Introduction to Data Flow Analysis

Reading: NNH 1.1-1.3, 1.7-1.8

17-654/17-765
Analysis of Software Artifacts
Jonathan Aldrich

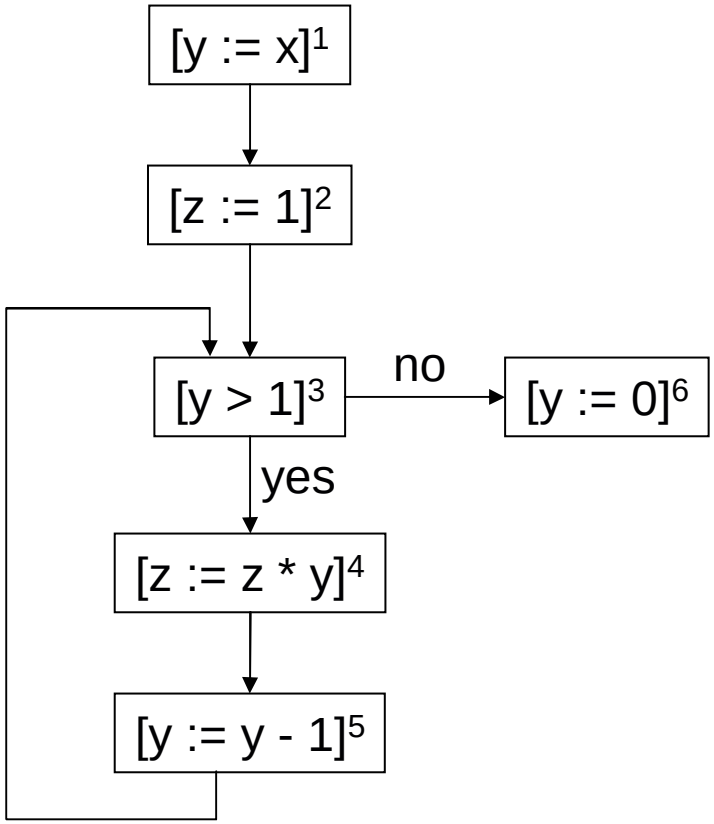
Example WHILE Program

$[y := x]^1$;
$[z := 1]^2$;
while $[y>1]^3$ do
    $[z := z * y]^4$;
    $[y := y – 1]^5$;
$[y := 0]^6$;

Computes the factorial function, with the input in x and the output in z

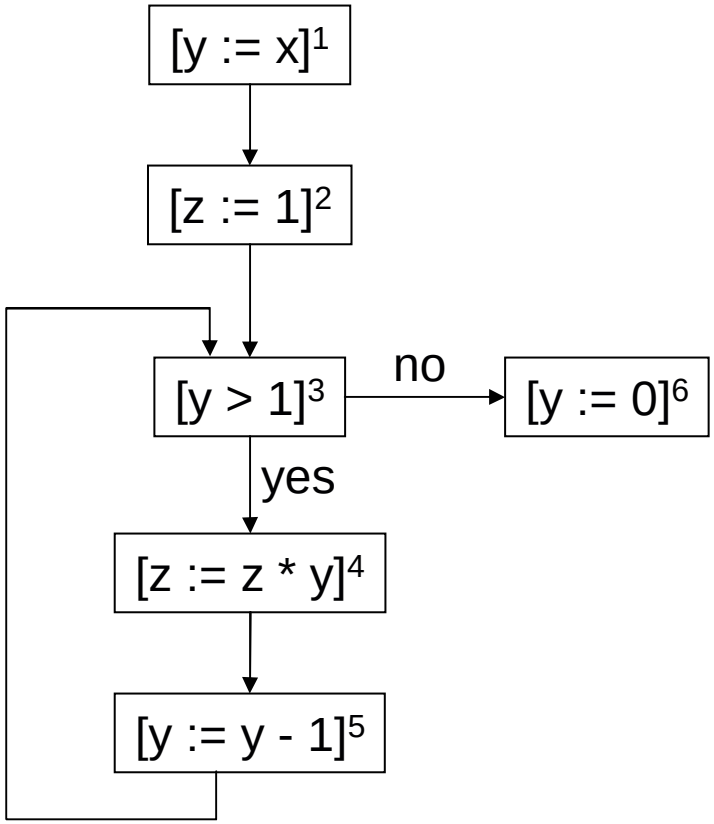
Data Flow Analysis

- View program as graph
  - Nodes are elementary blocks like assignments, if statements, etc.
  - Edges show control flow



