Eq. (3) does not depend on λ . So even when the arrival rate λ is large and the number of peers in the system becomes very large, the average sojourn time T_i (and also T) will not be affected in the steady state. This is an important observation since this indicates that the file-swarming system has a good scaling property: when one increases the arrival rate, the performance will not degrade. Since T_i is the average sojourn time for type i peers, i.e. it takes on average, T_i unit of time slots to download the next chunk when a peer holds i chunks, let us explore the relationships among the T_i ’s at the steady state.

Lemma 1. *The sojourn time is an increasing sequence, i.e. $1 \leq T_1 < T_2 < \dots < T_{K-1}$.*

Proof. According to Eq. (3) we have $q_i \leq 1$. Therefore, one can conclude that $T_i \geq 1$ for $i = 1, \dots, K - 1$. According to Eq. (1), when $i > i'$, $p_{i,j} \leq p_{i',j}$ holds for $j = 1, \dots, K - 1$ and $p_{i,j} < p_{i',j}$ holds for some j . So $q_i = \sum_{j=1}^{K-1} \frac{T_j}{T} p_{i,j} < \sum_{j=1}^{K-1} \frac{T_j}{T} p_{i',j} = q_{i'}$. Thus, we have $T_i > T_{i'}$ when $i > i'$. ■

Lemma 2. *The upper and lower bounds of T_i are*

$$\frac{1}{1 - \left[\left(\frac{1}{K-2+H_K} \right) \left(\frac{i}{K-i+1} \right) \right]^m} + O(K^{-2}) < T_i < \frac{1}{1 - \left[\left(\frac{1}{K-1} \right) \left(\frac{i}{K-i+1} \right) \right]^m},$$

where K is the number of chunks in \mathcal{F} and H_K is the K th harmonic number.

Proof. The sequence $\{t_j = T_j/T\}$ is increasing and the sequence $\{a_j = p_{i,j}\}$ in non-decreasing. From Chebyshev’s sum inequality, we have

$$\begin{aligned} q_i &> \frac{1}{K-1} \left(\sum_{j=1}^{K-1} \frac{T_j}{T} \right) \left(\sum_{j=1}^{K-1} p_{i,j} \right) = \frac{1}{K-1} \left(K-1 - \sum_{j=1}^i \frac{C_i^j}{C_K^j} \right) \\ &= 1 - \left(\frac{1}{K-1} \right) \left(\frac{i}{K-i+1} \right) \quad (\text{“Concrete Mathematics” [9], p. 174}). \end{aligned}$$

One can apply it to Eq. (3) and obtain the upper bound of T_i as claimed. For the lower bound of T_i , let us first derive an upper bound of T , which is

$$\begin{aligned} T &= \sum_{i=1}^{K-1} \frac{1}{1 - (1 - q_i)^m} \leq \sum_{i=1}^{K-1} \frac{1}{q_i} < \sum_{i=1}^{K-1} \frac{(K-1)(K-i+1)}{K(K-i)-1} \\ &= \frac{(K-1)^2}{K} + \frac{K^2-1}{K} \sum_{i=1}^{K-1} \frac{1}{Ki-1} = K-2 + H_K + O(K^{-1}). \end{aligned}$$

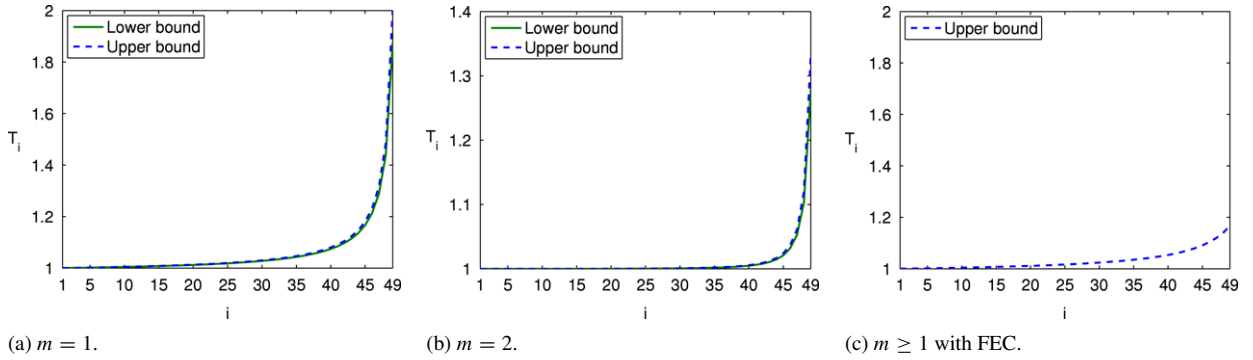


Fig. 2. Illustration on the last-piece problem: bounds of T_i for $m = 1, 2$ and $m \geq 1$ with FEC. $K = 50$ chunks.

We can apply it to Eq. (3) to obtain an upper bound of q_i as

$$\begin{aligned}
 q_i &= 1 - \sum_{j=1}^i \left(\frac{T_j}{T} \right) \left(\frac{C_i^j}{C_K^j} \right) < 1 - \sum_{j=1}^i \frac{C_i^j}{C_K^j} \left(\frac{1}{K - 2 + H_K + O(K^{-1})} \right) \\
 &= 1 - \left(\frac{1}{K - 2 + H_K} \right) \left(\frac{i}{K - i + 1} \right) + O(K^{-2}).
 \end{aligned}$$

With this upper bound of q_i , one can substitute it to Eq. (3) to obtain the lower bound of T_i as claimed. ■

Remark. The importance of the above two lemmas is that one can use them to understand the “last-piece” problem in P2P file-swarming systems. i.e. how long it takes for a peer to receive the last few chunks of the file since it becomes increasingly more difficult to find other peers that can help.

To illustrate this issue, let us consider the upper and lower bounds of T_i for a file with $K = 50$ chunks. The scenario is illustrated in Fig. 2(a) and (b). There are two important observations. First, one can observe that the upper and lower bounds are indeed *very tight*, which implies that we can use T_i to give a very accurate measure of the average file downloading time T . Second, one can observe that the sojourn times T_i are very close to 1 for $i \ll K - 1$, but when i approaches $K - 1$, Fig. 2(a) (and Fig. 2(b)) shows that both bounds approach 2 (approach 1.4) quickly. The increasing downloading time, especially for the last few chunks, depicts the last-piece problem. Intuitively, the reason for this problem is that it becomes more and more difficult for a peer to find other peers that are useful, especially when the peer is very close to finishing downloading the whole file. However, one can amend this problem, at least to a certain degree, by simply changing the parameter m . One can observe that when $m = 1$ (as shown in Fig. 2(a)), it costs 2 time slots on average to download the last chunk but when $m = 2$ (as shown in Fig. 2(b)), it only costs 1.4 time slots to obtain the last chunk. The reason is that when $m = 2$, peers can ask for more peers for bitmaps and thereby increase the chance to find useful peers. Given m , we can derive the bounds of T from Lemma 2.

Theorem 1. When $m = 1$, the average downloading time $T = K - 2 + H_K + O(\frac{\log^2 K}{K})$.

Proof. In the proof of Lemma 2, we have obtained $T < K - 2 + H_K + O(K^{-1})$. For the lower bound of T , let us denote $A = K - 2 + H_K$, then

$$\begin{aligned}
 T &= \sum_{i=1}^{K-1} T_i > \sum_{i=1}^{K-1} \frac{1}{1 - \frac{1}{A} \left(\frac{i}{K-i+1} \right)} + O(K^{-1}) \\
 &= \frac{A}{A+1} \left(K - 1 + \frac{K+1}{A+1} \sum_{j=1}^{K-1} \frac{1}{j} \right) + O\left(\frac{\log K}{K} \right) = K - 2 + H_K + O\left(\frac{\log^2 K}{K} \right).
 \end{aligned}$$

Combining the upper and lower bounds, Theorem 1 can be shown as claimed. ■

Remark. Note that when $m = 1$, the system corresponds to the “open and flat” case of the coupon system [15], in which the authors give an upper bound $T < K + O(\sqrt{K})$. However, the result in Theorem 1 states that

$T = K - 2 + H_K + O(\frac{\log^2 K}{K})$. We know that H_K is the K th harmonic number, $H_K = \log K + \gamma + O(K^{-1})$, where $\gamma = 0.5772\dots$ is the *Euler–Mascheroni constant*. Thus $T = K + \log K + O(1)$. Therefore, we obtain a tighter bound than [15].

