Applications of the Goulden-Jackson cluster method to counting Dyck paths by occurrences of subwords

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Abstract

Applications of the Goulden-Jackson cluster method to counting Dyck paths by occurrences of subwords

A dissertation presented to the Faculty of the
Graduate School of Arts and Sciences of Brandeis
University, Waltham, Massachusetts

by Chao-Jen Wang

Goulden and Jackson introduced the cluster method for counting words avoiding a prescribed set of subwords in [**14, 15**]. Noonan and Zeilberger [**17**] generalized it and wrote many Maple programs to implement the method and its extensions. We count Dyck paths according to the number of occurrences of certain patterns, using a variation of the Goulden-Jackson cluster method. We will give several examples of counting Dyck paths by occurrences of subwords and show how to use the cluster method to compute generating functions for those examples. Then we show more applications to count paths with bounded height by occurrences of subwords and more applications to count $r-$Dyck paths.

CHAPTER 1

Introduction

We first give some definitions and describe the general Goulden-Jackson cluster method with examples in Chapter 1. Then we give several examples of counting Dyck paths by occurrences of subwords and show how to use the cluster method to compute generating functions for those examples in Chapter 2. We apply the cluster method to count paths with bounded height by occurrences of subwords in Chapter 3. We show more applications to count $r-$Dyck paths in Chapter 4.

A *Dyck path* is a path in the first quadrant which begins at the origin. It ends at $(2n, 0)$ and consists of steps $(1, 1)$, called rises, and $(1, -1)$, called falls. We will refer to n as the *semilength* of the path. It is well-known that the number of all Dyck paths of semilength n is the nth Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Catalan number generating function is

$$C(x) = \sum_n c_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

We can encode each rise by a letter U for an up step and each fall by a letter D for a down step, obtaining the encoding of a Dyck path by a *Dyck word*.

A *Motzkin path* is a path in the first quadrant which begins at the origin. It ends at $(n, 0)$ and consists of steps $(1, 1)$, $(1, 0)$, and $(1, -1)$. Here n is the *length* of the path. The number of all Motzkin n-paths (paths with length n) is the nth Motzkin number m_n. The Motzkin number generating function is

$$M(x) = \sum_n m_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

We can encode $(1,1)$ by a letter U for an up step, $(1,-1)$ by a letter D for a down step, and $(1,0)$ by a letter F for a flat step, obtaining the encoding of a Motzkin path by a *Motzkin word*.

Goulden and Jackson introduced the cluster method for counting words avoiding a prescribed set of subwords in [**14, 15**]. Noonan and Zeilberger [**17**] generalized it and wrote many Maple programs to implement the method and its extensions. See also Stanley [**19**, Ch. 4, Ex. 14], Kupin and Yuster [**16**], and the references given there. We first illustrate the approach and describe the general Goulden-Jackson cluster method with an example:

Let $w = w_1 w_2 \cdots w_n$ be a word in an alphabet $A = \{a_1, a_2, \ldots, a_k\}$ and let A^* be the set of words made up by letters in A. Define the *length* of w as $l(w) = n$. A *marked subword* of w is a pair (i, v) such that

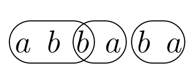
$$v = w_i w_{i+1} \cdots w_{i+l(v)-1}$$

where $l(v) \geq 2$. Here i indicates where the marked subword starts in w and v is the subword. A *marked word* is a word w together with a (possibly empty) set of marked subwords of w.

For example, the word *abbaba* together with the set of marked subwords

$$\{(1, abb), (3, ba), (5, ba)\}$$

is a marked word which we represent as



We can concatenate marked words in the obvious way. For example, concatenating

$$a\ \boxed{b\ a} \quad \text{and} \quad \boxed{b\ a}\ b\ a \qquad \text{gives} \quad a\ \boxed{b\ a}\boxed{b\ a}\ b\ a$$

A marked word is a *cluster* if it is not a concatenation of two nonempty marked words.

A marked word is the same as a word in the set of single letters and clusters. We can define $f(t)$ as the generating function for a set of marked words.

Given a set S of words of length at least 2, we may consider the generating function

$$f(t) = \sum_w w t^{n(w)}$$

where the sum runs over all words $w \in A^*$ and $n(w)$ is the number of occurrences of marked words in S in w. We think of the letters as noncommuting variables and t as commuting with these variables. It is easier to compute

$$\begin{aligned} f(1+t) &= \sum_w w(1+t)^{n(w)} \\ &= \sum_w w \sum_k \binom{n(w)}{k} t^k \\ &= \sum_w w \sum_{T \subseteq B} t^{|T|} \end{aligned}$$

where B is the set of occurrences of words in S in w.

So $f(1+t)$ is the sum of the weights of the marked words whose marked subwords are in S, where the weight of a marked word w is the underlying word times $t^{m(w)}$, where $m(w)$ is the number of marked subwords in w.

Therefore, we have

$$f(1+t) = (1 - a_1 - a_2 - \cdots - a_k - L(t))^{-1}$$

where $A = \{a_1, a_2, \ldots, a_k\}$ is an alphabet and $L(t)$ is the generating function for clusters.

More generally, we can use different weights for different words. By the same reasoning, we have the following result:

THEOREM 1. *Let* $A = \{a_1, a_2, \ldots, a_i\}$ *be an alphabet . Let* $S = \{v_1, v_2, \ldots, v_k\}$ *be a set of words of length at least 2. Let* $f(t_1, t_2, \ldots, t_k)$ *be the generating function for counting words in* A^* *by occurrences of* $v_1, v_2, ..., v_k$*. where we assign the weight* t_j *to* v_j*. Then*

$$f(1+t_1, 1+t_2, \ldots, 1+t_k) = (1 - a_1 - a_2 - \cdots - a_i - L(t_1, t_2, \ldots, t_k))^{-1}$$

where $L(t_1, t_2, \ldots, t_k)$ *is the generating function for clusters.*

For example, we want to count all words in $\{a, b, c\}^*$ by occurrences of ab (weighted t_1) and occurrences of bc (weighted t_2). Let $f(t_1, t_2)$ be the generating function

$$f(t_1, t_2) = \sum_w w t_1^i t_2^j$$

where i and j represent the number of occurrences of ab and bc in w. Consider

$$f(1+t_1, 1+t_2) = \sum_w w(1+t_1)^i (1+t_2)^j.$$

This counts all words in $\{a, b, c\}^*$ in which some occurrences of ab may be marked and some occurrences of bc may be marked. For example, a given word $w = ababc$ could contribute different marked words:

$a\ b\ a\ b\ c$	w together with empty set.
$(a\ b)\ a\ b\ c$	w together with $\{(1, ab)\}$.
$a\ b\ (a\ b)\ c$	w together with $\{(3, ab)\}$.

$a\ b\ a\ b\ c$	w together with $\{(1, ab), (3, ab)\}$.
$a\ b\ a\ b\ c$	w together with $\{(4, bc)\}$.
$a\ b\ a\ b\ c$	w together with $\{(1, ab), (4, bc)\}$.
$a\ b\ a\ b\ c$	w together with $\{(3, ab), (4, bc)\}$.
$a\ b\ a\ b\ c$	w together with $\{(1, ab), (3, ab), (4, bc)\}$.

Because $ababc$ contains two occurrences of ab and one occurrence of bc, its coefficient in the sum of the weights of the markings of $ababc$ is $(1+t_1)^2(1+t_2)$, which corresponds to the eight marked words above.

On the other hand, all of the marked words are made of letters and clusters:

$$a, \quad b, \quad c, \quad ab, \; bc, \; abc$$

Thus,

$$f(1+t_1, 1+t_2) = (1 - a - b - c - abt_1 - bct_2 - abct_1t_2)^{-1}. \tag{1}$$

Therefore, we can get the real generating function $f(t_1, t_2)$ by replacing t_1 with $t_1 - 1$ and t_2 with $t_2 - 1$ in equation (1). So

$$f(t_1, t_2) = (1 - a - b - c - ab(t_1 - 1) - bc(t_2 - 1) - abc(t_1 - 1)(t_2 - 1))^{-1}.$$

We can use the same method for counting more general sets of words. We can always reduce a problem of counting a set of words by occurrences of subwords to a problem of counting marked words. This approach will be useful whenever we can count the corresponding marked words.

We will apply the method to counting Dyck words. Suppose $S = \{UU, UDU\}$, and w is restricted to Dyck words, then a marked Dyck word could be

$$U\,U\,U\,D\,U\,U\,D\,U\,U\,D\,U\,U$$

It is not true that every concatenation of clusters is a marked Dyck word, so we need to do more work to count marked Dyck words by replacing each cluster with a new step. In this example, there are single down steps D, single up steps U, and clusters consisting of UU and UDU. So the problem reduces to counting paths with a more general set of steps that never go below the x-axis.

There is one complication. If, for example, S contains the word DU then we replace it with a flat step F, but this step cannot occur at height 0. This problem is no longer reduced to a problem of counting Motzkin paths. Instead, it reduces to a problem of counting Motzkin paths with no flat step at height 0. There are some special cases that are easier than the general case:

(1) Each cluster is equivalent to an up step U, a down step D, or a flat step F, and there is no restriction on where they occur. So the problem is equivalent to counting Dyck paths or Motzkin paths. Examples are $\{UUD\}$, $\{UD\}$, $\{UDU\}$, etc.

(2) Each cluster is equivalent to an up step U, a down step D, or a flat step F, but there are some restrictions on where they occur. We can derive generating functions for this case from some quadratic equations related to the Catalan, Motzkin or Narayana generaing function. Examples are $\{DU\}$ (cannot occur at height 0), $\{DDU\}$ (cannot occur at height 1 or 0), etc.

(3) The clusters are equivalent to steps that can go up by an arbitrary amount or down by at most 1 (or vice-versa). We can derive generating functions for this case from some functional equations which are sometimes quadratic or even of higher degree. So we may apply Lagrange inversion [**20**, Ch. 5, Page. 38] to solve them. Examples are $\{UU\}$, $\{DD\}$, $\{UUU\}$, etc.

CHAPTER 2

Examples of Counting Dyck Paths by Occurrences of Subwords

In the following examples, we compute a generating function g for counting Dyck paths by occurrences of subwords in which each marked word is counted with a weight $1+t$ and then we compute the real generating function h with a weight t by replacing t with $t-1$ in g.

2.1. Occurrences of UUD

Count Dyck paths by occurrences of UUD. For example, a marked Dyck word could be

$$U\ D\ \boxed{U\ U\ D}\ U\ U\ D\ D\ D$$

We assign to such a marked word the weight $x^i t^j$, where the semilength is i and there are j marked occurrences of UUD. To count all these marked words, we replace each occurrences of UUD by a new up step U'. So our example would be replaced with

$$U\ D\ U'\ U\ U\ D\ D\ D$$

Note that the original word is a marked Dyck word if and only if the new word (when U' is replaced by U) is a Dyck word. In this example, UUD is the only cluster. Since the occurrence of U in a Dyck path equals that of its semilength, we can count modified Dyck words where U has the weight x, D has the weight 1, and U' has the weight x^2t. Equivalently, we can count ordinary Dyck paths where each up step is weighted by $x + x^2t$. We set $u_1 = x + x^2t$ and $d_1 = 1$, where u_1 is the generating function for reducing up steps. Then we count Dyck paths with up steps weighted by u_1 and down steps weighted by d_1. To count these, we use a well-known decomposition called the first return decomposition.