Data Flow Equations (1)

- Transfer Functions
  - show how a statement affects data flow information



$RD_{exit}(1) = (RD_{entry}(1) \setminus \{(y,*)\}) \cup (y,1)$

$RD_{exit}(2) = (RD_{entry}(2) \setminus \{(z,*)\}) \cup (z,2)$

$RD_{exit}(3) = RD_{entry}(3)$

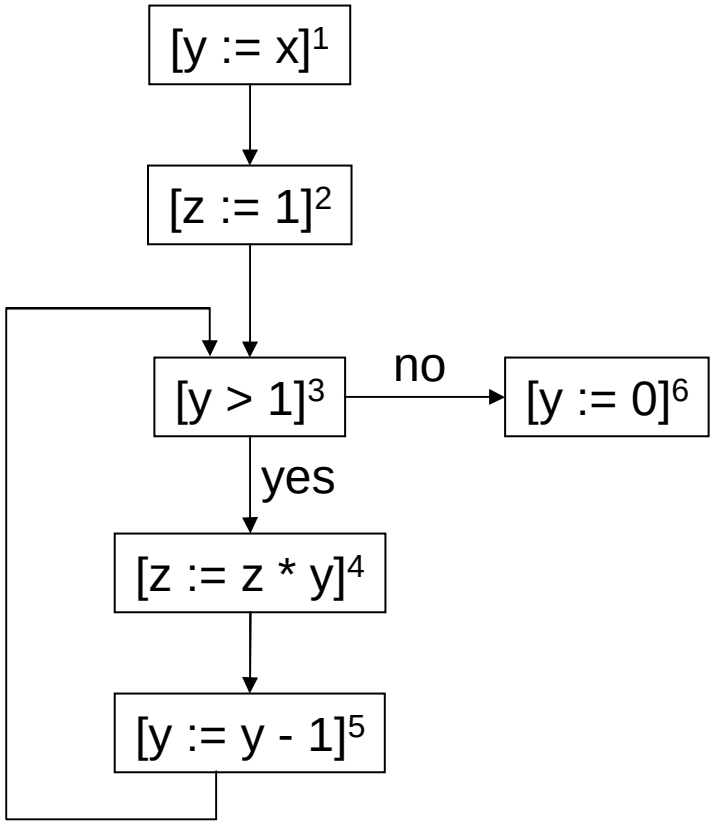
$RD_{exit}(4) = (RD_{entry}(4) \setminus \{(z,*)\}) \cup (z,4)$

$RD_{exit}(5) = (RD_{entry}(5) \setminus \{(y,*)\}) \cup (y,5)$

$RD_{exit}(6) = (RD_{entry}(6) \setminus \{(y,*)\}) \cup (y,6)$

Data Flow Equations (1)

- Pattern
  - Assignments
    - **kill** reaching defs for that var
    - **generate** new reaching def
  - All others
    - no change



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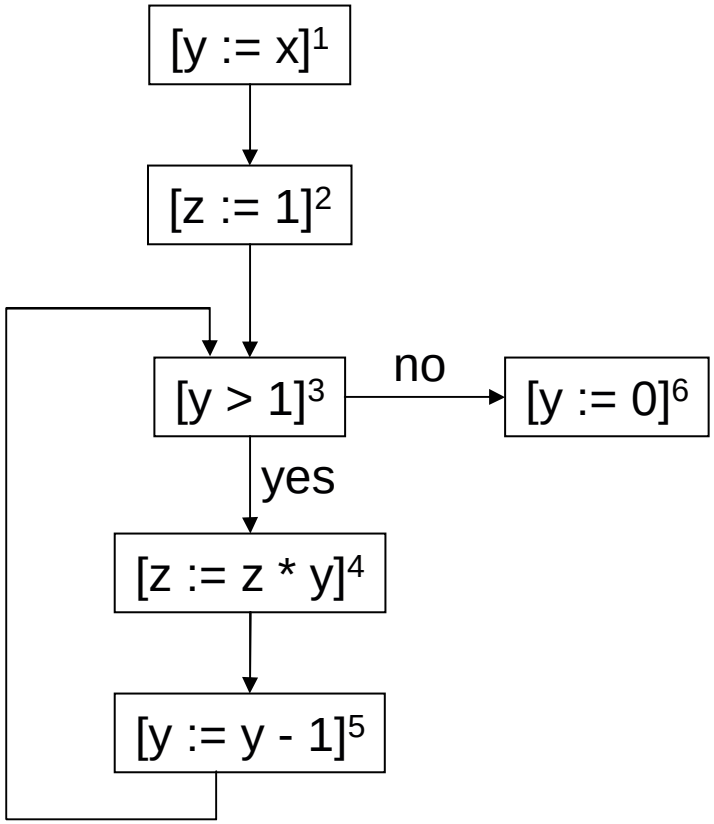
Data Flow Equations (2)