Similarly, we can derive the lower and upper bounds of T from Lemma 2 when $m \geq 2$. Due to the lack of space, we only show the derivation of the upper bound in the following theorem.

Theorem 2. When $m \geq 2$, the average downloading time $T < K + O(\frac{\log K}{K})$.

Proof.

$$\begin{aligned} T &< \sum_{i=1}^{K-1} \frac{1}{1 - \left[\left(\frac{1}{K-1}\right)\left(\frac{i}{K-i+1}\right)\right]^2} = K - 1 + \sum_{i=1}^{K-1} \frac{\left[\left(\frac{1}{K-1}\right)\left(\frac{i}{K-i+1}\right)\right]^2}{1 - \left[\left(\frac{1}{K-1}\right)\left(\frac{i}{K-i+1}\right)\right]^2} \\ &< K - 1 + \frac{4}{3} \sum_{i=1}^{K-1} \left[\left(\frac{1}{K-1}\right)\left(\frac{i}{K-i+1}\right)\right]^2 \\ &= K - 1 + \frac{4}{3(K-1)^2} \sum_{i=1}^{K-1} \left[\frac{(K+1)^2}{(K-i+1)^2} - \frac{2(K+1)}{K-i+1} + 1\right] \\ &\leq K - 1 + \frac{4}{3}(\zeta(2) - 1) + O\left(\frac{\log K}{K}\right) < K + O\left(\frac{\log K}{K}\right). \quad \blacksquare \end{aligned}$$

Remark. Since it is necessary to require at least $K - 1$ time slots to finish the downloading of the whole file \mathcal{F} , we can conclude by fetching multiple bitmaps (setting $m \geq 2$), the average downloading time is near-optimal. To see this, one can compare it with the result in Theorem 1, which states that it takes at least $K + \log(K) + O(1)$ time slots to finish the downloading, and we remove the $\log(K)$ term by getting more than one bitmap. Setting $m = 2$ is sufficient for achieving the near-optimal performance. This result is encouraging and insightful, it shows that due to the diversity of chunks held and the altruistic uploading for every peer, a “simple-design” can achieve very good performance.

3.2.2. Altruistic file-swarming with FEC

We have seen that by fetching bitmaps from multiple peers, the system performance can reach near-optimal levels. Here, we provide an *alternative approach* to reach the near-optimal performance by using the *forward error correction* (FEC) coding technique [21]. Given a file \mathcal{F} , one can encode the original K chunks to $Q = (1 + \alpha)K$ chunks with erasure codes before the distribution process. Any peer can reconstruct the original file \mathcal{F} after it receives *any* K distinct chunks of these Q chunks. This technique makes it unnecessary to download the “last” chunk and will ease the last-piece problem, making the system more efficient. To make this claim formally, we have the following theorem:

Theorem 3. For $m \geq 1$, using FEC with redundancy rate of $\alpha > 0$, the average downloading time $T_{\text{FEC}} < K - 2 + (1 + \alpha) \log \frac{1+\alpha}{\alpha} + O(K^{-1})$.

Proof. Note that FEC makes $p_{i,j} = 1 - C_i^j / C_Q^j$ when $1 \leq j \leq i \leq K - 1$ and all other equations remain the same. Similarly to the proof of Lemma 2, one can derive that $T_i < [1 - (\frac{1}{K-1})(\frac{i}{Q-i+1})]^{-1}$. So

$$\begin{aligned} T_{\text{FEC}} &< \sum_{i=1}^{K-1} \frac{1}{1 - \left(\frac{1}{K-1}\right)\left(\frac{i}{Q-i+1}\right)} = \frac{K-1}{K} \sum_{i=1}^{K-1} \left(1 + \frac{1}{\frac{K(Q-i+1)}{Q+1} - 1}\right) \\ &= \frac{(K-1)^2}{K} + \frac{(K-1)(Q+1)}{K^2} \sum_{j=Q-K+1}^{Q-1} \frac{1}{j} + O(K^{-1}) \\ &= K - 2 + (1 + \alpha) \log \frac{1+\alpha}{\alpha} + O(K^{-1}). \quad \blacksquare \end{aligned}$$

Remark. Compared with Theorem 1, the harmonic term H_K is replaced with the term $(1 + \alpha) \log \frac{1+\alpha}{\alpha}$. Note that, when $\alpha = 0.1$ (i.e. 10% redundancy), this term is less than 2.7. Thus, given a particular redundancy rate α , T_{FEC} is bounded by $K - 1$ plus a small constant. So by using FEC codes, even if a peer only contacts one other peer for a bitmap (i.e. $m = 1$), the average downloading time T can still approach the near-optimal value.

Gkantsidis and Rodriguez [8] declare that traditional P2P content distribution software such as BitTorrent usually suffers from last-piece problem and it could be settled by the network-coding technique they propose. In our model we have seen that there exists a last-piece problem as Fig. 2(a) and (b) shown. It takes about 2 time slots on average to download the last piece. To illustrate how FEC affects the last-piece problem, let us consider the upper bound of T_i for a file with $K = 50$ chunks again. By setting $\alpha = 0.1$ (i.e. 10% redundancy), we show the upper bound of T_i in Fig. 2(c). This bound holds for all $m \geq 1$. From Fig. 2(c), one can observe that the last-piece problem can be eased if we use the FEC technique to generate a few redundant chunks. This observation is helpful for the advanced P2P content distribution system design in the future.

4. Altruistic file-swarming system with download and upload constraints

In this section, we consider the file-swarming system where each peer has a limited bandwidth on the download and upload capacity. Note that this is a more realistic setting than the unlimited upload bandwidth assumption in Section 3 and the coupon replication system [15]. This is a very important point since in the current Internet, the bottleneck is not at the network core but rather at the edge, and usually the upload capacity of a host is indeed limited (e.g. ADSL or the cable system). To simplify our analysis, we only consider the case $m = 1$ (i.e. in each time slot, peer A will first contact *one* other peer randomly in the system to obtain its bitmap). If this peer can help peer A , peer A will request a useful chunk. It is possible that a peer may obtain multiple requests for a chunk. Due to the upload capacity constraint, this peer will only randomly pick one peer to upload. If peer A is chosen, then peer A can download one useful chunk within the current time slot. Otherwise, peer A will remain idle for the current time slot.

4.1. Model formulation

As in Section 3, let $p_{i,j}$ denote the probability that a type j peer is useful to the type i peer, $y_i(t)$ denote the number of type i peers in the system at time t . The total number of peers in the system at time t is $y(t) = \sum_{i=1}^{K-1} y_i(t)$. When a type j peer is requested by another peer for its bitmap, the probability that this request comes from a type i peer is $y_i(t)/y(t)$. Thus, the probability that the type j peer is useful to a peer who contacts it is $\beta_j(t) = \sum_{i=1}^{K-1} p_{i,j} y_i(t)/y(t)$.