Every nonempty Dyck path can be decomposed at the first return to x-axis. Every nonempty Dyck path can be factored as UG_1DG_2 where U is an up step, D is a down step, and G_1, G_2 are (possibly empty) Dyck paths. See Figure 2.1.

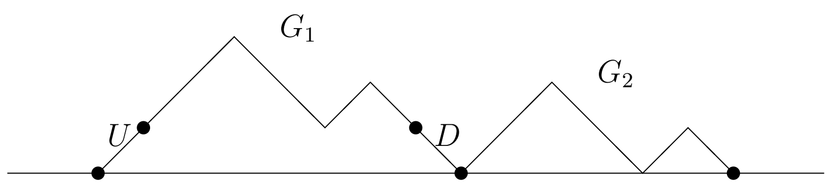


FIGURE 2.1. The first return decomposition for Dyck paths

So the generating function $g(x, t)$ with a weight $1 + t$ satisfies

$$g = 1 + u_1gd_1g$$

where 1 represents the empty path, and u_1gd_1g represents the decomposition for nonempty Dyck paths. Replacing u_1 by $x + x^2t$, and d_1 by 1, we get

$$g = 1 + (x + x^2t)g^2.$$

Solving for g, we get

$$\begin{aligned} g &= \frac{1 - \sqrt{1 - 4x(1 + xt)}}{2x(1 + xt)} \\ &= C(x + x^2t). \end{aligned}$$

As described in the introduction, g counts Dyck words where every occurrence of UUD is weighted by $1+t$. So the generating function h for Dyck words weighted by t^j where j is the number of occurrences of UUD, is obtained by replacing t with $t-1$ in g. So we get the real generating function h in which every occurrence of UUD is weighted by t:

$$\begin{aligned} h(x,t) &= \frac{1-\sqrt{1-4x(1-x+xt)}}{2x(1-x+xt)} \\ &= 1+x+(1+t)x^2+(1+4t)x^3 \\ &\quad + (1+11t+2t^2)x^4 + (1+26t+15t^2)x^5 + \cdots \end{aligned}$$

Here the coefficients are sequence A091156 in the Online Encyclopedia of Integer Sequences [**18**], where they are described as the number of Dyck paths of semilength n, having k long ascents (i.e, ascents of length at least 2). It is easy to see that every Dyck path having k long ascents has exactly k occurrences of UUD, since every long ascent must be followed by a down step D.

In particular, for $t=0$, we have

$$\begin{aligned} h(x,0) &= \frac{1}{1-x} \\ &= 1+x+x^2+x^3+\cdots \end{aligned}$$

This is the generating function of UUD-free Dyck paths (i.e. Dyck paths with no occurrences of UUD) with semilength weighted by x. The only UUD-free Dyck paths are of the form $(UD)^n$. Therefore, the coefficients of powers of x in $h(x,0)$ are all 1.

2.2. Occurrences of UDU

Suppose we now count Dyck paths by occurrences of UDU. The occurrences of UDU are weighted by t. In this example, we use the same approach as in the case of occurrences of UUD in section 2.1. We need to find the clusters first, since the clusters are no longer trivial.

In this case, the clusters are marked Dyck words of the form $U(DU)^i$ for $i = 1, 2, 3, \ldots$:

$$UDU,\ UDUDU,\ UDUDUDU,\ \ldots$$

So the cluster generating function $L(t)$ is

$$\begin{aligned} udut + ududut^2 + udududut^3 + \cdots &= \sum_{i\geq 1} u(du)^i t^i \\ &= \frac{u^2dt}{1-udt} \end{aligned}$$

where u and d are commuting variables.

We count these modified Dyck words where U has the weight x, D has the weight 1, and the cluster generating function is $\dfrac{x^2t}{1-xt}$. Since these clusters reduce to up steps, we can set $u_1 = x + \dfrac{x^2t}{1-xt} = \dfrac{x}{1-xt}$ to get that the generating function $g(x,t)$ with a weight $1+t$ satisfies

$$\begin{aligned} g &= 1 + u_1 g d_1 g \\ &= 1 + \left(\frac{x}{1-xt}\right) g^2 \end{aligned} \tag{2}$$

where u_1 represents reducing up steps and d_1 represents reducing down steps.

Notice that any functional equation of this from

$$g = 1 + ag^2$$

has the solution

$$g = \frac{1-\sqrt{1-4a}}{2a} = C(a).$$

Therefore, by equation (2), we get

$$\begin{aligned} g(x,t) &= C\left(\frac{x}{1-xt}\right) \\ &= \frac{1-\sqrt{1-4\left(\frac{x}{1-xt}\right)}}{2\left(\frac{x}{1-xt}\right)} \\ &= \frac{1-xt-\sqrt{(1-4x-xt)(1-xt)}}{2x}. \end{aligned}$$

Using the cluster method, we replace t by $t-1$, and we get the real generating function

$$\begin{aligned} h(x,t) &= \frac{1+x-xt-\sqrt{(1-3x-xt)(1+x-xt)}}{2x} \\ &= 1 + x + (t+1)x^2 + (2+2t+t^2)x^3 + (4+6t+3t^2+t^3)x^4 + \cdots \end{aligned}$$

Here the coefficients are sequence A091869 in the Online Encyclopedia of Integer Sequences [**18**]. Some related statistics have been studied by Sun [**21**].

In particular, for $t = 0$, we have

$$\begin{aligned} h(x,0) &= \frac{1+x-\sqrt{(1-3x)(1+x)}}{2x} \\ &= \frac{1+x-\sqrt{1-2x-3x^2}}{2x} \\ &= 1 + x + x^2 + 2x^3 + 4x^4 + 9x^5 + 21x^6 + 51x^7 + 127x^8 + \cdots \end{aligned}$$

This is the generating function of UDU-free Dyck paths with semilength weighted by x. Subtracting 1 from it and dividing by x gives the generating function for Motzkin numbers as shown by Donaghey and Shapiro [**11**]. So the number of UDU-free Dyck paths with semilength n is m_{n-1}, the $(n-1)$th Motzkin number.

We can also verify that

$$h(x,t) = 1 + \frac{x}{1-xt} M\left(\frac{x}{1-xt}\right) \tag{3}$$

where $M(x) = \dfrac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, the Motzkin number generating function.

Expanding the right side of equation (3), we have

$$\begin{aligned}
1 + \frac{x}{1-xt} M\left(\frac{x}{1-xt}\right) &= 1 + \frac{1-\left(\frac{x}{1-xt}\right) - \sqrt{1-2\left(\frac{x}{1-xt}\right)-3\left(\frac{x}{1-xt}\right)^2}}{2\left(\frac{x}{1-xt}\right)} \\
&= 1 + \frac{1-xt-x-\sqrt{(1-xt)^2-2x(1-xt)-3x^2}}{2x} \\
&= \frac{1+x-xt-\sqrt{(1-3x-xt)(1+x-xt)}}{2x} \\
&= h(x,t).
\end{aligned}$$

Using equation (3), we can get an explicit formula for the coefficients of $h(x,t)$ so that

$$\begin{aligned}
h(x,t) &= 1 + \frac{x}{1-xt} M\left(\frac{x}{1-xt}\right) \\
&= 1 + \sum_{i\geq 0} m_i \frac{x^{i+1}}{(1-xt)^{i+1}} \\
&= 1 + \sum_{i\geq 0} m_i x^{i+1} \sum_{k\geq 0} \binom{i+k}{k} (xt)^k
\end{aligned}$$

$$= 1 + \sum_{i \geq 0, k \geq 0} \binom{i+k}{k} m_i x^{i+k+1} t^k$$
$$= 1 + \sum_{n \geq 1, 0 \leq k \leq n-1} \binom{n-1}{k} m_{n-k-1} x^n t^k$$

where m_i is the ith Motzkin number.

A bijective proof for this formula has been given by Callan [**3**].

2.3. Occurrences of UD

Count Dyck paths by occurrences of UD. For example, a marked Dyck word could be

$$\boxed{U\ D}\ U\ U\ D\ U\ \boxed{U\ D}\ D\ D$$

We assign to such a marked word the weight $x^i t^j$ where the semilength is i and there are j marked occurrences of UD. To count all these marked words, we replace each occurrence of UD by a new flat step F. So our example would be replaced with

$$F\ U\ U\ D\ U\ F\ D\ D$$

Note that the original word is a marked Dyck word if and only if the new word (when UD is replaced by F) is a Motzkin word. In this example, UD is the only cluster. We count these modified Motzkin words where U has the weight x, D has the weight 1, and F has the weight xt. Every nonempty Motzkin path can start with a flat step F or an up step U. It can be decomposed into FG or UG_1DG_2, where G, G_1, G_2 are Motzkin paths. See Figure 2.2.

So the generating function $g(x, t)$ with a weight $1 + t$ satisfies

$$g = 1 + fg + ugdg \tag{4}$$

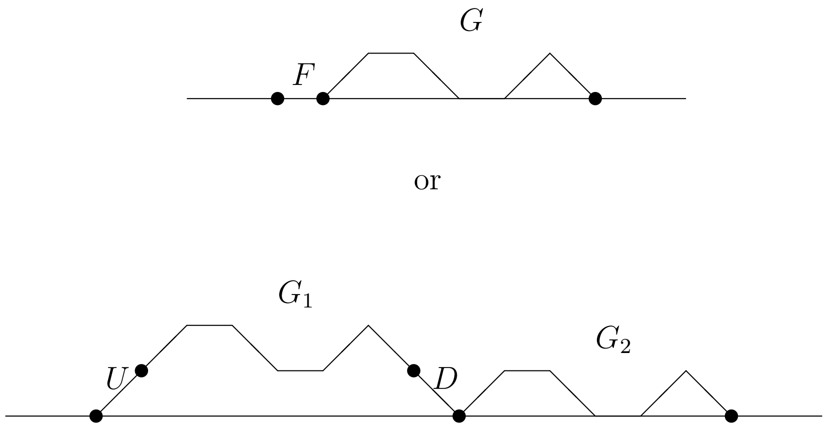


FIGURE 2.2. Two cases of decompositions for such path

where f represents reducing flat steps, u represents reducing up steps, and d represents reducing down steps. In this case, f represents a UD, u represents a single up step U, and d represents a single down step D.