- Flow equations
  - Show how analysis information flows from one statement to another



$RD_{entry}(1) = \{ (x,?), (y,?), (z,?) \}$

$RD_{entry}(2) = RD_{exit}(1)$

$RD_{entry}(3) = RD_{exit}(2) \cup RD_{exit}(5)$

$RD_{entry}(4) = RD_{exit}(3)$

$RD_{entry}(5) = RD_{exit}(4)$

$RD_{entry}(6) = RD_{exit}(3)$

Data Flow Solution

- Solution
  - A 12-tuple $\overrightarrow{RD}$
  - Such that $\overrightarrow{RD} = F(\overrightarrow{RD})$
  - Where *F* is derived from the equations at right
  - As small (precise) as possible
- *Fixed point* of *f*
  - A value v such that v = *f*(v)

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$RD_{exit}(3) = RD_{entry}(3)$

$RD_{exit}(4) = (RD_{entry}(4) \setminus \{(z,*)\}) \cup (z,4)$

$RD_{exit}(5) = (RD_{entry}(5) \setminus \{(y,*)\}) \cup (y,5)$

$RD_{exit}(6) = (RD_{entry}(6) \setminus \{(y,*)\}) \cup (y,6)$

$RD_{entry}(1) = \{ (x,?), (y,?), (z,?) \}$

$RD_{entry}(2) = RD_{exit}(1)$

$RD_{entry}(3) = RD_{exit}(2) \cup RD_{exit}(5)$

$RD_{entry}(4) = RD_{exit}(3)$

$RD_{entry}(5) = RD_{exit}(4)$

$RD_{entry}(6) = RD_{exit}(3)$

Equations as Functions

$$F(RD) = (F_{entry}(1)(\overrightarrow{RD}), RD_{exit}(1)(\overrightarrow{RD}),$$
$$\ldots,$$
$$F_{entry}(6)(\overrightarrow{RD}), RD_{exit}(6)(\overrightarrow{RD}))$$

where, for example,

$$F_{exit}(1)(\ldots, RD_{entry}(1), \ldots) = (RD_{entry}(1) \setminus \{(y,*)\}) \cup (y,1)$$

$$F_{entry}(3)(\ldots, RD_{exit}(2), \ldots, RD_{exit}(5), \ldots) = RD_{exit}(2) \cup RD_{exit}(5)$$

Computing a Fixed Point of *F*

- Start with the tuple $\overrightarrow{\mathrm{RD}_{\varnothing}} = (\varnothing, \varnothing, \varnothing, \ldots, \varnothing)$
- Let $F^0(x) = x$
- Let $F^{n+1}(x) = F(F^n(x))$

- Surprise!
  - Now find n such that $F^{n+1}(\overrightarrow{\mathrm{RD}_{\varnothing}}) = F^n(\overrightarrow{\mathrm{RD}_{\varnothing}})$
  - By definition $F^n(\overrightarrow{\mathrm{RD}_{\varnothing}})$ is a fixed point of F
  - Does this really work?

Computing the Fixed Point

	0	1	2	3	4	10
$RD_{entry}(1)$	∅	x?y?z?	→	→	→	x?y?z?
$RD_{exit}(1)$	∅	y1	x?y1z?	→	→	x?y1z?
$RD_{entry}(2)$	∅	∅	y1	x?y1z?	→	x?y1z?
$RD_{exit}(2)$	∅	z2	z2	y1z2		x?y1z2
$RD_{entry}(3)$	∅	∅	y5z2	y5z2z4		x?y1y5z2z4
$RD_{exit}(3)$	∅	∅	∅	y5z2		x?y1y5z2z4
$RD_{entry}(4)$	∅	∅	∅	∅		x?y1y5z2z4
$RD_{exit}(4)$	∅	z4	z4	z4		x?y1y5z4
$RD_{entry}(5)$	∅	∅	z4	z4		x?y1y5z4
$RD_{exit}(5)$	∅	y5	y5	y5z4		x?y5z4
$RD_{entry}(6)$	∅	∅	∅	∅		x?y1y5z2z4
$RD_{exit}(6)$	∅	y6	y6	y6		x?y6z2z4