Assume that peer A contacts peer B and B is of type j . Peer A finds that B is useful and sends B a request for a chunk. Let us consider the probability that A will be chosen by B for service. To derive this probability, we consider how many other peers contacted B for its bitmap. Since there are $y - 2$ peers (ignoring A and B) in the system selecting others to contact and B is contacted by a particular peer with probability $1/(y - 1)$ (each peer does not contact itself). Thus the number of peers that contacted B , denoted by the random variable R , is the number of successes in a sequence of $y - 2$ independent Bernoulli trials, or $R \sim \text{Bernoulli}(y - 2, \frac{1}{y-1})$. Since $y - 2$ is large and $(y - 2)/(y - 1) \sim 1$, R can be approximated as a Poisson random variable with mean 1, thus R has a probability mass function of $f_R(k) = e^{-1}/k!$, for $k \in \{0, 1, \dots\}$.

Assume $R = r$ (i.e. peer B was contacted by r peers for its bitmap). The probability that peer B is useful to a peer in R is $\beta_j(t)$. Thus B receives k requests for a chunk with probability $C_r^k \beta_j^k(t) (1 - \beta_j(t))^{r-k}$ for $k \leq r$. When A contacts B , finds B is useful and also sends B a request for a chunk, the probability that A is chosen by B for service is

$$\alpha_{j,r}(t) = \sum_{k=0}^r C_r^k \beta_j^k(t) (1 - \beta_j(t))^{r-k} \frac{1}{k+1} = \frac{1 - (1 - \beta_j(t))^{r+1}}{(r+1)\beta_j(t)}.$$

The system can be modeled as a Markov process $\mathbf{Y}(t) = \{y_i(t)\}_{i \in \{1, \dots, K-1\}}$. Again, it is easy to verify that $(\mathbf{Y}(t))_{t \geq 0}$ is a Markov process taking its values in \mathcal{Z}_+^{K-1} . The non-zero transition rates of this Markov process, for all $i \in \{1, \dots, K - 1\}$ is

$$\mathbf{Y} \longrightarrow \mathbf{Y} + e_1 \quad \text{with rate } \lambda,$$

$$\begin{aligned}
 Y &\longrightarrow Y - e_i + e_{i+1} \quad \text{with rate } y_i \sum_{j=1}^{K-1} \left[\frac{y_j}{y} p_{i,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r} \right], \quad i \in \{1, \dots, K-2\} \\
 Y &\longrightarrow Y - e_{K-1} \quad \text{with rate } y_{K-1} \sum_{j=1}^{K-1} \left[\frac{y_j}{y} p_{K-1,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r} \right].
 \end{aligned}$$

For a large population asymptotic regime, this density dependent jump Markov process converges to the solution of the system of differential equations

$$\begin{aligned}
 \frac{dy_1(t)}{dt} &= \lambda - y_1(t) \sum_{j=1}^{K-1} \left[\frac{y_j(t)}{y(t)} p_{1,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r}(t) \right], \\
 \frac{dy_i(t)}{dt} &= y_{i-1}(t) \sum_{j=1}^{K-1} \left[\frac{y_j(t)}{y(t)} p_{i-1,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r}(t) \right] \\
 &\quad - y_i(t) \sum_{j=1}^{K-1} \left[\frac{y_j(t)}{y(t)} p_{i,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r}(t) \right], \quad i = 2, \dots, K-1
 \end{aligned}$$

with some initial condition $Y(0)$.

4.2. Steady state analysis

We focus on the steady state performance and we are interested in its *equilibrium point*. In other words, the operating point wherein $dy_i/dt = 0$ for $1 \leq i \leq K-1$. Define T_i as the sojourn time for type i peer. It follows from Little’s theorem that $\lambda T_i = y_i$. Let the average file downloading time be $T = \sum_{j=1}^{K-1} T_j$, one can obtain the following equations at the equilibrium point:

$$\frac{1}{T_i} = \sum_{j=1}^{K-1} \left(\frac{T_j}{T} p_{i,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \alpha_{j,r} \right), \quad i = 1, 2, \dots, K-1 \tag{4}$$

where

$$\alpha_{j,r} = \frac{1 - (1 - \beta_j)^{r+1}}{(r+1)\beta_j} \quad \text{and} \quad \beta_j = \sum_{i=1}^{K-1} \frac{T_i}{T} p_{i,j}, \quad j = 1, 2, \dots, K-1.$$

In Section 3, we have shown that a file-swarming system that has only download capacity constraint is very efficient. With both download and upload capacity constraints, the performance of the system will not be as good. In this section, we seek to derive the bounds of T_i (and thereby T) to gain insight on how the upload capacity constraint can affect the system performance. Let us first state the upper and lower bounds of the sojourn time T_i .

Theorem 4. *The sojourn time T_i satisfies*

$$\frac{1}{1 - e^{-1}} + O\left(\frac{\log K}{K}\right) < T_i < \left[\frac{1}{1 - e^{-1}} \right] \left[\frac{1}{1 - \left(\frac{1}{K-1}\right) \left(\frac{i}{K-i+1}\right)} \right].$$

Proof. Because $\beta_j < 1, r \geq 0$, we have $\alpha_{j,r} \geq 1/(r+1)$. From Eq. (4), we use the same technique in proving the lower bound of q_i in Lemma 2:

$$\begin{aligned}
 \frac{1}{T_i} &\geq \sum_{j=1}^{K-1} \left(\frac{T_j}{T} p_{i,j} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \frac{1}{r+1} \right) = (1 - e^{-1}) \sum_{j=1}^{K-1} \frac{T_j}{T} p_{i,j} \\
 &> [1 - e^{-1}] \left[1 - \left(\frac{1}{K-1}\right) \left(\frac{i}{K-i+1}\right) \right].
 \end{aligned}$$

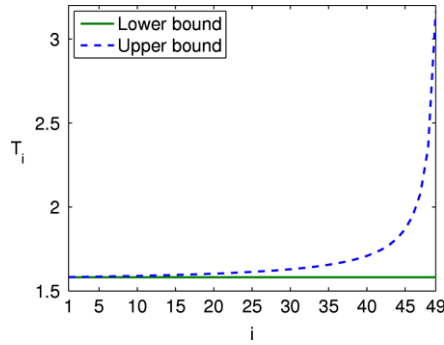


Fig. 3. Numerical results illustrated for the bounds of T_i for $m = 1$ when $K = 50$.

Therefore, the upper bound of T_i is obtained. For the lower bound of T_i , we have $\alpha_{j,r} \leq [1 + r(1 - \beta_j)]/(r + 1)$ because $\beta_j < 1$ and $r \geq 0$. Thus

$$\frac{1}{T_i} \leq \sum_{j=1}^{K-1} \left[\frac{T_j}{T} \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} \frac{1 + r(1 - \beta_j)}{r + 1} \right] = 1 - e^{-1} \sum_{j=1}^{K-1} \frac{T_j}{T} \beta_j < 1 - \frac{e^{-1}}{K-1} \sum_{j=1}^{K-1} \beta_j.$$

One can obtain an upper bound on the summation term as

$$\begin{aligned} \sum_{j=1}^{K-1} \beta_j &= \sum_{i=1}^{K-1} \frac{T_i}{T} \left(K - 1 - \frac{i}{K - i + 1} \right) = K - \frac{K + 1}{T} \sum_{i=1}^{K-1} \frac{T_i}{K - i + 1} \\ &> K - \frac{K + 1}{K - 1} \sum_{i=1}^{K-1} \frac{1}{(K - i + 1)(1 - e^{-1}) \left[1 - \left(\frac{1}{K-1} \right) \left(\frac{i}{K-i+1} \right) \right]} \\ &= K - \frac{H_K}{1 - e^{-1}} + O\left(\frac{\log K}{K}\right). \end{aligned}$$

Finally, the lower bound of T_i can be obtained as

$$\frac{1}{T_i} < 1 - \frac{e^{-1}}{K-1} \sum_{j=1}^{K-1} \beta_j < 1 - e^{-1} + O\left(\frac{\log K}{K}\right). \quad \blacksquare$$

Fig. 3 illustrates the upper and lower bounds of T_i for a file with $K = 50$ chunks and $m = 1$. Notice that the lower bound of T_i is rather loose since it is not related to the index i . Nevertheless, the spread of the bounds is tight for most values of T_i . Another observation is that for small values of i , T_i is not close to 1 any more as in the case of Section 3, but rather, close to $1/(1 - e^{-1})$. This performance degradation is contributed to the constraint on the upload capacity. In other words, if one limits the number of chunks that a peer can upload each time slot, it takes longer, on average, to obtain the file. Lastly, with the upper and lower bounds of T_i , one can derive the average downloading time T .

