Lecture Notes: A Dataflow Analysis Framework for WHILE3ADDR

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1 Defining a dataflow analysis

A dataflow analysis computes some dataflow information at each program point in the control flow graph. We thus start by examining how this information is defined. We will use σ to denote this information. Typically σ tells us something about each variable in the program. For example, σ may map variables to abstract values taken from some set L:

$$\sigma \in \mathit{Var} \to L$$

L represents the set of abstract values we are interested in tracking in the analysis. This varies from one analysis to another. For example, consider a *zero analysis*, which tracks whether each variable is zero or not at each program point (Thought Question: Why would this be useful?). For this analysis, we define L to be the set $\{Z, N, \top\}$. The abstract value Z represents the value 0, N represents all nonzero values. $\top$ is pronounced "top", and we define it more concretely later it in these notes; we use it as a question mark, for the situations when we do not know whether a variable is zero or not, due to imprecision in the analysis.

Conceptually, each abstract value represents a set of one or more concrete values that may occur when a program executes. We define an abstraction function α that maps each possible concrete value of interest to an abstract value:

$$\alpha : \mathbb{Z} \to L$$

For zero analysis, we define α so that 0 maps to Z and all other integers map to N:

$$\begin{aligned} \alpha_Z(0) &= Z \\ \alpha_Z(n) &= N \text{where } n \neq 0 \end{aligned}$$

The core of any program analysis is how individual instructions in the program are analyzed and affect the analysis state σ at each program point. We define this using *flow functions* that map the dataflow information at the program point immediately *before* an instruction to the dataflow information *after* that instruction. A flow function should represent the semantics of the instruction, but abstractly, in terms of the abstract values tracked by the analysis. We will link semantics to the flow function precisely when we talk about correctness of dataflow analysis. For now, to approach the idea by example, we define the flow functions f_Z for zero analysis on WHILE3ADDR as follows:

$$f_Z[\![x := 0]\!](\sigma) = [x \mapsto Z]\sigma \tag{1}$$
$$f_Z[\![x := n]\!](\sigma) = [x \mapsto N]\sigma \text{ where } n \neq 0 \tag{2}$$
$$f_Z[\![x := y]\!](\sigma) = [x \mapsto \sigma(y)]\sigma \tag{3}$$
$$f_Z[\![x := y \; op \; z]\!](\sigma) = [x \mapsto \top]\sigma \tag{4}$$
$$f_Z[\![\text{goto } n]\!](\sigma) = \sigma \tag{5}$$
$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!](\sigma) = \sigma \tag{6}$$

In the notation, the form of the instruction is an implicit argument to the function, which is followed by the explicit dataflow information argument, in the form $f_Z[\![I]\!](\sigma)$. (1) and (2) are for assignment to a constant. If we assign 0 to a variable x, then we should update the input dataflow information σ so that x maps to the abstract value Z. The notation $[x \mapsto Z]\sigma$ denotes dataflow information that is identical to σ except that the value in the mapping for x is updated to refer to Z. Flow function (3) is for copies from a variable y to another variable x: we look up y in σ, written $\sigma(y)$, and update σ so that x maps to the same abstract value as y.

We start with a generic flow function for arithmetic instructions (4). Arithmetic can produce either a zero or a nonzero value, so we use the abstract value $\top$ to represent our uncertainty. More precise flow functions are available based on certain instructions or operands. For example, if the instruction is subtraction and the operands are the same, the result will definitely be zero. Or, if the instruction is addition, and the analysis information tells us that one operand is zero, then the addition is really a copy and we can use a flow function similar to the copy instruction above. These examples could be written as follows (we would still need the generic case above for instructions that do not fit such special cases):

$$f_Z[\![x := y - y]\!](\sigma) = [x \mapsto Z]\sigma$$
$$f_Z[\![x := y + z]\!](\sigma) = [x \mapsto \sigma(y)]\sigma \quad \text{where } \sigma(z) = Z$$

Exercise 1. Define another flow function for some arithmetic instruction and certain conditions where you can also provide a more precise result than $\top$.

The flow function for branches ((5) and (6)) is trivial: branches do not change the state of the machine other than to change the program counter, and thus the analysis result is unaffected.