$RD_{exit}(1) = (RD_{entry}(1) \setminus \{(y,*)\}) \cup (y,1)$
$RD_{exit}(2) = (RD_{entry}(2) \setminus \{(z,*)\}) \cup (z,2)$
$RD_{exit}(3) = RD_{entry}(3)$
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$RD_{exit}(5) = (RD_{entry}(5) \setminus \{(y,*)\}) \cup (y,5)$
$RD_{exit}(6) = (RD_{entry}(6) \setminus \{(y,*)\}) \cup (y,6)$

$RD_{entry}(1) = \{ (x,?), (y,?), (z,?) \}$
$RD_{entry}(2) = RD_{exit}(1)$
$RD_{entry}(3) = RD_{exit}(2) \cup RD_{exit}(5)$
$RD_{entry}(4) = RD_{exit}(3)$
$RD_{entry}(5) = RD_{exit}(4)$
$RD_{entry}(6) = RD_{exit}(3)$

Finding the Fixed Point

- Why should we think we will find an *n* such that $F^{n+1}(\overrightarrow{\mathrm{RD}_{\varnothing}}) = F^{n}(\overrightarrow{\mathrm{RD}_{\varnothing}})$?

Monotone Functions

- f is *monotone* if $v \subseteq v'$ implies $f(v) \subseteq f(v')$
- Assertion: *F* is monotone
  - Intuition: preserves input/output relationship
  - Check a couple of cases
  - (1) $RD_{exit}(1) = (RD_{entry}(1) \setminus \{(y,*)\}) \cup (y,1)$
    - Assume $RD_{entry}(1) \subseteq RD'_{entry}(1)$
    - Then $(RD_{entry}(1) \setminus \{(y,*)\}) \subseteq (RD'_{entry}(1) \setminus \{(y,*)\})$
    - So $RD_{exit}(1) \subseteq RD'_{exit}(1)$

    - Would this also be true if we used $\subset$?

Monotone Functions

- f is *monotone* if $v \subseteq v'$ implies $f(v) \subseteq f(v')$
- Assertion: *F* is monotone
  - Intuition: preserves input/output relationship
  - Check a couple of cases
  - (1) $RD_{exit}(1) = (RD_{entry}(1) \setminus \{(y,*)\}) \cup (y,1)$
  - (2) $RD_{entry}(3) = RD_{exit}(2) \cup RD_{exit}(5)$
    - Assume $RD_{exit}(2) \subseteq RD'_{exit}(2)$
    - Assume $RD_{exit}(5) \subseteq RD'_{exit}(5)$
    - Then $RD_{exit}(2) \cup RD'_{exit}(5) \subseteq RD'_{exit}(2) \cup RD'_{exit}(5)$
    - So $RD_{entry}(3) \subseteq RD'_{entry}(3)$

Finding the Fixed Point

- Why should we think we will find an *n* such that $F^{n+1}(\overrightarrow{RD_\emptyset}) = F^n(\overrightarrow{RD_\emptyset})$?
  - *F* is monotone
  - Claim: $\forall n\ F^n(\overrightarrow{RD_\emptyset}) \subseteq F^{n+1}(\overrightarrow{RD_\emptyset})$
    - Base case: $\overrightarrow{RD_\emptyset} \subseteq F(\overrightarrow{RD_\emptyset})$
      - Since no tuple has smaller sets than $RD_\emptyset$
    - Inductive case:
      - Assume $F^{n-1}(\overrightarrow{RD_\emptyset}) \subseteq F^n(\overrightarrow{RD_\emptyset})$
      - *F* is monotone, so $F(F^{n-1}(\overrightarrow{RD_\emptyset})) \subseteq F(F^n(\overrightarrow{RD_\emptyset}))$
      - Equivalently, $F^n(\overrightarrow{RD_\emptyset}) \subseteq F^{n+1}(\overrightarrow{RD_\emptyset})$
  - Therefore, every application of F either:
    - Does not change $\overrightarrow{RD}$ (and so we have a fixed point)
    - Or increases the size of a set in $\overrightarrow{RD}$
  - The set of definitions is finite so the sets in $\overrightarrow{RD}$ cannot increase in size forever
  - Therefore the algorithm terminates with a fixed point at some finite *n*