Theorem 5. *The average downloading time T satisfies*

$$\frac{K}{1 - e^{-1}} + O(\log K) < T < \frac{1}{1 - e^{-1}}(K - 2 + H_K) + O(K^{-1}).$$

Proof. Given the upper bound of T_i , one can use the approach similar to Lemma 2 to derive that $T = \sum_{i=1}^{K-1} T_i < (K - 2 + H_K)/(1 - e^{-1}) + O(K^{-1})$. With the lower bound of T_i , we have

$$T = \sum_{i=1}^{K-1} T_i > (K - 1) \left[\frac{1}{1 - e^{-1}} + O\left(\frac{\log K}{K}\right) \right] = \frac{K}{1 - e^{-1}} + O(\log K). \quad \blacksquare$$

Compared with **Theorem 1**, the average downloading time has been scaled up by a factor of $1/(1 - e^{-1})$ when K is large. It is interesting to explore whether using FEC can improve the performance of the system. We have the following result.

Lemma 3. *When one uses FEC in this system, the bounds of T_i as specified in **Theorem 4** and the average downloading time T as specified in **Theorem 5** will remain the same.*

Proof. Similarly to Section 3.2.2, FEC will increase the value of $p_{i,j}$ and other equations remain the same. Thus the upper bound of T_i in **Theorem 4** still holds. Notice that we just replaced $p_{i,j}$ by 1 in the proof of the lower bound of T_i in **Theorem 4**. And $p_{i,j} \leq 1$ still holds even with FEC, thus the lower bound of T_i in **Theorem 4** also still holds. We know that **Theorem 5** is derived from 4 directly, thus the bounds in **Theorem 5** also remain the same. ■

Lemma 3 implies that FEC could not improve the performance very much. It can be explained as follows. The random peer selection policy may cause request collision since a peer may receive multiple chunk requests but can only serve one peer. Other peers requesting a chunk from the same peer will waste their time slot.

5. Incentive file-swarming via coordinated matching

From **Theorem 5**, one can observe that when there are both upload/download capacity constraints on cooperative peers and peers use a random peer selection policy, the average downloading time $T = \frac{K}{1-e^{-1}} + O(\log K)$, where the coefficient of the term K is $\frac{1}{1-e^{-1}} \approx 1.58$. The system performance degrades as compared with the file-swarming system without upload capacity constraint where the coefficient of term K is 1. The performance degradation can be explained as follows: the random peer selection may cause request collision since a peer may receive multiple chunk requests but can only serve one request. Therefore, some peers may waste the download opportunity and remain idle for a time slot. For the case of unlimited upload capacity, all requests can be satisfied, hence, the performance is better.

One may ask, in the system with both download and upload capacity constraints, can the system still achieve good performance by using peer selection algorithms other than the random policy? In the following, we show that by running a maximal matching algorithm (usually regarded as an “easy problem” with an efficient polynomial algorithm) at the beginning of every time slot, one can significantly improve the system performance. Also, we show that with built-in incentive mechanisms, this approach can also provide very good performance.

5.1. Without incentive mechanism

We assume that at the beginning of each time slot, every peer will run some distributed *maximal matching algorithm* [10], or obtains the help from some central server, so that peer A will find peer B as its neighbor while peer B will also find A as its neighbor. If the matching process is *independent* of the chunks held by each peer, then given peer A , the probability that peer B is of type i is y_i/y where y_i is the number of type i peers and y is the total number of peers in the system. At the current time slot, peer A can only communicate with peer B and vice versa and the matched peers can upload and download at most one chunk per time slot.

Let us first study the system without an incentive mechanism. When peer A and peer B are matched, peer A will help peer B iff peer A is useful to peer B (i.e. $\mathcal{F}_A \setminus \mathcal{F}_B \neq \emptyset$); similarly peer B will help peer A iff $\mathcal{F}_B \setminus \mathcal{F}_A \neq \emptyset$. Since neighbor selection is independent of peer type, we obtain the differential equations for the number of type i peers as

$$\frac{dy_i(t)}{dt} = \begin{cases} \lambda - y_1 \sum_{j=1}^{K-1} \frac{y_j(t)}{y(t)} p_{1,j} & i = 1 \\ y_{i-1}(t) \sum_{j=1}^{K-1} \frac{y_j(t)}{y(t)} p_{i-1,j} - y_i(t) \sum_{j=1}^{K-1} \frac{y_j(t)}{y(t)} p_{i,j} & i = 2, \dots, K - 1. \end{cases} \tag{5}$$

One finds that Eq. (5) is equivalent to the differential equations given in Eq. (2) where peers have unlimited upload capacity and $m = 1$. Thus, the asymptotic bounds given in **Theorem 1** still hold for this model, which implies that $T = K + \log(K) + O(1)$.

Remark. Both the download and upload capacity are one chunk per time slot, each peer has the same constraints as those in Section 4. However, we have better performance when matching is used instead of the random peer selection.

The random peer selection may cause request collision (i.e. a peer may receive multiple chunk requests but it can only serve one request due to its upload capacity), so the download capacities of the unserved peers are wasted. But if peers are matched at the beginning of each time slot, then the performance is greatly improved, approaching the performance of the random peer selection with *unlimited* upload capacity.

5.2. With incentive mechanism

Let us study the system with coordinated matching but with an incentive mechanism. Namely, given a pair of neighboring peers: peer *A* and peer *B*, both of them will perform chunk transfer *iff* both of them are useful to each other (i.e. $\mathcal{F}_A \setminus \mathcal{F}_B \neq \emptyset$ and $\mathcal{F}_B \setminus \mathcal{F}_A \neq \emptyset$). In this case, peer *A* and *B* will obtain one new chunk from each other in the current time slot. We use this model to capture the “*tit-for-tat*” incentive mechanism in the BT protocol. With this mechanism, the probability that a type *i* peer can exchange chunks with a type *j* peer is

$$p'_{i,j} = \begin{cases} 1 - \frac{C_i^j}{C_K^j} & 1 \leq j \leq i \leq K - 1, \\ 1 - \frac{C_j^i}{C_K^i} & 1 \leq i < j \leq K - 1. \end{cases} \tag{6}$$

Let us first state some important properties of $p'_{i,j}$.

Lemma 4. $p'_{i,j}$ has the following properties: (1) $p'_{i,j} = p'_{j,i}$; (2) $p'_{i,j} = p'_{K-j,K-i}$ and (3) $p'_{i,j}$ is an increasing function of *j* when $j \leq i$, and $p'_{i,j}$ is a decreasing function of *j* when $j \geq i$.

Proof. The proof of property (1) is trivial. To prove property (2), we consider the following three cases:

- Case 1: $1 \leq j \leq i$: we have $p'_{i,j} = 1 - C_i^j/C_K^j = 1 - C_{K-j}^{K-i}/C_K^{K-i}$. $j \leq i$ implies $K - i \leq K - j$, therefore, $p'_{K-j,K-i} = 1 - C_{K-j}^{K-i}/C_K^{K-i} = 1 - C_{K-j}^{K-i}/C_K^{K-i}$. So we obtain $p'_{i,j} = p'_{K-j,K-i}$
- Case 2: $i < j \leq K - 1$: We have $p'_{i,j} = p'_{j,i} = p'_{K-i,K-j} = p'_{K-j,K-i}$.