Replacing f by xt, u by 1, and d by x, we solve equation (4) to get

$$g = \frac{1 - xt - \sqrt{(1-xt)^2 - 4x}}{2x}.$$

Using the cluster method, we replace t by $t-1$ in g and get the real generating function

$$h(x,t) = \frac{1 + x - xt - \sqrt{(1+x-xt)^2 - 4x}}{2x}. \tag{5}$$

In particular, for $t = 0$, we have $h(x,0) = 1$. This is the generating function of Dyck paths with no peak, UD. As we know, the empty path is the only Dyck path with no peak. Note that h is a generating function for the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ satisfying

$$h(x,t) = 1 + \sum_{n,k=1}^{\infty} N(n,k) x^n t^k. \tag{6}$$

2.4. Occurrences of DU

Now we look at an example of type 2. Suppose we count Dyck paths by occurrences of DU weighted by t. We use the same approach as the case of occurrences of UD in section 2.3. However, here we want to count Motzkin paths with no flat steps at height 0. So a generating function $g(x, t)$ with a weight $1 + t$ for counting these modified Motzkin paths where U has the weight x, D has the weight 1, and F (when DU is replaced by F) has the weight xt, satisfies

$$g = 1 + fg + ugdg \tag{7}$$

where f, u, and d are commuting variables. This is the same as equation (4) in section 2.3. In this case, f represents DU, u represents a single up step U, and d represents a single down step D. However, DU cannot occur at height 0. So the problem can reduce to one of counting Motzkin paths with no flat step at height 0.

Let g be the generating function for counting Motzkin paths with no height restriction. Let g_1 be the generating function for counting Motzkin paths with no flat step at height 0. Every nonempty Motzkin path with no flat step at height 0 can be factored as $UGDG_1$ at the first return, where G is a Motzkin path with no height restriction and G_1 is a Motzkin path with no flat step at height 0. See Figure 2.3.

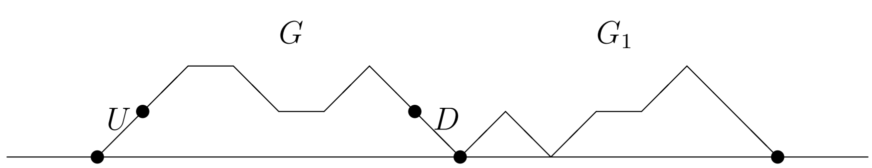


FIGURE 2.3. Every nonempty Motzkin path with no flat step at height 0 can be factored as $UGDG_1$ at the first return

Then g_1 satisfies

$$g_1 = 1 + ugdg_1.$$

So we can write

$$g_1 = \frac{1}{1-ugd}. \tag{8}$$

Replacing f by tx, u by 1, and d by x, we solve equation (7) to get

$$g = \frac{1 - xt - \sqrt{(1-xt)^2 - 4x}}{2x}.$$

Then we substitute this for g in equation (8) and solve for g_1. We get

$$g_1 = \frac{1}{1-xg} = \frac{2}{1 + xt + \sqrt{(1-xt)^2 - 4x}}.$$

Using the cluster method, we replace t by $t-1$ in g_1 to get the real generating function

$$\begin{aligned} h_1(x,t) &= \frac{2}{1 + x(t-1) + \sqrt{(1-x(t-1))^2 - 4x}} \\ &= \frac{1 + x(t-1) - \sqrt{(1-x(t-1))^2 - 4x}}{2xt}. \end{aligned} \tag{9}$$

In particular, for $t = 0$, we have

$$\begin{aligned} h_1(x,0) &= \frac{2}{1 - x + 1 - x} \\ &= \frac{1}{1-x} \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

This is the generating function of Dyck paths with no valley, DU. The only possible Dyck paths with no valley are of the form U^nD^n. Therefore, all coefficients of $h_1(x,0)$ are 1. Note that $h_1(x,t)$ is also a generating function for the Narayana numbers:

From equation (9), (5) and (6), we have

$$th_1 - t + 1 = h = 1 + \sum_{n,k=1}^{\infty} N(n,k)x^n t^k.$$

So,

$$h_1 = 1 + \sum_{n,k=1}^{\infty} N(n,k)x^n t^{k-1}. \tag{10}$$

We can see (10) directly, since a nonempty path with k peaks has $k-1$ valleys.

2.5. Occurrences of DDU

Suppose we now count Dyck paths by occurrences of DDU weighted by t. In this example, the only cluster, DDU, can be reduced to a down step. Using the same approach as in the case of occurrences of UUD in section 2.1, we can start from a generating function g with no height restriction and with a weight $1+t$ which satisfies

$$g = 1 + u_1 g d_1 g \tag{11}$$

where $u_1 = u$ and $d_1 = d + ddut$, a single down step D or a reducing down step DDU. Because there is a height restriction that DDU can only start from a height not less than 2, we can elevate g by using the same approach from equation (8) in section 2.4 to get a generating function g_1 under the height restriction and with a weight $1+t$ which satisfies

$$g_1 = \frac{1}{1-ugd}. \tag{12}$$

In equation (11), we replace u by x, and d by 1 to get

$$g = 1 + x(1+xt)g^2.$$

Solving for g,

$$\begin{aligned} g &= C(x(1+xt)) \\ &= \frac{1-\sqrt{1-4x(1+xt)}}{2x(1+xt)}. \end{aligned}$$

This is the same g as in section 2.1. We can substitute this for g in equation (12) to get

$$\begin{aligned} g_1 &= \frac{1}{1-xg} \\ &= \frac{2(1+xt)}{1+2xt+\sqrt{1-4x(1+xt)}}. \end{aligned}$$

Replacing t by $t-1$ in g_1, we get the real generating function

$$h_1(x,t) = \frac{2(1+x(t-1))}{1+2x(t-1)+\sqrt{1-4x(1+x(t-1))}}.$$

In particular, for $t=0$, we have

$$\begin{aligned} h_1(x,0) &= \frac{2-2x}{1-2x+\sqrt{1-4x+4x^2}} \\ &= \frac{1-x}{1-2x} \\ &= 1+\frac{x}{1-2x} \\ &= 1+\sum_{n=1}^{\infty} 2^{n-1}x^n. \end{aligned} \tag{13}$$

This is the generating function of DDU-free Dyck paths with semilength weighted by x. There are 2^{n-1} different DDU-free Dyck paths with semilength n. We can see equation (13) directly, since the DDU-free Dyck paths can be written as the form

$$U^{a_1}DU^{a_2}DU^{a_3}D\cdots U^{a_{k-1}}DU^{a_k}D^{n-k-1}$$

where a_i are positive integers, and $\sum_{i=1}^{k} a_i = n$. Therefore, the number of the DDU-free Dyck paths with semilength n is equal to 2^{n-1}, the number of compositions of n, if $n \geq 1$. See Callan [**5**].

We can also verify that

$$h_1(x,t) = 1 + \frac{x}{1-2x} C\left(\frac{x^2t}{(1-2x)^2}\right) \tag{14}$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, the Catalan number generating function.

Expanding each side of equation (14), we have for the left-hand side

$$\begin{aligned}
h_1(x,t) &= \frac{2(1+x(t-1))}{1+2x(t-1)+\sqrt{1-4x(1+x(t-1))}} \\
&= \frac{2(1+x(t-1))\left(1+2x(t-1)-\sqrt{1-4x(1+x(t-1))}\right)}{(1+2x(t-1))^2-(1-4x(1+x(t-1)))} \\
&= \frac{2(1+x(t-1))\left(1+2x(t-1)-\sqrt{1-4x(1+x(t-1))}\right)}{4xt(1+x(t-1))} \\
&= \frac{1+2x(t-1)-\sqrt{1-4x(1+x(t-1))}}{2xt}
\end{aligned}$$

and the right-hand side

$$\begin{aligned}
1 + \frac{x}{1-2x} C\left(\frac{x^2t}{(1-2x)^2}\right) &= 1 + \frac{x}{1-2x} \frac{1-\sqrt{1-4\left(\frac{x^2t}{(1-2x)^2}\right)}}{2\left(\frac{x^2t}{(1-2x)^2}\right)} \\
&= 1 + \frac{1-\sqrt{1-\frac{4x^2t}{(1-2x)^2}}}{\frac{2xt}{1-2x}}
\end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{1 - 2x - \sqrt{(1-2x)^2 - 4x^2 t}}{2xt} \\
&= \frac{1 + 2x(t-1) - \sqrt{1 - 4x(1 + x(t-1))}}{2xt} \\
&= h_1(x,t).
\end{aligned}$$

We can get an explicit formula for the coefficients of $h_1(x,t)$ so that

$$\begin{aligned}
h_1(x,t) &= 1 + \frac{x}{1-2x} C\left(\frac{x^2 t}{(1-2x)^2}\right) \\
&= 1 + \frac{x}{1-2x} \sum_{k \geq 0} c_k t^k \left(\frac{x^2}{(1-2x)^2}\right)^k \\
&= 1 + x \sum_{k \geq 0} c_k t^k 2^{-2k} \frac{(2x)^{2k}}{(1-2x)^{2k+1}} \\
&= 1 + x \sum_{k \geq 0} c_k t^k 2^{-2k} \sum_{n \geq 0} \binom{n}{2k} (2x)^n \\
&= 1 + x \sum_{n \geq 0, k \geq 0} 2^{n-2k} \binom{n}{2k} c_k x^n t^k \\
&= 1 + \sum_{n \geq 1, k \geq 0} 2^{n-2k-1} \binom{n-1}{2k} \frac{1}{k+1} \binom{2k}{k} x^n t^k
\end{aligned}$$

where c_k is the kth Catalan number.

There is a bijective proof for this formula given by Callan [**3**]. Some related problems have been studied by Deutsch [**8**] and Sun [**22**].

2.6. Occurrences of DD

Now we look at an example of type 3. In these problems, we need to count paths with steps that go up by 1 and down by any amount. We cannot use the approach of applying the first return decomposition (see Figure 2.1), so we use another decomposition.

We consider paths with steps U, D_0, D_1, $\ldots$, D_i where U is an up step and D_j is a step that goes down by j. Every nonempty such path can be factored as $G_1UG_2UG_3\cdots UG_{i+1}D_i$, where each G_j is a path which ends on the same height as the height of its starting point and never goes below the height. See Figure 2.4.

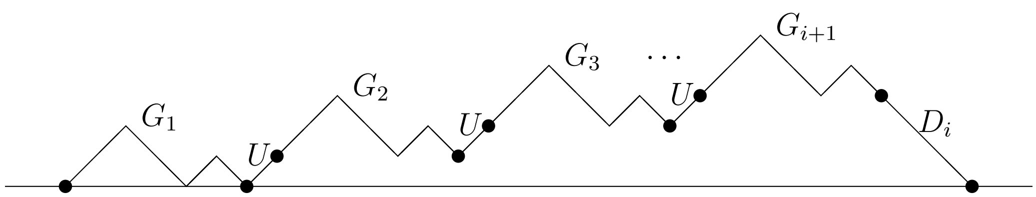


FIGURE 2.4. Decomposition for such path

So the generating function for such paths g satisfies

$$g = 1 + \sum_{i=0}^{\infty} u^i g^{i+1} d_i = 1 + g \sum_{i=0}^{\infty} (ug)^i d_i \tag{15}$$

where u represents a single up step and d_i represents a step that goes down by i.