Precision

- Is $F^n(\overrightarrow{\mathrm{RD}_\varnothing})$ the least fixed point?
  - i.e., the fixed point with the smallest sets?
- Yes. Proof:
  - Consider some other fixed point $\overrightarrow{\mathrm{RD}_{\mathrm{fix}}}$
  - $\overrightarrow{\mathrm{RD}_\varnothing} \subseteq \overrightarrow{\mathrm{RD}_{\mathrm{fix}}}$
  - Since F is monotone, $F(\overrightarrow{\mathrm{RD}_\varnothing}) \subseteq F(\overrightarrow{\mathrm{RD}_{\mathrm{fix}}})$
  - By induction $F^n(\overrightarrow{\mathrm{RD}_\varnothing}) \subseteq F^n(\overrightarrow{\mathrm{RD}_{\mathrm{fix}}})$
  - But $\mathrm{RD}_{\mathrm{fix}}$ is a fixed point so $\overrightarrow{\mathrm{RD}_{\mathrm{fix}}} = \mathrm{F}(\overrightarrow{\mathrm{RD}_{\mathrm{fix}}}) = F^n(\overrightarrow{\mathrm{RD}_{\mathrm{fix}}})$
  - Therefore $F^n(\overrightarrow{\mathrm{RD}_\varnothing}) \subseteq \overrightarrow{\mathrm{RD}_{\mathrm{fix}}}$
  - Therefore $F^n(\overrightarrow{\mathrm{RD}_\varnothing})$ is the least fixed point of F

Efficient Algorithms

- Computing $F^n(\overrightarrow{\mathrm{RD}_\varnothing})$ is slow
  - 10 iterations
  - Each iteration recomputes each member of $\overrightarrow{\mathrm{RD}_\varnothing}$
  - Few members of $\overrightarrow{\mathrm{RD}_\varnothing}$ change each iteration
- Optimization: Chaotic Iteration
  - Recompute one member of $\overrightarrow{\mathrm{RD}_\varnothing}$ at a time
  - Guess a member that is likely to change
  - Can compute fixed point in 17 iterations, one recomputation per iteration (vs. 12 before)

Chaotic Iteration

$RD_{1..n} = \emptyset$

while $RD_j \neq F_j(RD_{1..n})$ for some j

do $RD_j := F_j(RD_{1..n})$

- How to choose j?
  - Later!

Chaotic Iteration

$RD_{1..n} = \emptyset$
while $RD_j \neq F_j(RD_{1..n})$ for some j
  do $RD_j := F_j(RD_{1..n})$

- Properties
  - If chaotic iteration terminates, $RD_{1..n}$ is a fixed point of F
    - Proof: termination implies $\overrightarrow{RD} = F(\overrightarrow{RD})$
  - That fixed point is a least fixed point
    - Proof: $\overrightarrow{RD} \subseteq F^n(\overrightarrow{RD_\emptyset})$ is an invariant of the algorithm
  - Chaotic iteration terminates for monotone F and finite sets $RD_{1..n}$
    - Proof: F is monotone and so $\overrightarrow{RD}$ is increasing

Chaotic Iteration Example

Iter	Position	Value
0	--	$\vec{\varnothing}$
1	entry(1)	x?y?z?
2	exit(1)	x?y1z?
3	entry(2)	x?y1z?
4	exit(2)	x?y1z2
5	entry(3)	x?y1z2
6	exit(3)	x?y1z2
7	entry(4)	x?y1z2
8	exit(4)	x?y1z4
9	entry(5)	x?y1z4
10	exit(5)	x?y5z4
11	entry(3)	x?y1y5z2z4
12	exit(3)	x?y1y5z2z4
13	entry(4)	x?y1y5z2z4
14	exit(4)	x?y1y5z4
15	entry(5)	x?y1y5z4
16	entry(6)	x?y1y5z2z4
17	exit(6)	x?y6z2z4

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Comparison to Naïve Algorithm