To prove property (3), let us consider the following cases:

- Case 1: $1 \leq j' < j \leq i \leq K - 1$:

$$p'_{i,j} - p'_{i,j'} = \left(1 - \frac{C_i^j}{C_K^j}\right) - \left(1 - \frac{C_i^{j'}}{C_K^{j'}}\right) = \left(1 - \frac{C_{K-j}^{K-i}}{C_K^{K-i}}\right) - \left(1 - \frac{C_{K-j'}^{K-i}}{C_K^{K-i}}\right) > 0.$$

- Case 2: $i \leq j' < j \leq K - 1$: Since $K - j < K - j' \leq K - i$, we have

$$p'_{i,j} - p'_{i,j'} = p'_{K-i,K-j} - p'_{K-i,K-j'} < 0. \quad \blacksquare$$

To simplify our notation, let us denote $w_{i,j} = p'_{i,j} + p'_{i,K-j}$ ($i, j = 1, \dots, K - 1$). It is easy to show that $w_{i,j} = w_{i,K-j} = w_{K-i,j} = w_{j,i}$.

Lemma 5. For a given *i*, $w_{i,j}$ is an increasing (or decreasing) function of *j* for $j \leq K/2$ (for $j \geq K/2$).

Proof. Consider $i \leq K/2$ first, in this case,

- (1) $j \leq i$, we have

$$\begin{aligned} w_{i,j} - w_{i,j-1} &= p'_{i,j} + p'_{i,K-j} - (p'_{i,j-1} + p'_{i,K-j+1}) \\ &= (p'_{i,j} - p'_{i,j-1}) + (p'_{i,K-j} - p'_{i,K-j+1}) > 0. \end{aligned}$$

- (2) $i < j \leq K/2$, we have

$$\begin{aligned} w_{i,j} - w_{i,j-1} &= \left(1 - \frac{C_j^i}{C_K^i} + 1 - \frac{C_{K-j}^i}{C_K^i}\right) - \left(1 - \frac{C_{j-1}^i}{C_K^i} + 1 - \frac{C_{K-j+1}^i}{C_K^i}\right) \\ &= \frac{1}{C_K^i} [C_{K-j}^{i-1} - C_{j-1}^{i-1}] > 0. \end{aligned}$$

Combine case (1) and (2), we know when $i \leq K/2$, $w_{i,j}$ is increasing if $j \leq K/2$. Since $w_{i,j} = w_{i,K-j}$, $w_{i,j}$ is decreasing if $j \geq K/2$. Because $w_{i,j} = w_{K-i,j}$, the above results hold for $i > K/2$. ■

Lemma 6. $T_i = T_{K-i}$.

Proof. We take a reverse view in the steady state so that (1) we regard the departure as arrival; (2) if peer A’s storage is \mathcal{F}_A , we just imagine there is no peer A but its complementary peer \bar{A} with storage $\mathcal{F}_{\bar{A}} = \mathcal{F} \setminus \mathcal{F}_A$. So originally T_i is the average time for peer A to stay in type i (i.e. with i chunks), but now the average time for peer \bar{A} to stay in type $(K - i)$: $T'_i = T_{K-i}$. From Lemma 4 we know $p'_{i,j} = p'_{K-i,K-j}$. So the “reversed system” is identical to the original system which implies $T'_i = T_i$. Thus we obtain $T_i = T_{K-i}$. ■

Similarly to the steady state analysis in previous section, we have the equations for T_i :

$$\frac{1}{T_i} = \sum_{j=1}^{K-1} \frac{T_j}{T} p'_{i,j}, \quad i = 1, 2, \dots, K - 1, \tag{7}$$

where $T = \sum_{i=1}^{K-1} T_i$.

Lemma 7. For $i \leq K/2$, T_i is a decreasing sequence: $2 > T_1 > T_2 > \dots > T_{\lfloor K/2 \rfloor}$.

Proof. Let $1 \leq i' < i \leq \lfloor K/2 \rfloor$. Base on Lemma 6, we have

$$\frac{1}{T_i} = \sum_{j=1}^{K-1} \frac{T_j}{T} p'_{i,j} = \frac{1}{2} \sum_{j=1}^{K-1} \frac{T_j}{T} (p'_{i,j} + p'_{i,K-j}) = \frac{1}{2} \sum_{j=1}^{K-1} \frac{T_j}{T} w_{i,j}.$$

Similarly, $\frac{1}{T_{i'}} = \frac{1}{2} \sum_{j=1}^{K-1} \frac{T_j}{T} w_{i',j}$. From Lemma 5 and $w_{i,j} = w_{j,i}$, we have

$$\frac{1}{T_i} - \frac{1}{T_{i'}} = \sum_{j=1}^{K-1} \frac{T_j}{T} (w_{i,j} - w_{i',j}) = \sum_{j=1}^{K-1} \frac{T_j}{T} (w_{j,i} - w_{j,i'}) > 0.$$

Thus $T_i < T_{i'}$, and the upper bound of T_1 is

$$T_1 = \frac{2}{\sum_{j=1}^{K-1} \frac{T_j}{T} w_{1,j}} < \frac{2}{\sum_{j=1}^{K-1} \frac{T_j}{T} w_{1,1}} = \frac{2}{w_{1,1}} = 2. \quad \blacksquare$$

Theorem 6. Using the incentive mechanism stated above, the bounds on the average downloading time T are

$$K - 4 + 2H_K + O\left(\frac{\log K}{K}\right) \leq T \leq K - 2 + 4H_K + O\left(\frac{\log K}{K}\right).$$

Proof. Based on Lemmas 5–7, we have

$$\begin{aligned} \frac{1}{T_i} &= \sum_{j=1}^{K-1} \frac{T_j}{T} p'_{i,j} = \frac{1}{2} \sum_{j=1}^{K-1} \frac{T_j}{T} w_{i,j} \leq \frac{1}{K-1} \sum_{j=1}^{K-1} \frac{T_j}{T} \cdot \sum_{j=1}^{K-1} p'_{i,j} = \frac{1}{K-1} \sum_{j=1}^{K-1} p'_{i,j} \\ &= \frac{1}{K-1} \left[\sum_{j=1}^i \left(1 - \frac{C_i^j}{C_K^j}\right) + \sum_{j=i+1}^{K-1} \left(1 - \frac{C_j^i}{C_K^i}\right) \right] \\ &= 1 - \frac{1}{K-1} \left[\frac{i}{K-i+1} + \frac{K-i}{i+1} - \frac{1}{C_K^i} \right]. \end{aligned}$$

Therefore, we obtain the lower bound of T as

$$\begin{aligned}
 T &= \sum_{i=1}^{K-1} T_i \geq \sum_{i=1}^{K-1} \frac{1}{1 - \frac{1}{K-1} \left[\frac{i}{K-i+1} + \frac{K-i}{i+1} - \frac{1}{C_K^i} \right]} > \sum_{i=1}^{K-1} \frac{1}{1 - \frac{1}{K-1} \left[\frac{i}{K-i+1} + \frac{K-i}{i+1} - 1 \right]} \\
 &= \sum_{i=1}^{K-1} \left[\frac{K-1}{K+2} + \frac{K^2-1}{K^2+2K} \left(\frac{1}{i} + \frac{1}{K-i} \right) \right] = K - 4 + 2H_K + O\left(\frac{\log K}{K}\right).
 \end{aligned}$$