However, we can provide a better flow function for conditional branches if we distinguish the analysis information produced when the branch is taken or not taken. To do this, we extend our notation once more in defining flow functions for branches, using a subscript to the instruction to indicate whether we are specifying the dataflow information for the case where the condition is true (T) or when it is false (F). For example, to define the flow function for the true condition when testing a variable for equality with zero, we use the notation $f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_T(\sigma)$. In this case we know that x is zero so we can update σ with the Z lattice value. Conversely, in the false condition we know that x is nonzero:

$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_T(\sigma) = [x \mapsto Z]\sigma$$
$$f_Z[\![\text{if } x = 0 \text{ goto } n]\!]_F(\sigma) = [x \mapsto N]\sigma$$

Exercise 2. Define a flow function for a conditional branch testing whether a variable $x < 0$.

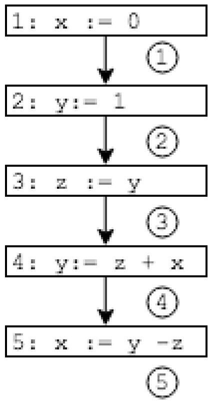
2 Running a dataflow analysis

The point of developing a dataflow analysis is to compute information about possible program states at each point in a program. For example, for of zero analysis, whenever we divide some expression by a variable x, we might like to know whether x must be zero (the abstract value Z) or may be zero (represented by $\top$) so that we can warn the developer.

2.1 Straightline code

Consider the following simple program (left), with its control flow graph (middle):

$$\begin{array}{ll} 1: & x := 0 \\ 2: & y := 1 \\ 3: & z := y \\ 4: & y := z + x \\ 5: & x := y - z \end{array}$$



	x	y	z
1	Z		
2	Z	N	
3	Z	N	N
4	Z	N	N
5	$\top$	N	N

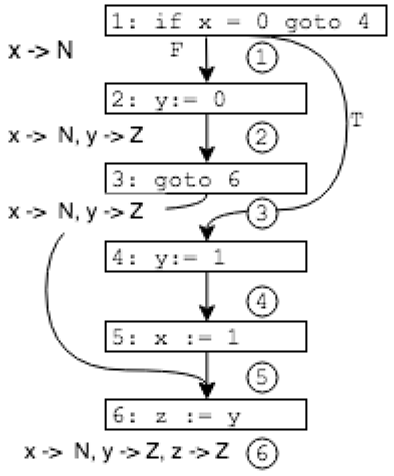
We simulate running the program in the analysis, using the flow function to compute, for each instruction in turn, the dataflow analysis information *after* the instruction from the information we had *before* the instruction. For such simple code, it is easy to track the analysis information using a table with a column for each program variable and a row for each program point (right, above). The information in a cell tells us the abstract value of the column's variable immediately after the instruction at that line (corresponding the the program points labeled with circles in the CFG).

Notice that the analysis is imprecise at the end with respect to the value of x. We were able to keep track of which values are zero and nonzero quite well through instruction 4, using (in the last case) the flow function that knows that adding a variable known to be zero is equivalent to a copy. However, at instruction 5, the analysis does not know that y and z are equal, and so it cannot determine whether x will be zero. Because the analysis is not tracking the exact values of variables, but rather approximations, it will inevitably be imprecise in certain situations. However, in practice, well-designed approximations can often allow dataflow analysis to compute quite useful information.

2.2 Alternative paths: Example

Things get more interesting in WHILE3ADDR code that contains `if` statements. In this case, there are two possible paths through the program. Consider the following simple example (left), and its CFG (middle). I have begun by analyzing one path through the program (the path in which the branch is not taken):

1 :	if $x = 0$ goto 4
2 :	$y := 0$
3 :	goto 6
4 :	$y := 1$
5 :	$x := 1$
6 :	$z := y$

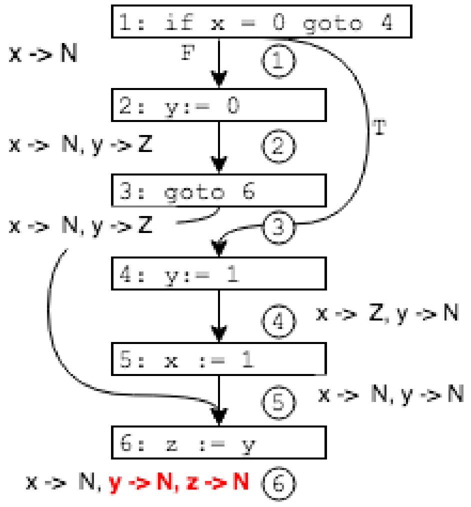


	x	y	z
1	Z_T, N_F		
2	N	Z	
3	N	Z	
4			
5			
6	N	Z	Z

In the table above, the entry for x on line 1 indicates the different abstract values produced for the true and false conditions of the branch. We use the false condition (x is nonzero) in analyzing instruction 2. Execution proceeds through instruction 3, at which point we jump to instruction 6. We have not yet analyzed a path through lines 4 and 5.