	0	1	2	3	4	10
$RD_{entry}(1)$	∅	x?y?z?	→	→	→	x?y?z?
$RD_{exit}(1)$	∅	y1	x?y1z?	→	→	x?y1z?
$RD_{entry}(2)$	∅	∅	y1	x?y1z?	→	x?y1z?
$RD_{exit}(2)$	∅	z2	z2	y1z2		x?y1z2
$RD_{entry}(3)$	∅	∅	y5z2	y5z2z4		x?y1y5z2z4
$RD_{exit}(3)$	∅	∅	∅	y5z2		x?y1y5z2z4
$RD_{entry}(4)$	∅	∅	∅	∅		x?y1y5z2z4
$RD_{exit}(4)$	∅	z4	z4	z4		x?y1y5z4
$RD_{entry}(5)$	∅	∅	z4	z4		x?y1y5z4
$RD_{exit}(5)$	∅	y5	y5	y5z4		x?y5z4
$RD_{entry}(6)$	∅	∅	∅	∅		x?y1y5z2z4
$RD_{exit}(6)$	∅	y6	y6	y6		x?y6z2z4

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Constant Folding

- A program optimization
  - Replaces computation with constants
  - Can use reaching definitions
- Notation
  - RD $\vdash$ S $\triangleright$ S'
    - "Given reaching definitions RD, statement S can be transformed into S' "
  - $\frac{C}{T}$
    - Transformation T is legal if condition(s) C hold
  - FV(exp)
    - The variables mentioned in exp
  - exp[y$\mapsto$n]
    - Replace all occurrences of y in exp with n

Constant Folding Rules

$$[ass_1] \quad \mathsf{RD} \vdash [x := a]^\ell \ \rhd \ [x := a[y \mapsto n]]^\ell$$
$$\text{if} \begin{cases} y \in FV(a) \ \wedge \ (y, ?) \notin \mathsf{RD}_{entry}(\ell) \ \wedge \\ \forall (z, \ell') \in \mathsf{RD}_{entry}(\ell) : (z = y \Rightarrow [\cdots]^{\ell'} \text{ is } [y := n]^{\ell'}) \end{cases}$$

$$[ass_2] \quad \mathsf{RD} \vdash [x := a]^\ell \ \rhd \ [x := n]^\ell$$
$$\text{if } FV(a) = \emptyset \ \wedge \ a \notin \mathbf{Num} \ \wedge \ a \text{ evaluates to } n$$

$$[seq_1] \quad \frac{\mathsf{RD} \vdash S_1 \ \rhd \ S_1'}{\mathsf{RD} \vdash S_1; S_2 \ \rhd \ S_1'; S_2}$$

$$[seq_2] \quad \frac{\mathsf{RD} \vdash S_2 \ \rhd \ S_2'}{\mathsf{RD} \vdash S_1; S_2 \ \rhd \ S_1; S_2'}$$

$$[if_1] \quad \frac{\mathsf{RD} \vdash S_1 \ \rhd \ S_1'}{\mathsf{RD} \vdash \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2 \ \rhd \ \texttt{if } [b]^\ell \texttt{ then } S_1' \texttt{ else } S_2}$$

$$[if_2] \quad \frac{\mathsf{RD} \vdash S_2 \ \rhd \ S_2'}{\mathsf{RD} \vdash \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2 \ \rhd \ \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2'}$$

$$[wh] \quad \frac{\mathsf{RD} \vdash S \ \rhd \ S'}{\mathsf{RD} \vdash \texttt{while } [b]^\ell \texttt{ do } S \ \rhd \ \texttt{while } [b]^\ell \texttt{ do } S'}$$

Taken from Nielson, Nielson, and Hankin, page 27

Example

$[x:=10]^1; [y:=x+10]^2; [z:=y+10]^3$

- $RD_{enter}(2) = \{ (x,1), (y,?), (z,?) \}$
- $RD \vdash [y:=x+10]^2 \triangleright [y:=10+10]^2$ *by* $[ass_1]$
- Thus:
  - $RD \vdash [x:=10]^1; [y:=x+10]^2; [z:=y+10]^3 \triangleright$ $[x:=10]^1; [y:=10+10]^2; [z:=y+10]^3$ *by [seq] rules*

Example

RD $\vdash [x:=10]^1; [y:=x+10]^2; [z:=y+10]^3$

$\triangleright [x:=10]^1; [y:=10+10]^2; [z:=y+10]^3$ *by* $[ass_1]$

$\triangleright [x:=10]^1; [y:=20]^2; [z:=y+10]^3$ *by* $[ass_2]$

$\triangleright [x:=10]^1; [y:=20]^2; [z:=20+10]^3$ *by* $[ass_1]$

$\triangleright [x:=10]^1; [y:=20]^2; [z:=30]^3$ *by* $[ass_2]$