According to Eq. (7) and Lemma 7, we have

$$\begin{aligned}
 \frac{1}{T_i} &= \sum_{j=1}^i \frac{T_j}{T} \left(1 - \frac{C_j^j}{C_K^j} \right) + \sum_{j=i+1}^{K-1} \frac{T_j}{T} \left(1 - \frac{C_j^i}{C_K^i} \right) = 1 - \left(\sum_{j=1}^i \frac{T_j}{T} \frac{C_j^j}{C_K^j} + \sum_{j=i+1}^{K-1} \frac{T_j}{T} \frac{C_j^i}{C_K^i} \right) \\
 &> 1 - \frac{T_1}{T} \left(\sum_{j=1}^i \frac{C_j^j}{C_K^j} + \sum_{j=i+1}^{K-1} \frac{C_j^i}{C_K^i} \right) > 1 - \frac{2}{K-1} \left(\frac{i}{K-i+1} + \frac{K-i}{i+1} \right).
 \end{aligned}$$

Thus for $i = 3 \dots K - 3$ (assuming $K \geq 5$), we have

$$\begin{aligned}
 T_i &< \frac{1}{1 - \frac{2}{K-1} \left(\frac{i}{K-i+1} + \frac{K-i}{i+1} \right)} = \frac{(K-1)(K-i+1)(i+1)}{K^2i - K^2 - Ki^2 + 3Ki - 3i^2 - 2K - 1} \\
 &< \frac{(K-1)(K-i+1)(i+1)}{K^2i - K^2 - Ki^2 + K} = \frac{K-1}{K} + \frac{2(K-1)}{K-2} \left(\frac{1}{K-i-1} + \frac{1}{i-1} \right).
 \end{aligned}$$

Thus the upper bound of T is

$$\begin{aligned}
 T &= \sum_{i=1}^{K-1} T_i < 4T_1 + \sum_{i=3}^{K-3} \left[\frac{K-1}{K} + \frac{2(K-1)}{K-2} \left(\frac{1}{K-i-1} + \frac{1}{i-1} \right) \right] \\
 &< 8 + \frac{(K-1)(K-5)}{K} + \frac{4(K-1)}{K-2} (H_{K-4} - 1) + O(K^{-1}) \\
 &= K + 4H_K - 2 + O\left(\frac{\log K}{K}\right). \quad \blacksquare
 \end{aligned}$$

Remark. Given the upper and lower bounds in Theorem 6, one can conclude that when an incentive mechanism is employed to enhance fairness, the performance of the file-swarming system still achieves better than the random peer selection policy in Section 5 wherein no fairness is guaranteed and free riders can benefit from peers’ altruistic service. Therefore, it is important for a system to help peers avoid a waste of download capacity (request collision). By the assistance of the peer-matching mechanism (such as the coordinated matching presented) even if the upload and download capacity is tightly constrained, the system can still provide good performance with a fairness guarantee.

6. Simulation

In this section, we carry out simulations to (1) validate our analytical results and (2) obtain other performance measures such as probability distribution of the file downloading time. Unless we state otherwise, the arrival process of peers is a Poisson process with $\lambda = 2.0$. Since the system is slotted, peers arriving at time slot t will obtain the initial chunk and will start participating in the file-swarming process at the beginning of time slot $t + 1$. The file that will be shared by all peers has $K = 200$ chunks. We also have results for $K = 500$, but due to the lack of space we mainly discuss the case $K = 200$.

Experiment 1: The goal of this experiment is to validate the analytical results in Section 3 and to illustrate the probability density function of the file download time. For this experiment, we set $m = 1$ or equivalently, this corresponds to the coupon model [15].

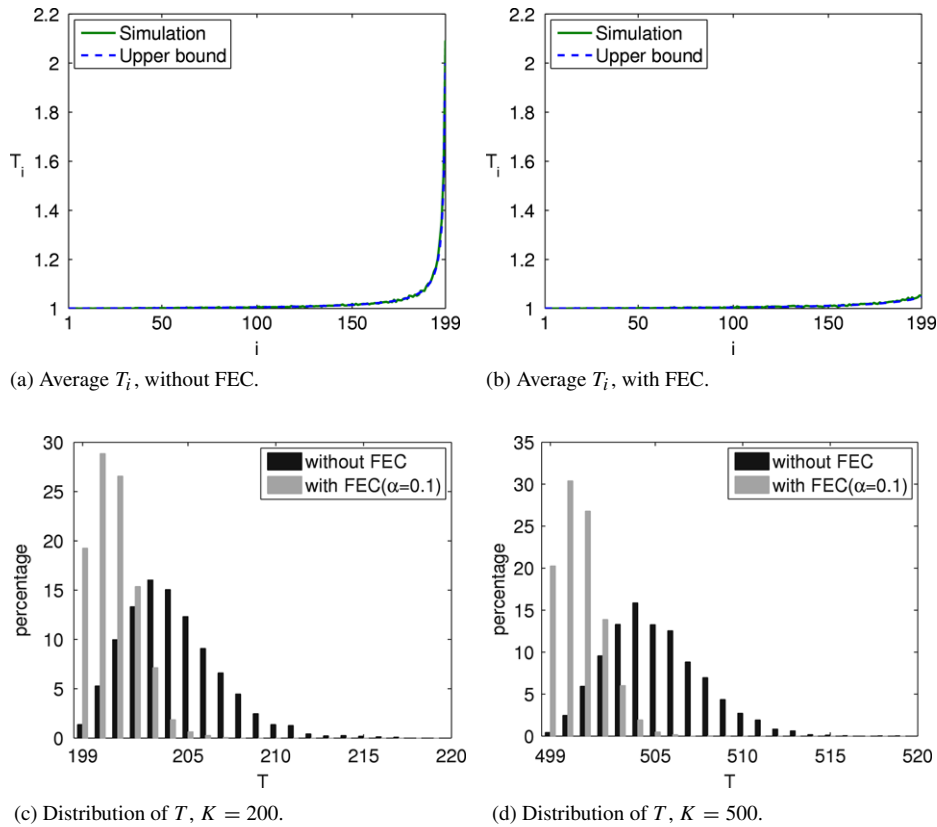


Fig. 4. T_i and T for $m = 1$, constraint on download capacity only.

Fig. 4(a) presents the average sojourn time T_i for a file with $K = 200$ chunks. We compare the simulation results and the analytical results.² This indicates that our analytical result is very accurate. Fig. 4(b) illustrates T_i under a similar setting but we enable the FEC with 10% redundancy (i.e., $\alpha = 0.1$). One can conclude that the analytical model is again very accurate and that using FEC can resolve the last-piece problem. Fig. 4(c)–(d) illustrate the probability density function for the average file downloading time T , with and without using FEC, for $K = 200$ and 500 respectively. When $K = 200$ ($K = 500$) without using FEC, the analytical average file downloading time is $T = 203.88$ ($T = 504.79$), and the simulation average file downloading time is $T = 204.11$ ($T = 504.99$). When $K = 200$ ($K = 500$) and FEC is enabled, the analytical average file downloading time is $T = 200.64$ ($T = 500.64$) while simulation average file downloading time is $T = 200.72$ ($T = 500.64$). We can observe that by using FEC, not only one can reduce the average T but also the variance of T .

Experiment 2: This experiment is to validate the results in Section 3 when $m > 1$. According to our analysis, there is not much difference between $m = 2$ and $m > 2$ since the average downloading time T will be bounded by K . For this experiment, we set $m = 2$. Fig. 5(a) presents the average sojourn time T_i without FEC. The simulation results are similar to the analytical results again. Comparing Figs. 5(a) and 4(a), one can find that the last-piece problem is not so severe for $m = 2$. T_i rises only for the last five chunks. If we deploy FEC ($\alpha = 0.1$) together with $m = 2$, the last-piece problem can be resolved and this is illustrated in Fig. 5(b). Notice that we only give out a loose upper bound of T_i in Fig. 5(b), which is also the upper bound of the system without FEC in Fig. 5(a). Now we examine the probability density function of T in Fig. 5(c). Without FEC, 50% of peers finished in $K - 1$ time slots and 80% of peers finished in less than or equal to K time slots. After we enable the FEC with $\alpha = 0.1$, 96% of peers finished in $K - 1$ time slots and all finished in less than or equal to K time slots. One can conclude that the average downloading time T is close to the optimal value of 199 (or $K - 1$), and the variance of T is also reduced. When $K = 200$ ($K = 500$) and without

² For the analytical results, since the spread of the bounds is very tight, we simply plot the upper bound of T_i .