Turning to that alternative path, we can start by analyzing instructions 4 and 5 as if we had taken the true branch at instruction 1:

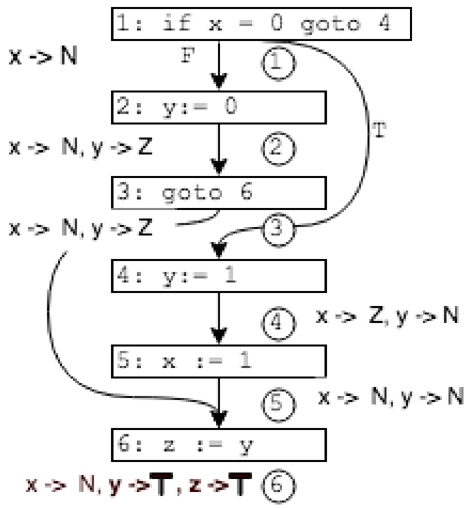
	x	y	z	
1	Z_T, N_F			
2	N	Z		
3	N	Z		
4	Z	N		
5	N	N		
6	N	Z	Z	*note: incorrect!*



We have a dilemma in analyzing instruction 6. We already analyzed it with respect to the previous path, assuming the dataflow analysis we computed from instruction 3, where x was nonzero and y was zero. However, we now have conflicting information from instruction 5: in this case, x is still nonzero, but y is also nonzero in this case.

We resolve this dilemma by combining the abstract values computed along the two paths for y and z. The incoming abstract values at line 6 for y are N and Z. We can represent this uncretainty with the abstract value $\top$, indicating that we do know know if y is zero or not at this instruction, because of the uncertainty about how we reached this program location. We can apply similar logic in the case of x, but because x is nonzero on both incoming paths we can maintain our knowledge that x is nonzero. Thus, we should reanalyze instruction 5 assuming the dataflow analysis information $\{x \mapsto N, y \mapsto \top\}$. The results of our final analysis are shown below:

	x	y	z	
1	Z_T, N_F			
2	N	Z		
3	N	Z		
4	Z	N		
5	N	N		
6	N	$\top$	$\top$	*corrected*



2.3 Join

We generalize the procedure of combining analysis results along multiple paths by using a *join* operation, $\sqcup$. When taking two abstract values $l_1, l_2 \in L$, the result of $l_1 \sqcup l_2$ is an abstract value l_j that generalizes both l_1 and l_2.

To precisely define what "generalizes" means, we define a partial order $\sqsubseteq$ over abstract values, and say that l_1 and l_2 are at least as precise as l_j, written $l_1 \sqsubseteq l_j$. Recall that a partial order is any relation that is:

- reflexive: $\forall l : l \sqsubseteq l$
- transitive: $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$
- anti-symmetric: $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$

A set of values L that is equipped with a partial order $\sqsubseteq$, and for which the least upper bound of any two values in that ordering $l_1 \sqcup l_2$ is unique and is also in L, is called a *join-semilattice.* Any join-semilattice has a maximal element $\top$ (pronounced "top"). We require that the abstract values used in dataflow analyses form a join-semilattice. We will use the term lattice for short; as we will see below, this is the correct terminology for most dataflow analyses anyway. For zero analysis, we define the partial order with $Z \sqsubseteq \top$ and $N \sqsubseteq \top$, where $Z \sqcup N = \top$.