Now consider the special case of counting Dyck paths by occurrences of DD weighted by t. Look at the generating function for all possible reducing down steps, including a single down step and clusters consisting of DD's:

$$D,\ DD,\ DDD, \ldots$$

So, the generating function for reducing down steps d_i is

$$d + d^2t + d^3t^2 + \cdots = \frac{d}{1-dt}.$$

This is a problem in which we count paths reduced to paths with steps up by 1, and down by any amount. We may keep track of the height of paths. Let

$$\phi(z) = \sum_{i=0}^{\infty} z^i d_i = \frac{dz}{1-dzt}.$$

Then, we have

$$g = 1 + g\phi(ug).$$

Replacing u by x and d by 1, we have

$$g = 1 + \frac{xg^2}{1 - xgt}.$$

Solving for g, we get the generating function g with a weight $1 + t$

$$g = \frac{1 + xt - \sqrt{(1 - xt)^2 - 4x}}{2x(t + 1)}.$$

Using the cluster method, we replace t by $t - 1$, and we get the real generating function h with a weight t

$$h(x, t) = \frac{1 + x(t - 1) - \sqrt{(1 - x(t - 1))^2 - 4x}}{2xt}. \qquad (16)$$

From equations (9) and (16), we can see that the generating functions are the same for counting paths by occurrences of DD and DU, as is well known [**7, 8**].

2.7. Occurrences of DDD

Now we can look at another example of type 3. Suppose we count Dyck paths by occurrences of DDD weighted by t. We can use the same approach as in the case of occurrences of DD in section 2.6. The generating function g with a weight $1 + t$ satisfies

$$g = 1 + \sum_{i=0}^{\infty} u^i g^{i+1} d_i = 1 + g \sum_{i=0}^{\infty} (ug)^i d_i. \qquad (17)$$

The reducing down steps are a single down step and clusters consisting of DDD's

$$D, DDD, DDDD, DDDDD, \ldots$$

So, the generating function for reducing down steps d_i is

$$d + \frac{d^3 t}{1 - dt - d^2 t}.$$

Let

$$\phi(z) = dz + \frac{d^3 z^3 t}{1 - dzt - d^2 z^2 t} = \sum_{i=0}^{\infty} z^i d_i$$

where d_i is the contribution from reducing steps that go down by i.

From equation (17), we have

$$g = 1 + g\phi(ug).$$

Replacing u by x and d by 1, we have

$$\phi(ug) = xg + \frac{x^3 g^3 t}{1 - xgt - x^2 g^2 t}.$$

So,

$$g = 1 + g\left(xg + \frac{x^3 g^3 t}{1 - xgt - x^2 g^2 t}\right).$$

This looks cubic, but turns out to be quadratic.

Solving for g, we get the generating function g with a weight $1 + t$

$$g = \frac{1 + xt - \sqrt{1 - 2xt + x^2 t^2 - 4x + 4x^2 t}}{2x(t + 1 - xt)}.$$

Using the cluster method, we replace t by $t-1$, and we get the real generating function h with a weight t

$$h(x,t) = \frac{1 + xt - x - \sqrt{1 - 2xt + x^2 t^2 - 2x + 2x^2 t - 3x^2}}{2x(t - xt + x)}. \tag{18}$$

In particular, for $t = 0$, we have

$$\begin{aligned} h(x,0) &= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + \cdots \end{aligned}$$

This is the generating function of DDD-free Dyck paths with semilength weighted by x. This is also the generating function for the Motzkin numbers. There is a bijection from UUU-free Dyck n-paths to Motzkin n-paths given by Callan [**3**]. It is easy to see that DDD-free Dyck n-paths have the same distribution as UUU-free Dyck n-paths when writing a path in reverse order.

2.8. Occurrences of UUD and UDD

Now we will give an example of counting Dyck paths according to the occurrences of two subwords. Suppose they are UUD weighted by s and UDD weighted by t. In this case, the only nontrivial cluster consisting of UUD and UDD is $UUDD$. This is an example of type 2. So we use the same approach as in the case of UD in section 2.3. The generating function g with weights $1 + s$ and $1 + t$ satisfies

$$\begin{aligned} g &= 1 + fg + u_1 g d_1 g \\ &= 1 + u^2 d^2 stg + (u + uuds)g(d + uddt)g \end{aligned}$$

where $f = u^2 d^2 st$ represents a flat step, $u_1 = u + uuds$ represents up steps, $d_1 = d + uddt$ represents down steps.

Replacing u by x and d by 1, and solving for g, we get

$$g = \frac{1 - x^2 st - \sqrt{1 - 2x^2 st + x^4 s^2 t^2 - 4x - 4x^2 s - 4x^2 t - 4x^3 st}}{2x(1 + xs)(1 + xt)}$$

$$= \frac{1 - x^2st - \sqrt{(1 - x^2st)^2 - 4x(1 + xs)(1 + xt)}}{2x(1 + xs)(1 + xt)}.$$

Using the cluster method, we replace s by $s - 1$ and t by $t - 1$, and get the real generating function h with weights s and t

$$h = \frac{1 - x^2(s-1)(t-1) - \sqrt{(1 - x^2(s-1)(t-1))^2 - 4x(1 - x + xs)(1 - x + xt)}}{2x(1 - x + xs)(1 - x + xt)}.$$

In particular, letting $s = t$ gives

$$\begin{aligned} h(x,t,t) &= \frac{1 + x - xt - \sqrt{(1 + x - xt)^2 - 4x}}{2x(1 - x + xt)} \\ &= 1 + x + (1 + t^2)x^2 + (1 + 4t^2)x^3 + (1 + 10t^2 + 2t^3 + t^4)x^4 + \cdots \end{aligned}$$

Here the coefficients are sequence A127155 in the Online Encyclopedia of Integer Sequences [**18**], where they are described as the number of Dyck paths of semilength n having a total of k long ascents and long descents. It is easy to see that every Dyck path having a total of k long ascents and long descents has a total of k occurrences of UUD and UDD, since every long ascent is followed by a down step D and every long descent is preceded with an up step U.

2.9. Occurrences of UU and UDD

Now we give another example to count Dyck paths according to the occurrences of the given subwords UU weighted by s and UDD weighted by t. The semilength is weighted by x. This is an example in which we count paths reduced to paths with steps down by 1 or up by any amount. See Figure 2.5.

We consider paths with steps D_1, U_0, U_1, …, U_i where D_1 is the reducing down step that goes down by 1 and U_j is the step that goes up by j. Every nonempty such path can be factored as $U_iG_1D_1G_2D_1G_3 \cdots G_iD_1G_{i+1}$, where each G_j is a path which ends on the same height as that of its starting point and never goes below the height. See Figure 2.5.

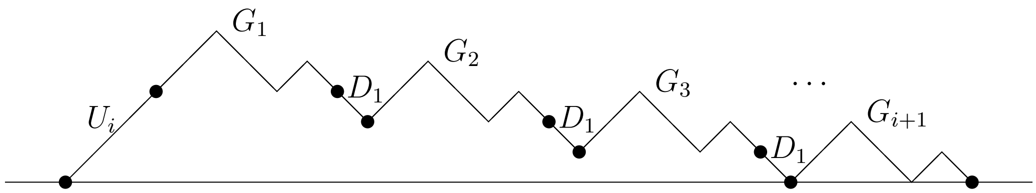


FIGURE 2.5. Decomposition for such path

So the generating function g with weights $1+s$ for UU and $1+t$ for UDD satisfies

$$g = 1 + \sum_{i=0}^{\infty} u_i g^{i+1} {d_1}^i$$

where u_i represents reducing steps that go up by i, and d_1 represents reducing down steps that go down by 1. The generating function for reducing down steps, including only a single down step and a UDD is

$$d_1 = d + uddt.$$

Therefore, we can see that the clusters consisting of UU and UDD are of the form U^j for $j = 2, 3, 4, \ldots$ or $U^k DD$ for $k = 1, 2, 3, \ldots$. So the generating function for reducing up steps, including a single up step and the clusters consisting of UU and UDD, is obtained by subtracting d_1 from the generating function of all possible reducing steps:

$$\begin{aligned} u + d + uddt + \frac{u^2 s}{1 - us}(1 + ddt) - d_1 &= u + \frac{u^2 s}{1 - us}(1 + ddt) \\ &= \frac{u + u^2 d^2 st}{1 - us}. \end{aligned}$$

So, let

$$\phi(z) = \sum_{i=0}^{\infty} u_i z^i = \frac{uz + u^2 d^2 st}{1 - uzs}$$

where z is the weight for height.

Then we have

$$g = 1 + g\phi(gd_1).$$

Replacing u by x, d by 1, and d_1 by $1 + xt$, we get

$$\phi(gd_1) = \phi((1+xt)g) = \frac{x(1+xt)g + x^2st}{1 - x(1+xt)gs}.$$

So,

$$g = 1 + \left(\frac{x(1+xt)g + x^2st}{1 - x(1+xt)gs}\right) g.$$

Simplifying , we get a quadratic equation

$$x(1+xt)(s+1)g^2 - (1+xs)g + 1 = 0.$$

Solving for g, we get the generating function g with weights $1 + s$ and $1 + t$

$$g = \frac{1 + xs - \sqrt{(1+xs)^2 - 4x(1+xt)(s+1)}}{2x(1+xt)(s+1)}.$$

Using the cluster method, we replace s by $s - 1$ and t by $t - 1$, and get the real generating function h with weights s and t

$$h(x,s,t) = \frac{1 - x + xs - \sqrt{1 - 2xs - 2x + x^2s^2 + 2x^2s + x^2 - 4x^2st}}{2xs(1 - x + xt)}. \tag{19}$$

Notice that h satisfies

$$1 - (1 - x + xs)h + xs(1 - x + xt)h^2 = 0.$$

Then, we have

$$h - xsh^2 = 1 + xh - xsh - x^2sh^2 + x^2sth^2$$

$$h(1 - xsh) = (1 - xsh)(1 + xh) + x^2sth^2.$$

So, h also satisfies

$$h = 1 + xh + \frac{x^2sth^2}{1 - xsh}. \tag{20}$$

It would be interesting to find a direct proof of (20). We can do this using a decomposition of Deutsch [**6**]. See Figure 2.6.