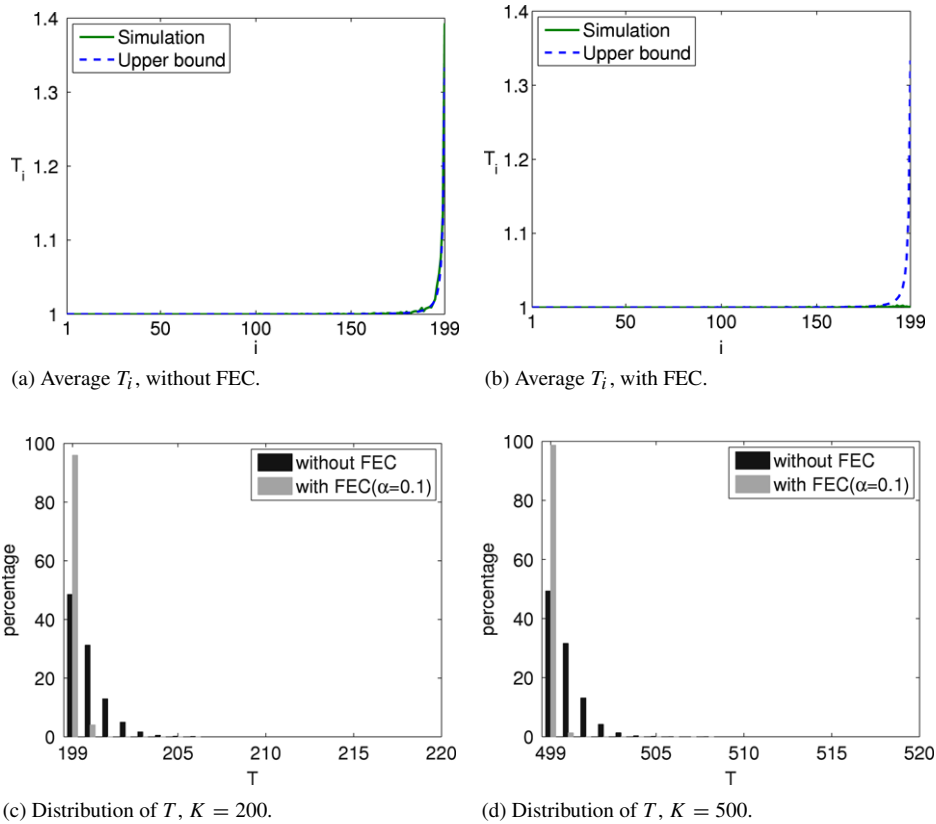


Fig. 5. T_i and T for $m = 2$, constraint on download capacity only.

FEC, our analysis gives an upper bound of the average downloading time $T \leq 200$ ($T \leq 500$), and the simulation is $T = 199.83$ ($T = 499.78$). After using FEC with $\alpha = 0.1$, the analytical upper bound of T still holds, while the simulation gives $T = 199.04$ ($T = 499.01$).

Experiment 3: This experiment is to validate the altruistic system with download and upload capacity constraints in Section 4. We consider $m = 1$ in our analysis, thus we set $m = 1$ in this simulation. Fig. 6(a) presents the average sojourn time T_i without FEC. The simulation results and the analytical results match very well, i.e. our theoretical upper bound is very tight. Comparing Fig. 6(a) and (b), we observe that FEC eases the last-piece problem, but most of T_i remain the same and they cannot approach 1 even with FEC. The reason is that the performance degradation is due to the request collision but not the last-piece problem. Also note that when we have uploaded and downloaded the capacity constraints, the variance on T is significantly larger than in the previous experiments. This can be confirmed by Fig. 6(c): the downloading time T varies in a wide range, from 275 to 375, and using FEC does not reduce the variance very much. When $K = 200$ ($K = 500$) and without FEC, our analytical bound of the average downloading time is $T \leq 322.53$ ($T \leq 798.56$), while the simulation gives $T = 319.99$ ($T = 793.67$). With FEC, the upper bound still holds, and the simulation result is $T = 316.06$ ($T = 791.01$); these show that using FEC in this type of system cannot improve T very much.

Experiment 4: This experiment is to validate the coordinated matching system with the incentive mechanism as described in Section 5. Fig. 7(a) presents the average sojourn time T_i without FEC. One can observe that the gap between the simulation results and our analytical upper bound is small. Also, one can observe both the *last-piece problem* and *first-piece problem*³ in our analytical bound and simulation result. The *first-piece problem* can be explained as follows. When a peer has very few chunks, it can hardly help other peers. Due to the incentive mechanism, it is difficult for this peer to obtain service from others. We can observe that FEC does well in easing the

³ This problem is reported as first block problem in [14] by measurement study as the slow startup due to choking.

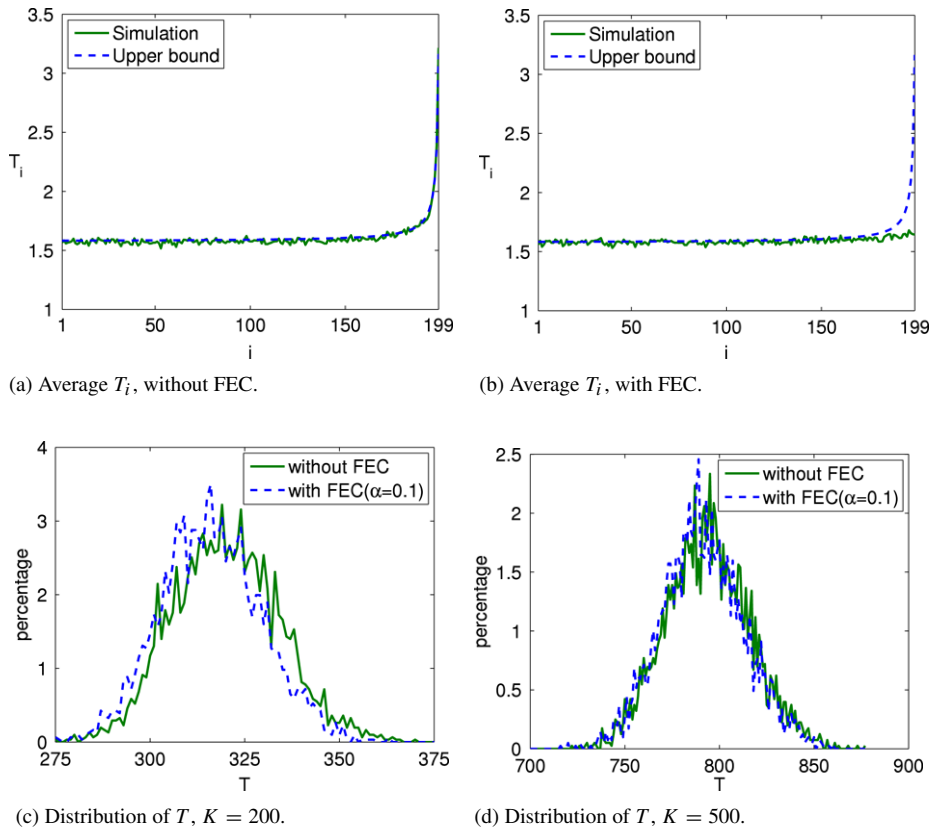


Fig. 6. T_i and T for $m = 1$, constraint on upload and download capacity.

last-piece problem, but it is not so good at easing the *first-piece problem*, as Fig. 7(b) has indicated. From Fig. 7(c) and (d), one can observe that the average and variance of file downloading time can be reduced when FEC is deployed. Another important observation is that when FEC is deployed, the performance measures of T (both for the average and variance) are significantly improved as compared with the results in Experiment 3 wherein both systems are under the upload and download capacity constraints. When $K = 200$ ($K = 500$) and without FEC, our analysis gives an upper bound of the average downloading time $T \leq 221.50$ ($T \leq 525.17$), and the simulation is $T = 211.78$ ($T = 513.10$). After using the FEC with $\alpha = 0.1$, the analytical upper bound of T still holds, and the simulation gives $T = 203.90$ ($T = 503.77$). This validates our analytical models.