We have now introduced and considered all the elements necessary to define a dataflow analysis:

- a lattice $(L, \sqsubseteq)$
- an abstraction function α
- initial dataflow analysis assumptions σ_0
- a flow function f

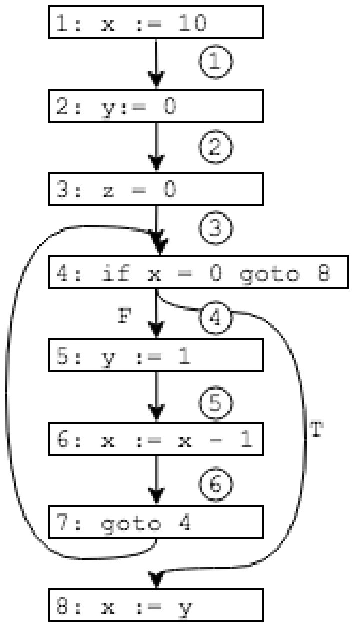
Note that the theory of lattices answers a side question that comes up when we begin analyzing the first program instruction: what should we assume about the value of input variables (like x on program entry)? If we do not know anything about the value x can be, a good choice is to assume it can be anything; That is, in the initial environment σ_0, input variables like x are mapped to $\top$.

2.4 Dataflow analysis of loops

We now consider WHILE3ADDR programs with loops. While an `if` statement produces two paths that diverge and later join, a loop produces an potentially unbounded number of program paths.

Despite this, we would like to analyze looping programs in bounded time. Let us examine how through the following simple looping example:[1]

1 :	$x := 10$
2 :	$y := 0$
3 :	$z := 0$
4 :	if $x = 0$ goto 8
5 :	$y := 1$
6 :	$x := x - 1$
7 :	goto 4
8 :	$x := y$



	x	y	z
1	N		
2	N	Z	
3	N	Z	Z
4	Z_T, N_F	Z	Z
5	N	N	Z
6	$\top$	N	Z
7	$\top$	N	Z
8			

The right-hand side above shows the straightforward straight-line analysis of the path that runs the loop once. We must now re-analyze instruction 4. This should not be surprising; it is analogous to the one we encountered earlier, merging paths after an `if` instruction. To determine the analysis information at instruction 4, we join the dataflow analysis information flowing in from instruction 3 with the dataflow analysis information flowing in from instruction 7. For x we have $N \sqcup \top = \top$. For y we have $Z \sqcup N = \top$. For z we have $Z \sqcup Z = Z$. The information for instruction 4 is therefore unchanged, except that for y we now have $\top$.

We can now choose between two paths once again: staying within the loop, or exiting out to instruction 8. We will choose (arbitrarily, for now) to stay within the loop, and consider instruction 5. This is our second visit to instruction 5, and we have new information to consider: since we have gone through the loop, the assignment $y := 1$ has been executed, and we have to assume that y may be nonzero coming into instruction 5. This is accounted for by the latest update to instruction 4's analysis information, in which y is mapped to $\top$. Thus the information for instruction 4 describes both possible paths. We must update the analysis information for instruction 5 so it does so as well. In this case, however, since the instruction assigns 1 to y, we still know that y is nonzero after it executes. In fact, analyzing the instruction again with the updated input data does not change the analysis results for this instruction.

A quick check shows that going through the remaining instructions in the loop, and even coming back to instruction 4, the analysis information will not change. That is because the flow functions are deterministic: given the same input analysis information and the same instruction, they will produce the same output analysis information. If we analyze instruction 6, for example, the input analysis information from instruction 5 is the same input analysis information we used when analyzing instruction 6 the last time around. Thus, instruction 6's output information will not change, and so instruction 7's input information will not change, and so on. No matter which

[1] I provide the CFG for reference but omit the annotations in the interest of a cleaner diagram.

instruction we run the analysis on, anywhere in the loop (and in fact before the loop), the analysis information will not change.

We say that the dataflow analysis has reached a *fixed point*.[2] In mathematics, a fixed point of a function is a data value v that is mapped to itself by the function, i.e. $f(v) = v$. In this analysis, the mathematical function is the flow function, and the fixed point is a tuple of the dataflow analysis values at each program point. If we invoke the flow function on the fixed point, the analysis results do not change (we get the same fixed point back).

Once we have reached a fixed point of the function for this loop, it is clear that further analysis of the loop will not be useful. Therefore, we will proceed to analyze statement 8. The final analysis results are as follows:

	x	y	z	
1	N			
2	N	Z		
3	N	Z	Z	
4	Z_T, N_F	$\top$	Z	*updated*
5	N	N	Z	*already at fixed point*
6	$\top$	N	Z	*already at fixed point*
7	$\top$	N	Z	*already at fixed point*
8	Z	$\top$	Z	

Quickly simulating a run of the program program shows that these results correctly approximate actual execution. The uncertainty in the value of x at instructions 6 and 7 is real: x is nonzero after these instructions, except the last time through the loop, when it is zero. The uncertainty in the value of y at the end shows imprecision in the analysis: this loop always executes at least once, so y will be nonzero. However, the analysis (as currently formulated) cannot tell this for certain, so it reports that it cannot tell if y is zero or not. This is safe—it is always correct to say the analysis is uncertain—but not as precise as would be ideal.