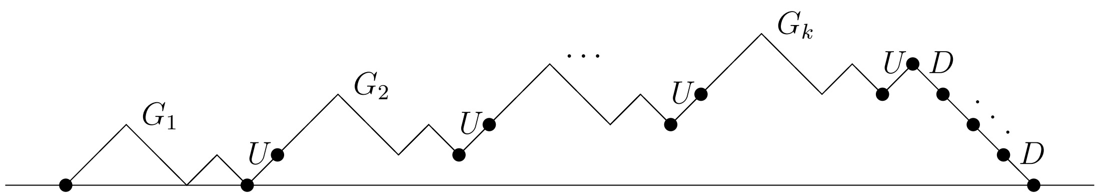


FIGURE 2.6. Deutsch's decomposition for Dyck paths

Let G be a nonempty Dyck path. Suppose there are exactly k consecutive down steps after the last up step in G. Then G can be factored uniquely as $G_1UG_2U \cdots G_kUD^k$, where each G_j is a Dyck path. Let h be the generating function for counting all Dyck paths. Then this decomposition shows that h satisfies

$$h = 1 + \sum_{k=1}^{\infty} (hu)^k d^k.$$

Now we count Dyck paths according to the occurrences of UU weighted by s and UDD weighted by t. If $k = 1$, the occurrences of UU and UDD in G are the same as those in G_1. If $k \geq 2$, then every UU or UDD in each G_j occurs in G. Moreover, for $1 \leq j \leq k-1$ the U between G_j and G_{j+1} is followed by another U, giving additional $k - 1$ occurrences of UU and there is one extra UDD from the last up step followed by at least two down steps.

So the generating function h with weights s for UU and t for UDD satisfies

$$h = 1 + hud + \sum_{k=2}^{\infty} s^{k-1} t (hu)^k d^k$$

Replacing u by x and d by 1, we get

$$h = 1 + xh + \frac{x^2 sth^2}{1 - xsh}.$$

We can apply Lagrange inversion [**20**, Ch. 5, Page. 38] to get an explicit formula for the coefficients of $h(x, s, t)$.

Theorem 2. *Let $h(x, s, t)$ be the generating function for counting Dyck paths by occurrences of UU(weighted s) and UDD(weighted t). Then*

$$h(x, s, t) = \sum_{n,i,j} \frac{1}{n+1} \binom{n+1}{i+1, j, n-i-j} \binom{i-1}{i-j} x^n s^i t^j$$

where the sum runs over all nonnegative integers for n, i, and j.

Proof. In order to apply Lagrange inversion, we can add a dummy variable z to equation (20) getting

$$h = z \left(1 + xh + \frac{x^2 sth^2}{1 - xsh} \right).$$

By Lagrange inversion [**20**, Ch. 5, Page. 38], we have

$$\begin{aligned}
[z^n]h^k &= \frac{k}{n}[y^{n-k}] \left(1 + xy + \frac{x^2 sty^2}{1 - xsy} \right)^n \\
&= \frac{k}{n}[y^{n-k}] \sum_{\substack{i,j,m \\ i+j+m=n}} \binom{n}{i, j, m} (xy)^m \left(\frac{x^2 sty^2}{1 - xsy} \right)^j \\
&= \frac{k}{n}[y^{n-k}] \sum_{\substack{i,j,m,l \\ i+j+m=n \\ 2j+m+l=n-k}} \binom{n}{i, j, m} (xy)^m (x^2 sty^2)^j \binom{j+l-1}{l} (xsy)^l
\end{aligned}$$

$$= \frac{k}{n} \sum_{\substack{i,j,m,l \\ i+j+m=n \\ 2j+m+l=n-k}} \binom{n}{i,j,m} x^m (x^2st)^j \binom{j+l-1}{l} (xs)^l.$$

Replacing n with $i+j+m$ and l with $n-k-2j-m = i-k-j$, we get a power series for h^k in z,

$$h(x,s,t,z)^k = \sum_{i,j,m} \frac{k}{i+j+m} \binom{i+j+m}{i,j,m} \binom{i-k-1}{i-k-j} x^m (x^2st)^j (xs)^{i-k-j} z^n.$$

If we set $z=1$,

$$\begin{aligned} h(x,s,t)^k &= \sum_{i,j,m} \frac{k}{i+j+m} \binom{i+j+m}{i,j,m} \binom{i-k-1}{i-k-j} x^m (x^2st)^j (xs)^{i-k-j} \\ &= \sum_{i,j,m} \frac{k}{i+j+m} \binom{i+j+m}{i,j,m} \binom{i-k-1}{i-k-j} x^{i+j+m-k} s^{i-k} t^j. \end{aligned}$$

Replacing the variables, i by $i+k$, and m by $n-i-j$, we get

$$h(x,s,t)^k = \sum_{n,i,j} \frac{k}{n+k} \binom{n+k}{i+k, j, n-i-j} \binom{i-1}{i-j} x^n s^i t^j.$$

In particular for $k=1$, we have

$$h(x,s,t) = \sum_{n,i,j} \frac{1}{n+1} \binom{n+1}{i+1, j, n-i-j} \binom{i-1}{i-j} x^n s^i t^j. \tag{21}$$

Here, for all nonnegative integers n, i, and j, the coefficient of $x^n s^i t^j$ is nonzero and equal to

$$\frac{1}{n+1} \binom{n+1}{i+1, j, n-i-j} \binom{i-1}{i-j}$$

for $n \geq i+j$ and $i \geq j$ and is 0 otherwise. □

In equation (19), setting $s = t$ gives

$$h(x,t,t) = \frac{1 - x + xt - \sqrt{1 - 2xt - 2x + 2x^2t + x^2 - 3x^2t^2}}{2xt(1 - x + xt)}$$
$$= 1 + x + (1 + t^2)x^2 + (1 + 3t^2 + t^3)x^3 + (1 + 6t^2 + 4t^3 + 3t^4)x^4 + \cdots$$

This is counting Dyck paths by the sum of occurrences of UU and UDD (weighted t). Here the coefficients are sequence A124926 in the Online Encyclopedia of Integer Sequences [**18**].

In particular, we can find the generating function for Dyck paths with the same semilength and sum of occurrences of UU and UDD by setting $s = t$ and letting $i = n - j$ in equation (21). We obtain

$$\sum_{n,j} \frac{1}{n+1} \binom{n+1}{j} \binom{n-j-1}{n-2j} x^n t^n$$

Here the coefficients are sequence A005043 in the Online Encyclopedia of Integer Sequences [**18**]. They are called the Riordan numbers (or ring numbers). Let r_n be the n-th Riordan number defined by

$$r_n = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n+1} \binom{n+1}{j} \binom{n-j-1}{n-2j}$$

with $r_0 = 1$.

The Riordan numbers count Motzkin paths containing no flatsteps at ground level. Deutsch [**18**] gave the interpretation that the Riordan number is equal to the number of Dyck paths of semilength n with no ascents of length 1 (an ascent in a Dyck path is a maximal string of up steps).

It is known that

$$r_n + r_{n+1} = m_n \tag{22}$$

where m_n is the nth Motzkin number. We can give a combinatorial interpretation to equation (22).

We know that m_n counts UDU-free Dyck paths with semilength $n+1$ by the result in section 2.2. We can also see that a Dyck path with the same semilength and sum of occurrences of UU and UDD is equivalent to a Dyck path with no UDU that does not end in UD, because every U must be followed by U or DD.

Consider UDU-free Dyck paths with semilength $n+1$. We can separate them into two cases.

(1) Paths that end with UD: Removing the UD at the end gives Dyck n-paths with no UDU that do not end with UD. These are counted by r_n. It is impossible to get Dyck n-paths with no UDU that do end with UD, since we start from UDU-free Dyck paths with semilength $n+1$. They are not allowed to end with $UDUD$.

(2) Paths that don't end with UD: These are counted by r_{n+1}.

2.10. Occurrences of UUU and UDD

Now count Dyck paths according to the occurrences of two subwords, UUU weighted by s and UDD weighted by t. The semilength is weighted by x. We use the same approach as in the case in section 2.9. The generating function g with weights $1+s$ for UUU and $1+t$ for UDD satisfies

$$g = 1 + \sum_{i=0}^{\infty} u_i g^{i+1} {d_1}^i$$

where u_i represents reducing steps that go up by i, and d_1 represents reducing down steps that go down by 1. The generating function for reducing down steps, including only a single down step and a UDD is

$$d_1 = d + uddt.$$

Therefore, we can see that the clusters consisting of UUU and UDD are of the form U^j for $j = 3, 4, 5, \ldots$ or $U^k DD$ for $k = 1, 3, 4, 5, \ldots$. So, the generating function for reducing up steps, including a single up step and the clusters consist of UUU and UDD, is obtained by subtracting d_1 from the generating function of all possible reducing steps:

$$\begin{aligned} u + d + uddt + \frac{u^3 s}{1 - us - u^2 s}(1 + ddt) - d_1 &= u + \frac{u^3 s(1 + ddt)}{1 - us - u^2 s} \\ &= \frac{u - u^2 s + u^3 d^2 st}{1 - us - u^2 s}. \end{aligned}$$

So, let

$$\phi(z) = \sum_{i=0}^{\infty} u_i z^i = \frac{uz - u^2 z^2 s + u^3 d^2 zst}{1 - uzs - u^2 z^2 s}.$$

Then we have

$$g = 1 + g\phi(gd_1).$$

Replacing u by x, d by 1, and d_1 by $1 + xt$, we get

$$\phi(gd_1) = \phi((1 + xt)g) = \frac{x(1 + xt)g - x^2(1 + xt)^2 g^2 s + x^3(1 + xt)gst}{1 - x(1 + xt)gs - x^2(1 + xt)^2 g^2 s}.$$

So,

$$g = 1 + \left(\frac{x(1 + xt)g - x^2(1 + xt)^2 g^2 s + x^3(1 + xt)gst}{1 - x(1 + xt)gs - x^2(1 + xt)^2 g^2 s}\right) g.$$

Simplifying, we get a quadratic equation

$$x(1 + xt)(xs - s - 1)g^2 - (1 + xs + x^2 st)g + 1 = 0.$$

Solving for g, we get the generating function g with weights $1+s$ and $1+t$

$$g = \frac{1 + xs + x^2st - \sqrt{(1 + xs + x^2st)^2 - 4x(1 + xt)(xs - s - 1)}}{2x(1 + xt)(xs - s - 1)}.$$

Using the cluster method, we replace s by $s-1$ and t by $t-1$, and get the real generating function h with weights s and t

$$h(x,s,t) = \frac{\begin{array}{l} 1 - x + xs + x^2(s-1)(t-1) - \\ \qquad \sqrt{(1 - x + xs + x^2(s-1)(t-1))^2 - 4x(1 - x + xt)(xs - x - s)} \end{array}}{2x(1 - x + xt)(xs - x - s)}.$$

2.11. Occurrences of DU^{k}DU

Barnabei, Bonetti, and Silimbani [**1**, Proposition 7] showed that the two statistics, number of occurrences of DDD and number of occurrences of DU^kDU, where k is any positive integer, are equidistributed on Dyck n-paths.