7. Related work

There are numerous empirical studies on the BT protocol, for instance, [1,4,11,14,19]. Izal et al. [11] present the traffic information on peers behavior collected during a five-month period. Pouwelse et al. [19] study the availability, the integrity, the flash crowd effect and the download performance from a trace which was collected for eight months. Erman et al. [4] study the session interarrival times, sizes and durations and propose to use the hyper-exponential distribution to model the session interarrivals, and use the log-normal distribution to fit session durations and sizes. Legout et al. [14] evaluate the two core components of BitTorrent: choking and the rarest first algorithm and claim that they are enough to guarantee the efficiency and viability. Bindal and Cao [1] report great variability of downloading time and claim that instead of network bandwidth, “*close neighbor set*” (i.e. those peers in a stable data-exchange relationship) is the major contributing factor for the variability. However, a major limitation of these empirical studies is that the data collected is usually from a local view (i.e. the tracker log or a modified client), and the behavior is very time-dependent. Therefore, it is not an easy task to understand the efficiency of the BT protocol simply based on empirical studies.

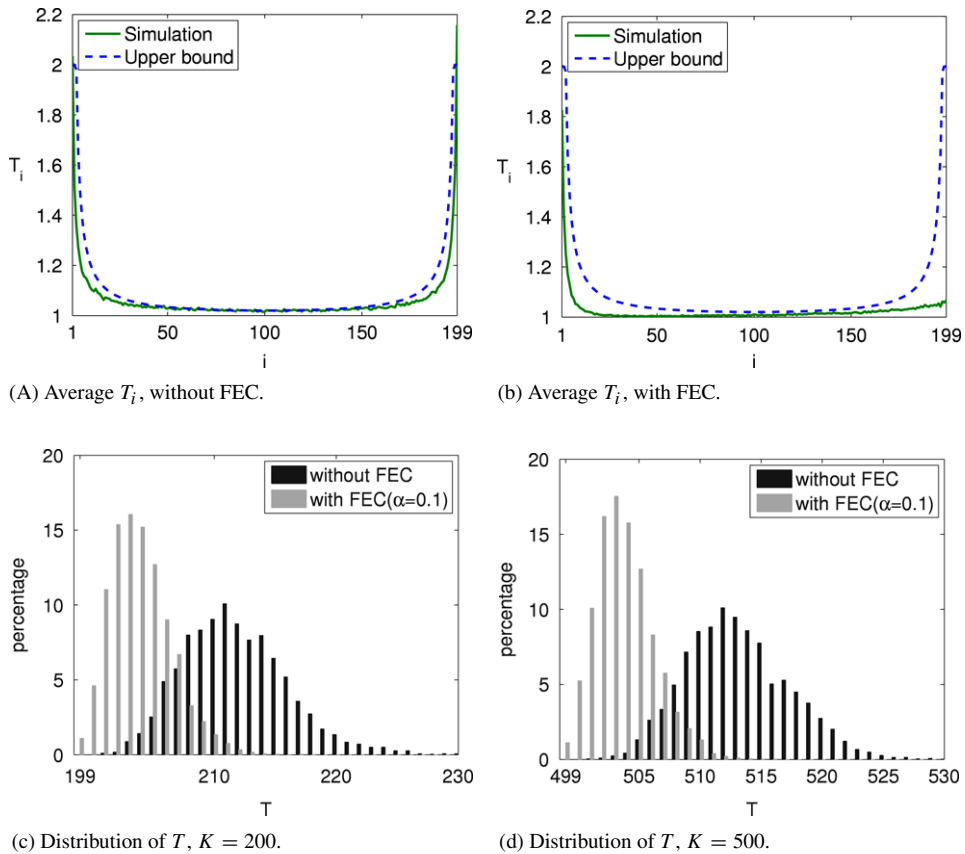


Fig. 7. T_i and T for coordinated matching, incentive mechanism, with constraint on upload and download capacity.

There are also several analytical studies of BT protocol. Yang and de Veciana [23] study the service capacity of BT protocols. Their result indicates that the service capacity of BT protocols increases exponentially at the beginning and scales well with the number of peers, thus providing fast downloading independent of demand rate. Qiu and Srikant [20] extend the coarse-grain Markovian model in [23] by providing an analytical solution to a fluid model in steady state which shows high scalability and stability of BT protocols. Our work differs from [20,23] in that we provide a detailed probabilistic model to capture the peers’ diversity (in terms of downloading progress) and show the change of downloading speed during the whole session. We also analyze the peer selection and chunk selection which are not considered in [20,23]. Fan et al. [5] also generalize Qiu’s model by dividing peers into three types according to the number of chunks they hold. Our work extends the number of types from 3 to $K - 1$ so as to capture the system more accurately. Under the assumption that “uplink is the only constraint”, Mundinger et al. [18] propose a deterministic scheduling algorithm to achieve the optimal makespan which requires global knowledge. Sanghavi et al. [22] also propose a gossip-like randomized algorithm requiring only local knowledge. Both studies in [18,22] are orthogonal to ours as they only consider the “closed system” where no new peer will arrive during the file dissemination while we consider an “open system” which new peers are joining in, according to the Poisson process. The work that is closely related to our study is [15]. In that paper, the authors provide a detailed probabilistic model to investigate the stability and effectiveness of a P2P file-swarming system. Their results state that even by the “random chunk selection” policy, the system throughput is still asymptotically optimal. Our paper improves and extends the result in [15] by providing tighter asymptotic bounds and relaxing its assumption of unlimited upload capacity. Moreover, we study the peer selection by both random selection and coordinated matching policies. Gaeta et al. [7] also use a probabilistic model to study the large-scale P2P network but they are focusing on a searching strategy. There are some other analytical studies in fairness to BT besides performance modeling. In [2,16,17], the authors present a mathematical analysis on service differentiation in resource allocation for P2P networks. In [6], the authors present a mathematical framework to

study the tradeoff between performance and fairness in BT-like systems. In [24], the authors present the first analytical model of BT-like systems and quantify the tradeoff between scalability and QoS support for multimedia streaming applications.

8. Conclusion

In this paper, we propose a probabilistic model which generalizes the model in [15] to capture the basic properties of a file-swarming system. Under the same assumption as [15] (i.e. unlimited upload capacity), we first improve its asymptotic bound of the average downloading time. Then we provide two different approaches, namely fetching multiple bitmaps and using an FEC code, to help the system achieve near-optimal performance. Besides showing that the FEC code can also remedy the last-piece problem, we also remove the assumption of “unlimited upload capacity” and analyze the performance under the random peer selection algorithm. Since the performance deteriorates due to request collision, we propose a matching scheme to improve performance. We show that under coordinated matching, if peers are altruistic, the system performance can achieve as good as the system with unlimited upload capacity. Even when the system deploys a certain incentive mechanism (tit-for-tat), the average downloading time is still good. The result suggests that the performance of a P2P file-swarming system does not depend critically on altruistic peers, but rather, due to the diversity of peers’ stored data, so the system can perform well.

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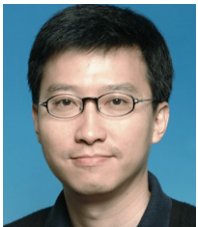
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