The benefit of analysis, however, is that we can gain correct information about all possible executions of the program with only a finite amount of work. In our example, we only had to analyze the loop statements at most twice each before reaching a fixed point. This is a significant improvement over the actual program execution, which runs the loop 10 times. We sacrificed precision in exchange for coverage of all possible executions, a classic tradeoff in static analysis.

How can we be confident that the results of the analysis are correct, besides simulating every possible run of a (possibly very complex) program? The intuition behind correctness is the invariant that at each program point, the analysis results approximate all the possible program values that could exist at that point. If the analysis information at the beginning of the program correctly approximates the program arguments, then the invariant is true at the beginning of program execution. One can then make an inductive argument that the invariant is preserved. In particular, when the program executes an instruction, the instruction modifies the program's state. As long as the flow functions account for every possible way that instruction can modify state, then at the analysis fixed point they will have correctly approximated actual program execution. We will make this argument more precise in a future lecture.

[2]Sometimes abbreviated in one word as fixpoint.

2.5 A convenience: the $\bot$ abstract value and complete lattices

As we think about defining an algorithm for dataflow analysis more precisely, a natural question comes up concerning how instruction 4 is analyzed in the example above. On the first pass, we analyzed it using the dataflow information from instruction 3, but on the second pass we had to consider dataflow information from both instruction 3 and instruction 7.

It is more consistent to say that analyzing an instruction always uses the incoming dataflow analysis information from all instructions that could precede it. That way, we do not have to worry about following a specific path during analysis. However, for instruction 4, this requires a dataflow value from instruction 7, even if instruction 7 has not yet been analyzed. We could do this if we had a dataflow value that is always ignored when it is joined with any other dataflow value. In other words, we need a abstract dataflow value $\bot$ (pronounced "bottom") such that $\bot \sqcup l = l$.

$\bot$ plays a dual role to the value $\top$: it sits at the bottom of the dataflow value lattice. For all l, we have the identity $l \sqsubseteq \top$ and correspondingly $\bot \sqsubseteq l$. There is an greatest lower bound operator *meet*, $\sqcap$, which is dual to $\sqcup$. The meet of all dataflow values is $\bot$.

A set of values L that is equipped with a partial order $\sqsubseteq$, and for which both least upper bounds $\sqcup$ and greatest lower bounds $\sqcap$ exist in L and are unique, is called a *complete lattice*.

The theory of $\bot$ and complete lattices provides an elegant solution to the problem mentioned above. We can initialize σ at every instruction in the program, except at entry, to $\bot$, indicating that the instruction there has not yet been analyzed. We can then *always* merge all input values to a node, whether or not the sources of those inputs have been analysed, because we know that any $\bot$ values from unanalyzed sources will simply be ignored by the join operator $\sqcup$, and that if the dataflow value for that variable will change, we will get to it before the analysis is completed.

3 Analysis execution strategy

The informal execution strategy outlined above considers all paths through the program, continuing until the dataflow analysis information reaches a fixed point. This strategy can be simplified. The argument for correctness outlined above implies that for correct flow functions, it doesn't matter how we get to the analysis fixed point. This is sensible: it would be surprising if analysis correctness depended on which branch of an if statement we explored first! It is in fact possible to run the analysis on program instructions in any order we choose. As long as we continue doing so until the analysis reaches a fixed point, the final result will be correct. The simplest correct algorithm for executing dataflow analysis can therefore be stated as follows:

```
for Instruction i in program
    input[i] = ⊥
input[firstInstruction] = initialDataflowInformation

while not at fixed point
    pick an instruction i in program
    output = flow(i, input[i])
    for Instruction j in sucessors(i)
        input[j] = input[j] ⊔ output
```

Although in the previous presentation we have been tracking the analysis information immediately after each instruction, it is more convenient when writing down the algorithm to track the analysis information immediately before each instruction. This avoids the need for a distinguished location before the program starts (the start instruction is not analyzed).

In the code above, the termination condition is expressed abstractly. It can easily be checked, however, by running the flow function on each instruction in the program. If the results of analysis do not change as a result of analyzing any instruction, then it has reached a fixed point.