We have already found the generating function for the number of occurrences of DDD in equation (18) in section 2.7. Now we use the cluster method to count Dyck paths according to the occurrences of DU^kDU weighted by t, where k is any positive integer. The semilength is weighted by x. This is also an example in which we count paths reduced to paths with steps down by 1, and up by any amount. We use the same decomposition as used in Figure 2.5.

The clusters consisting of DU^kDU are

$DU^{k_1}DU^{k_2}\cdots DU^{k_j}\cdots DU$ where k_j is any positive integer and $j = 1, 2, 3, \ldots$

So the generating function for reducing up steps, including a single up step and the clusters consisting of DU^kDU is

$$u+\frac{d\frac{u}{1-u}dut}{1-d\frac{u}{1-u}t}=u+\frac{u^2d^2t}{1-u-udt}.$$

So, let

$$\phi(z)=\sum_{i=0}^{\infty}u_iz^i=uz+\frac{u^2d^2t}{1-uz-udt}$$

where z is the weight for height. The generating function $g(x,t)$ with a weight $1+t$ for DU^kDU satisfies

$$g=1+\sum_{i=0}^{\infty}u_ig^{i+1}d^i$$

where u_i represents reducing steps that go up by i, and d represents a single down step. Then we have

$$g=1+g\phi(gd).$$

Replacing u by x, and d by 1, we get

$$\phi(gd)=\phi(g)=xg+\frac{x^2t}{1-xg-xt}.$$

So,

$$g=1+g\left(xg+\frac{x^2t}{1-xg-xt}\right).$$

Solving for g, we get the generating function g with weights $1+t$

$$g=\frac{1-xt-\sqrt{1-2xt+x^2t^2+4x^2t-4x}}{2x}. \tag{23}$$

However, there is a height restriction whereby the clusters consisting of DU^kDU cannot occur at height 0. By elevating g, we get a generating function $g_1(x,t)$ under the restriction which satisfies

$$g_1 = \frac{1}{1-ugd}. \tag{24}$$

Replacing u by x and d by 1 in equation (24), we substitute for g to get g_1. We get

$$g_1 = \frac{1}{1-xg} = \frac{2}{1+xt+\sqrt{1-2xt+x^2t^2+4x^2t-4x}}.$$

Using the cluster method, we replace t by $t-1$ and get the real generating function $h_1(x,t)$ for occurrences of DU^kDU

$$\begin{aligned} h_1(x,t) &= \frac{2}{1+xt-x+\sqrt{1-2xt+x^2t^2-2x+2x^2t-3x^2}} \\ &= \frac{2\left(1+xt-x-\sqrt{1-2xt+x^2t^2-2x+2x^2t-3x^2}\right)}{4xt-4x^2t+4x^2} \\ &= \frac{1+xt-x-\sqrt{1-2xt+x^2t^2-2x+2x^2t-3x^2}}{2x(t-xt+x)}. \end{aligned} \tag{25}$$

From equation (18) and (25), we can see that the generating functions for the number of occurrences of DDD and the number of occurrences of DU^kDU, where k is any positive integer, are the same.

2.12. Occurrences of UUU and UD

Now we want to count Dyck paths according to the occurrences of two subwords, UUU weighted by s and UD weighted by t. The semilength is weighted by x. This is also an example of type 3. In this problem, we need to count paths with steps that go down by 1 or up by any amount. We use the same approach as applied in section 2.9. We can see that the clusters consisting of UUU and UD are of the form U^j for $j = 3, 4, 5, \ldots$ or U^kD for $k = 1, 3, 4, 5, \ldots$ So, the generating function for reducing up steps, including a single up step and the clusters consist of UUU and UD, is

$$\begin{aligned} u + \frac{u^3 s}{1 - us - u^2 s} + udt + \frac{u^3 dst}{1 - us - u^2 s} &= (u + udt)\left(1 + \frac{u^2 s}{1 - us - u^2 s}\right) \\ &= \frac{(u + udt)(1 - us)}{1 - us - u^2 s}. \end{aligned}$$

So the generating function g for such path with weights $1 + s$ for UUU and $1 + t$ for UD satisfies

$$g = 1 + \sum_{i=0}^{\infty} u_i g^{i+1} d^i$$

where u_i represents reducing steps that go up by i, and d represents down steps that go down by 1. So, let

$$\phi(z) = \sum_{i=0}^{\infty} u_i z^i = \frac{(uz + udt)(1 - uzs)}{1 - uzs - u^2 z^2 s}. \tag{26}$$

Then we have

$$g = 1 + g\phi(gd).$$

Replacing u by x, d by 1, we get

$$g = 1 + g\frac{(xg + xt)(1 - xgs)}{1 - xgs - x^2 g^2 s}.$$

Simplifying, we get a quadratic equation

$$x(1 + s - xs - xst)g^2 - (1 + xs - xt)g + 1 = 0.$$

Solving for g, we get the generating function g with weights $1 + s$ and $1 + t$

$$\begin{aligned} g &= \frac{1 + xs - xt - \sqrt{(1 + xs - xt)^2 - 4x(1 + s - xs - xst)}}{2x(1 + s - xs - xst)} \\ &= \frac{1 + xs - xt - \sqrt{1 - 2xs - 2xt + x^2 s^2 + 2x^2 st + x^2 t^2 + 4x^2 s - 4x}}{2x(1 + s - xs - xst)}. \end{aligned} \tag{27}$$

We replace s by $s-1$, t by $t-1$ in equation (27) and get the real generating function h with weights s for UUU and t for UD

$$h(x,s,t) = \frac{1+xs-xt-\sqrt{1-2xs-2xt+x^2s^2+2x^2st-4x^2t+x^2t^2}}{2x(s+xt-xst)}.$$

In particular, letting $s=t$ gives

$$\begin{aligned} h(x,t,t) &= \frac{1-\sqrt{1-4xt-4x^2t+4x^2t^2}}{2x(t+xt-xt^2)} \\ &= 1+tx+(t+t^2)x^2+(4t^2+t^3)x^3+(2t^2+11t^3+t^4)x^4 \\ &\quad + (15t^3+26t^4+t^5)x^5+\cdots \end{aligned}$$

Here the coefficients are sequence A091156 with rows reversed in the Online Encyclopedia of Integer Sequences [**18**]. In section 2.1 we counted Dyck paths by UUD and got these coefficients in reverse order. We can see that the number of Dyck n-paths having exactly k occurrences of UUD is equal to the number of Dyck n-paths having a total of $n-k$ UUU and UD, since the sum of the numbers of UUD, UUU, and UD is always equal to the semilength n in a Dyck path.

2.13. Occurrences of UUU and DU

Now count Dyck paths according to the occurrences of two subwords, UUU weighted by s and DU weighted by t. This is an example of counting paths with steps that go down by 1 or up by any amount.

The clusters are almost the same as those in section 2.12. We can see that the clusters consisting of UUU and DU are of the form U^j for $j=3,4,5,\ldots$ or DU^k for $k=1,3,4,5,\ldots$. However, we cannot use the same decomposition as the one in section 2.12. It is not allowed to have a cluster of the form DU^k for $k=1,3,4,5,\ldots$ at height 0.

We discuss a more general path–counting problem as presented in section 2.9. We count paths reduced to paths with steps down by 1 or up by any amount. Let us consider the path with a step that goes up by i is weighted v_i if it starts on the x-axis and u_i if it starts at height > 0. Every nonempty such path with these weights can be factored as $U_i G_1 D G_2 D \cdots G_i D G'$, where G' is a path with these weights and each G_j is a path with no height restriction(U_i is only weighted by u_i). See Figure 2.7.

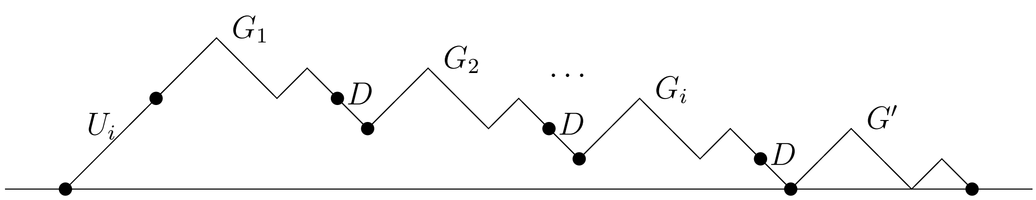


Figure 2.7. Decomposition for such path

Let g' be the generating function for these paths. Then

$$g' = 1 + \sum_{i=0}^{\infty} v_i (gd)^i g'.$$

Let

$$\psi(z) = \sum_{i=0}^{\infty} v_i z^i.$$

Then

$$g' = 1 + \psi(gd) g'.$$

So

$$g' = \frac{1}{1 - \psi(gd)}. \tag{28}$$

Now we apply these formulas to the problem of counting such paths by UUU and DU. We can see that the clusters consisting of UUU and DU are of the form U^j for $j = 3, 4, 5, \ldots$ or DU^k for $k = 1, 3, 4, 5, \ldots$ So applying the results of previous paragraph, we have

$$\phi(z) = \sum_{i=0}^{\infty} u_i z^i = \frac{(uz + dut)(1 - uzs)}{1 - uzs - u^2 z^2 s}$$

and the generating function g for each G_j which satisfies

$$g = 1 + g\phi(gd).$$

From equation (26), we can see that the generating function g is the same as the g in section 2.12 (UUU and UD). However, it is not allowed that DU^k occur at height 0. So $\psi(z)$ counts only clusters whose underlying is of the form U^j for $j = 3, 4, 5, \ldots$ as applied in section 2.7 with D replaced by U. So, the generating function for these reducing up steps is

$$u + \frac{u^3 s}{1 - us - u^2 s}.$$

Then

$$\begin{aligned}\psi(z) &= \sum_{i=0}^{\infty} v_i z^i \\ &= uz + \frac{u^3 z^3 s}{1 - uzs - u^2 z^2 s}.\end{aligned}$$

By equation (28) and replacing u by x and d by 1, we have

$$\begin{aligned}g' &= \frac{1}{1 - \psi(gd)} \\ &= \frac{1}{1 - \left(xg + \dfrac{x^3 g^3 s}{1 - xgs - x^2 g^2 s}\right)} \\ &= \frac{1 - xgs - x^2 g^2 s}{1 - xg - xgs}. \qquad (29)\end{aligned}$$

We substitute g in equation (27) for g in equation (29) and simplify to get g' :

$$g' = \frac{\begin{array}{l} 1 + xs + xt + 2xst - 2x^2st - 2x^2st^2 \\ \qquad - \sqrt{1 - 2xs - 2xt + x^2s^2 + 2x^2st + x^2t^2 + 4x^2s - 4x} \end{array}}{2x(1+t)(1+s-xs-xst)}.$$

We replace s by $s-1$ and t by $t-1$, and get the real generating function h with weights s for UUU and t for DU

$$\begin{aligned} h(x,s,t) &= \frac{\begin{array}{l} 1 - xs - xt - 2x^2t + 2x^2t^2 + 2xst + 2x^2st - 2x^2st^2 \\ \qquad - \sqrt{1 - 2xs - 2xt + x^2s^2 + 2x^2st - 4x^2t + x^2t^2} \end{array}}{2x(s+xt-xst)} \\ &= 1 + x + (1+t)x^2 + (3t+s+t^2)x^3 + (4ts+2t+6t^2+s^2+t^3)x^4 + \cdots \end{aligned}$$

Marilena Barnabei, Flavio Bonetti, and Matteo Silimbani [**2**] obtained the same generating function by a different approach.