How do we know the algorithm will terminate? The intuition is as follows. We rely on the choice of an instruction to be fair, so that each instruction is eventually considered. As long as the analysis is not at a fixed point, some instruction can be analyzed to produce new analysis results. If our flow functions are well-behaved (technically, if they are monotone, as we will discuss in a future lecture) then each time the flow function runs on a given instruction, either the results do not change, or they get become more approximate (i.e. they are higher in the lattice). Later runs of the flow function consider more possible paths through the program and therefore produce a more approximate result which considers all these possibilities. If the lattice is of finite height—meaning there are at most a finite number of steps from any place in the lattice going up towards the $\top$ value—then this process must terminate eventually. More concretely: once an abstract value is computed to be $\top$, it will stay $\top$ no matter how many times the analysis is run. The abstraction only flows in one direction.

Although the simple algorithm above always terminates and results in the correct answer, it is still not always the most efficient. Typically, for example, it is beneficial to analyze the program instructions in order, so that results from earlier instructions can be used to update the results of later instructions. It is also useful to keep track of a list of instructions for which there has been a change since the instruction was last analyzed in the result dataflow information of some predecessor. Only those instructions need be analyzed; reanalyzing other instructions is useless since their input has not changed. Kildall captured this intuition with his worklist algorithm, described in pseudocode as:

```
for Instruction i in program
    input[i] = ⊥
input[firstInstruction] = initialDataflowInformation
worklist = { firstInstruction }

while worklist is not empty
    take an instruction i off the worklist
    output = flow(i, input[i])
    for Instruction j in succs(i)
        if output ⋢ input[j]
            input[j] = input[j] ⊔ output
            add j to worklist
```

The algorithm above is very close to the generic algorithm declared previously, except for the worklist that chooses the next instruction to analyze and determines when a fixed point is reached.

We can reason about the performance of this algorithm as follows. We only add an instruction to the worklist when the input data to some node changes, and the input for a given node can only change h times, where h is the height of the lattice. Thus we add at most $n * h$ nodes to the worklist, where n is the number of instructions in the program. After running the flow function for a node, however, we must test all its successors to find out if their input has changed. This test is done once for each edge, for each time that the source node of the edge is added to the worklist: thus at most $e * h$ times, where e is the number of control flow edges in the successor graph between instructions. If each operation (such as a flow function, $\sqcup$, or $\sqsubseteq$ test) has cost $O(c)$, then the overall cost is $O(c * (n + e) * h)$, or $O(c * e * h)$ because n is bounded by e.

The algorithm above is still abstract: We have not defined the operations to add and remove instructions from the worklist. We would like adding to the work list to be a set addition operation, so that no instruction appears in it multiple times. If we have just analysed the program with respect to an instruction, analyzing it again will not produce different results.

That leaves a choice of which instruction to remove from the worklist. We could choose among several policies, including last-in-first-out (LIFO) order or first-in-first-out (FIFO) order. In practice, the most efficient approach is to identify the strongly-connected components (i.e. loops) in the control flow graph of components and process them in topological order, so that loops that are nested, or appear in program order first, are solved before later loops. This works well because we do not want to do a lot of work bringing a loop late in the program to a fixed point, then have to redo that work when dataflow information from an earlier loop changes.

Within each loop, the instructions should be processed in reverse postorder, the reverse of the order in which each node is last visited when traversing a tree. Consider the example from Section 2.2 above, in which instruction 1 is an `if` test, instructions 2-3 are the then branch, instructions 4-5 are the else branch, and instruction 6 comes after the `if` statement. A tree traversal might go as follows: 1, 2, 3, 6, 3 (again), 2 (again), 1 (again), 4, 5, 4 (again), 1 (again). Some instructions in the tree are visited multiple times: once going down, once between visiting the children, and once coming up. The postorder, or order of the last visits to each node, is 6, 3, 2, 5, 4, 1. The reverse postorder is the reverse of this: 1, 4, 5, 2, 3, 6. Now we can see why reverse postorder works well: we explore both branches of the if statement (4-5 and 2-3) before we explore node 6. This ensures that we do not have to reanalyze node 6 after one of its inputs changes.

Although analyzing code using the strongly-connected component and reverse postorder heuristics improves performance substantially in practice, it does not change the worst-case performance results described above.