CHAPTER 3

Counting Dyck Paths with Bounded Height

In the following examples, we apply the cluster method to count paths with bounded height by occurrences of subwords.

3.1. Occurrences of UD

We want to count Dyck paths of height at most k, by occurrences of peaks UD weighted by t. We can use the same approach as in section 2.3. We can replace each peak by a new flat step F. Then this problem is equivalent to the problem of counting modified Motzkin paths of height at most k with no flat step at height k. It is not allowed to have a flat step at height k in these modified Motzkin paths, otherwise there will be a peak reaching height $k+1$ in the original Dyck paths.

Let $g_k(x,t)$ be the generating function for such Motzkin paths with flat steps weighted t which correspond to peaks weighted $1+t$. We can use the same decomposition as in Figure 2.2 even with bounded heights. Then g_k satisfies

$$g_{k+1} = 1 + fg_{k+1} + ug_k dg_{k+1} \quad \text{and} \quad g_0 = 1$$

where f represents a flat step F, u represents a single up step U, and d represents a single down step D. This can be written as

$$g_{k+1} = \frac{1}{1 - f - ug_k d}. \tag{30}$$

Then g_k can be written as a continued fraction

$$g_k = \cfrac{1}{1-f-u\cfrac{1}{1-f-u\cfrac{1}{\ddots\cfrac{}{1-f-u\cfrac{1}{1-f-ug_0d}d}d}d}d}$$

We define p_k by the linear recurrence equation. We will show that $g_k = p_k/p_{k+1}$.

$$p_{k+2} = (1-f)p_{k+1} - udp_k \quad \text{with} \quad p_0 = 1, p_1 = 1 \tag{31}$$

Let

$$G_k = \frac{p_k}{p_{k+1}}.$$

Dividing both sides of equation (31) by p_{k+1}, we get

$$\frac{1}{G_{k+1}} = 1 - f - udG_k \quad \text{with} \quad G_0 = 1.$$

Comparing this with equation (30) shows that $G_k = g_k$. Then let

$$P(z) = \sum_{k=0}^{\infty} p_k z^k.$$

We can solve equation (31) to get

$$P(z) = \frac{1 + z - (1-f)z}{1 - (1-f)z + udz^2.} \tag{32}$$

We replace f by xt, u by x, and d by 1 to get

$$P(z) = \frac{1+z-(1-xt)z}{1-(1-xt)z+xz^2}.$$

Using the cluster method, we replace t by $t-1$ to get

$$\begin{aligned}
\hat{P}(z) &= \frac{1+z-(1-x(t-1))z}{1-(1-x(t-1))z+xz^2} \\
&= 1+z+(1-xt)z^2+(1-2xt-x^2t+x^2t^2)z^3+\cdots \\
&= \sum_{k=0}^{\infty} \hat{p}_k z^k.
\end{aligned}$$

Then the real generating function h_k for Dyck paths with heights at most k and weight t for UD is given by

$$h_k(x,t) = \frac{\hat{p}_k}{\hat{p}_{k+1}}.$$

The formula for counting Dyck paths of bounded height as a quotient of these polynomials is well known.

The first few values for $\hat{p}_k$ are

$$\hat{p}_0 = 1$$

$$\hat{p}_1 = 1$$

$$\hat{p}_2 = 1-xt$$

$$\hat{p}_3 = 1-2xt-x^2t+x^2t^2$$

$$\hat{p}_4 = 1-3xt-2x^2t+3x^2t^2-x^3t+2x^3t^2-x^3t^3$$

$$\hat{p}_5 = 1-4xt-3x^2t+6x^2t^2-2x^3t+6x^3t^2-4x^3t^3-x^4t+3x^4t^2-3x^4t^3+x^4t^4.$$

We can find an explicit formula for $\hat{p}_k$. We start from subtracting 1 from $\hat{P}(z)$, then we get

$$\begin{aligned}
\hat{P}(z) - 1 &= \frac{1 - xz + xtz}{1 - z - xz + xtz + xz^2} - 1 \\
&= \frac{z - xz^2}{1 - z - xz + xz^2 + xtz} \\
&= \frac{z(1 - xz)}{(1 - z)(1 - xz) + xtz} \\
&= \frac{z}{(1 - z)} \cdot \frac{1}{1 + \dfrac{xtz}{(1 - z)(1 - xz)}} \\
&= \frac{z}{1 - z} \sum_{i=0}^{\infty} (-1)^i \frac{(xtz)^i}{(1 - z)^i (1 - xz)^i} \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{x^i t^i z^{i+1}}{(1 - z)^{i+1} (1 - xz)^i} \\
&= \sum_{i,l,m} (-1)^i x^i t^i z^{i+1} \binom{i + l}{l} z^l \binom{i + m - 1}{m} (xz)^m \\
&= \sum_{i,l,m} (-1)^i \binom{i + l}{l} \binom{i + m - 1}{m} x^{i+m} t^i z^{i+1+l+m}.
\end{aligned}$$

Replacing the variables, m by $n - i$ and l by $k - n - 1$, we get

$$\hat{P}(z) - 1 = \sum_{n,i,k} (-1)^i \binom{i + k - n - 1}{i} \binom{n - 1}{n - i} x^n t^i z^k.$$

So, for $k \geq 1$, we have

$$\hat{p}_k = \sum_{n=0}^{k-1} \sum_{i=0}^{n} (-1)^i \binom{i + k - n - 1}{i} \binom{n - 1}{n - i} x^n t^i.$$

Note that setting $t = 1$ gives

$$\begin{aligned}
\hat{p}_k &= \sum_{n=0}^{k-1}\sum_{i=0}^{n}(-1)^i\binom{i+k-n-1}{i}\binom{n-1}{n-i}x^n \\
&= \sum_{n=0}^{k-1}\sum_{i=0}^{n}\binom{n-k}{i}\binom{n-1}{n-i}x^n \\
&= \sum_{n=0}^{k-1}\binom{2n-k-1}{n}x^n \\
&= \sum_{n=0}^{k-1}(-1)^n\binom{-2n+k+1+n-1}{n}x^n \\
&= \sum_{n=0}^{k-1}(-1)^n\binom{k-n}{n}x^n \\
&= \sum_{n=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^n\binom{k-n}{n}x^n.
\end{aligned}$$

Let us look more closely at the case of Dyck paths of height at most 2. Here we have

$$\begin{aligned}
h_2(x,t) &= \frac{\hat{p}_2}{\hat{p}_3} \\
&= \frac{1-xt}{1-2xt-x^2t+x^2t^2} \\
&= \frac{1}{1-xt}\cdot\frac{1}{1-\dfrac{x^2t}{(1-xt)^2}} \\
&= \sum_{n=0}^{\infty}\frac{(x^2t)^n}{(1-xt)^{2n+1}} \\
&= \sum_{n=0}^{\infty}x^{2n}t^n\sum_{i=0}^{\infty}\binom{2n+i}{i}x^it^i \\
&= \sum_{n,i}\binom{2n+i}{i}x^{2n+i}t^{n+i}.
\end{aligned}$$

Replacing the variables l by $2n+i$ and m by $n+i$, we get

$$h_2(x,t) = \sum_{l,m} \binom{l}{2l-2m} x^l t^m. \tag{33}$$

We can give a combinatorial interpretation for equation (33). We can start from a Dyck path with semilength l and exactly m peaks and height at most 2. We add an extra down step in front of it and an extra up step after it. The modified path can be decomposed as

$$(DU)^{i_1}(UD)^{i_2}(DU)^{i_3}(UD)^{i_4}\cdots(DU)^{i_{2k+1}}$$

where each i_s is a positive integer and $i_1+i_2+\cdots+i_{2k+1} = l+1$. Conversely, any path with such a decomposition is a modified path from a Dyck path with semilength l and height at most 2. See Figure 3.1. We call the components $(DU)^{i_{2j+1}}$ odd components and components $(UD)^{i_{2j}}$ even components. We can think of counting these paths as counting compositions of $l+1$ with $2k+1$ parts. We know that the number of such paths is the the number of compositions of $l+1$ with $2k+1$ parts, which is $\binom{l}{2k}$.

For example, the corresponding composition for the path in Figure 3.1 is

$$9 = 3+2+2+1+1.$$

We want to find the connection between k, l and the number m of peaks. An odd component $(DU)^i$ contributes i to the semilength and $i-1$ to the number of peaks. On the other hand, an even component $(UD)^j$ contributes j to the semilength and also j to the number of peaks. So the difference between the sum $i_1+i_2+\cdots+i_{2k+1}$ and the number of peaks is $k+1$, the number of odd components. Therefore, we have $l+1-m = k+1$. So $k = l-m$ and thus, the number of Dyck path with semilength

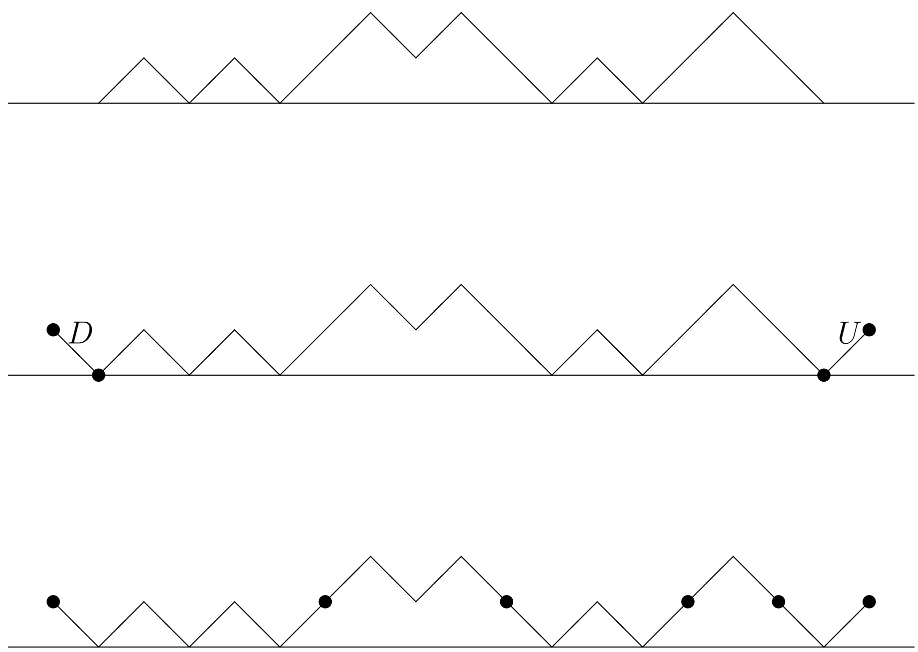


FIGURE 3.1. Decomposition for modified paths

l and m peaks and height at most 2 is

$$\binom{l}{2k} = \binom{l}{2l-2m}.$$

3.2. Occurrences of UDU

Count Dyck paths with bounded height by occurrences of UDU weighted by t. We can use the same approach as in section 2.2. This problem is equivalent to the problem of counting modified Dyck paths of height at most k.

Let $g_k(x,t)$ be the generating function for such paths with bounded height k. The clusters are of the form $U(DU)^i$ for $i = 1, 2, 3, \ldots$

So, the cluster generating function is

$$udut + ududut^2 + udududut^3 + \cdots = \frac{u^2dt}{1-udt}.$$

Since these clusters reduce to up steps, we can set $u_1 = u + \frac{u^2dt}{1-udt} = \frac{u}{1-udt}$ to get the generating function $g_k(x,t)$ with a weight $1+t$ which satisfies

$$g_{k+1} = 1 + u_1 g_k d g_{k+1} \quad \text{and} \quad g_0 = 1.$$

This can be written as

$$g_{k+1} = \frac{1}{1-u_1 g_k d}. \tag{34}$$

Then g_k can be written as a continued fraction

$$g_k = \cfrac{1}{1-u_1\cfrac{1}{1-u_1\cfrac{1}{1-u_1\cfrac{\ddots}{1-u_1\cfrac{1}{1-u_1g_0d}d}d}d}d}$$

We want to find g_k and we can use a similar approach as in section 3.1. Define p_k by the linear recurrence equation as equation (31). So setting $f=0$ and replacing u with u_1 in equation (32), we get $g_k = p_k/p_{k+1}$ where

$$P(z) := \sum p_k z^k = \frac{1}{1-z+\frac{x}{1-xt}z^2}.$$

This continued fraction is the same as the previous one in section 3.1 with $f = 0$ and u replace with u_1. So

$$g_k = \frac{p_k}{p_{k+1}}$$

and

$$P(z) = \sum_{k=0}^{\infty} p_k z^k.$$

Then we can solve a linear recurrence equation by substituting 0 for f and u_1 for u in equation (32) to get

$$P(z) = \frac{1}{1 - z + u_1 d z^2}.$$

We replace u_1 by $\dfrac{x}{1-xt}$ and d by 1 to get

$$P(z) = \frac{1}{1 - z + \dfrac{x}{1-xt} z^2}.$$

Using the cluster method, we replace t by $t-1$ in $P(z)$ to get

$$\begin{aligned}\hat{P}(z) &= \sum_{k=0}^{\infty} \hat{p}_k z^k \\ &= \frac{1}{1 - z + \dfrac{x}{1-xt+x} z^2} \\ &= 1 + z + \frac{1-xt}{1-xt+x} z^2 + \frac{1-xt-x}{1-xt+x} z^3 + \cdots\end{aligned}$$

Then the real generating function h_k for Dyck paths with heights at most k and weight t for UDU is given by

$$h_k(x,t) = \frac{\hat{p}_k}{\hat{p}_{k+1}}$$

where $\hat{p}_k$ is given by the coefficient of $\hat{P}(z)$.

The first few values for $\hat{p}_k$ are

$$\hat{p}_0 = 1$$

$$\hat{p}_1 = 1$$

$$\hat{p}_2 = \frac{1-xt}{1-xt+x} = 1 - \frac{x}{1-xt+x}$$

$$\hat{p}_3 = \frac{1-xt-x}{1-xt+x} = 1 - 2\left(\frac{x}{1-xt+x}\right)$$

$$\hat{p}_4 = \frac{1-2xt+x^2t^2-x+x^2t-x^2}{(1-xt+x)^2} = 1 - 3\left(\frac{x}{1-xt+x}\right) + \left(\frac{x}{1-xt+x}\right)^2$$

$$\hat{p}_5 = \frac{1-2xt+x^2t^2-2x+2x^2t}{(1-xt+x)^2} = 1 - 4\left(\frac{x}{1-xt+x}\right) + 3\left(\frac{x}{1-xt+x}\right)^2$$

We can find an explicit formula for $\hat{p}_k$:

$$\begin{aligned}
\hat{P}(z) &= \frac{1}{1-z+\dfrac{x}{1-xt+x}z^2} \\
&= \sum_{l=0}^{\infty}\left(\frac{-xz^2}{1-xt+x}+z\right)^l \\
&= \sum_{l,m}\binom{m}{l}(-1)^l z^{2l}\left(\frac{x}{1-xt+x}\right)^l z^{m-l} \\
&= \sum_{l,m}(-1)^l\binom{m}{l}\left(\frac{x}{1-xt+x}\right)^l z^{m+l}.
\end{aligned}$$

Replacing the variables, m by $k-l$, we get

$$\hat{P}(z) = \sum_{l,k}(-1)^l\binom{k-l}{l}\left(\frac{x}{1-xt+x}\right)^l z^k.$$

For the coefficients of $\hat{P}(z)$, if $l > \frac{k}{2}$, we have

$$\binom{k-l}{l} = 0.$$

So, we have

$$\hat{p}_k = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^l \binom{k-l}{l} \left(\frac{x}{1-xt+x} \right)^l .$$

In particular, for $k = 2$, we have

$$\begin{aligned} h_2(x,t) &= \frac{\hat{p}_2}{\hat{p}_3} \\ &= \frac{1-xt}{1-xt-x} \\ &= 1 + \frac{x}{1-x(t+1)} \\ &= 1 + \sum_{n=1}^{\infty} x^n (t+1)^{n-1} \\ &= 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \binom{n-1}{m} x^n t^m. \end{aligned} \tag{35}$$

We can give a combinatorial interpretation for equation (35). We can start from a Dyck path with semilength n and exactly m occurrences of UDU and height at most 2. We can use the same decomposition as Figure 3.1. We add an extra down step in front of the path and an extra up step after it. The modified path can be decomposed as

$$(DU)^{i_1}(UD)^{i_2}(DU)^{i_3}(UD)^{i_4} \cdots (DU)^{i_{2k+1}}$$

where each i_s is a positive integer and $i_1 + i_2 + \cdots + i_{2k+1} = n + 1$.

Conversely, any path with such a decomposition is a modified path from a Dyck path with semilength n and height at most 2. We can think of counting these paths as counting compositions of $n+1$ with $2k+1$ parts. We know that the number of such paths is the the number of compositions of $n+1$ with $2k+1$ parts, which is $\binom{n}{2k}$.

Then, we want to find the connection between k, n and the number m of occurrences of UDU. An even component $(UD)^j$ contributes j to the semilength and $j-1$ to the occurrences of UDU.

On the other hand, an odd component $(DU)^i$ contributes i to the semilength and $i-1$ to the occurrences of UDU except the last odd component. Therefore, we can separate into two cases.

If $i_{2k+1} = 1$, the last odd component $(DU)^{i_{2k+1}}$ contributes i_{2k+1} to the semilength and $i_{2k+1} - 1$ to the occurrences of UDU. So the difference between the sum $i_1 + i_2 + \cdots + i_{2k+1}$ and the number of occurrences of UDU is $2k+1$. Therefore, we have $n+1-m = 2k+1$. The compositions we are counting are compositions of $n+1$ with $2k+1$ parts in which the last part is 1. Deleting the last part gives a composition of n with $2k$ parts, and here are $\binom{n-1}{2k-1}$ of them.

So $2k = n - m$ and

$$\binom{n-1}{2k-1} = \binom{n-1}{n-m-1} = \binom{n-1}{m}.$$

Note that this only applies when $n-m$ is even.

If $i_{2k+1} \geq 2$, the last odd component $(DU)^{i_{2k+1}}$ contributes i_{2k+1} to the semilength but $i_{2k+1} - 2$ to the occurrences of UDU, since the last UDU in the modified path is not in the original path. So the difference between the sum $i_1 + i_2 + \cdots + i_{2k+1}$ and the number of UDU is $2k+2$. Therefore we have $n+1-m = 2k+2$. The compositions we are counting are compositions of $n+1$ with $2k+1$ parts in which the last part is at least 2. Subtracting 1 from the last part gives a composition of n with $2k+1$ parts, and here are $\binom{n-1}{2k}$ of them.

So $2k = n - m - 1$ and

$$\binom{n-1}{2k} = \binom{n-1}{n-m-1} = \binom{n-1}{m}.$$

Note that this only applies when $n - m$ is odd.

Thus, combining the results of these two cases, we get that the number of Dyck path with semilength n and m occurrences of UDU and height at most 2 is

$$\binom{n-1}{m}.$$

CHAPTER 4

Applications to r-Dyck paths

In the following examples, we apply the cluster method to count paths with up steps U that go up by 1 and down steps D that go down by an arbitrary number, r. We define an r-*Dyck path* to be a path with up steps U that go up by 1 and down steps D that go down by r.

4.1. Occurrences of UD

We count r-Dyck paths by occurrences of UD (weighted t). In this case, the only cluster is UD. We may use the same approach as section 2.6. We consider paths with steps that go up by 1, U, down by r, D_r, and down by $r-1$, D_{r-1}. Here a U in such a path corresponds to a U in a r-Dyck path, weighted by x, a D_r in such a path corresponds to a D in a r-Dyck path, weighted by 1, and a D_{r-1} in such a path corresponds to a UD in a r-Dyck path, weighted by t. So, by equation (15) that we obtained in section 2.6, the generating function for such paths g satisfies

$$g = 1 + u^{r-1}g^r d_{r-1} + u^r g^{r+1} d_r.$$

Replacing u by x, d_r by 1, and d_{r-1} by xt, we get

$$\begin{aligned} g &= 1 + x^{r-1}g^r xt + x^r g^{r+1} \\ &= 1 + x^r g^r t + x^r g^{r+1}. \end{aligned}$$

Using the cluster method, we replace t by $t-1$ and get the real generating function $h(x,t)$ which satisfies

$$h = 1 + x^r h^r (t-1) + x^r h^{r+1}. \tag{36}$$

Let

$$h = 1 + tH.$$

We substitute this in equation (36) to get

$$\begin{aligned} 1 + tH &= 1 + x^r (1+tH)^r (t-1) + x^r (1+tH)^{r+1} \\ tH &= x^r (1+tH)^r (t - 1 + 1 + tH) \\ H &= x^r (1+tH)^r (1+H). \end{aligned}$$

Set

$$x^r = z.$$

By Lagrange inversion [**20**, Ch. 5, Page. 38], we have

$$\begin{aligned} [z^n]H^k &= \frac{k}{n}[y^{n-k}]((1+ty)^r(1+y))^n \\ &= \frac{k}{n}[y^{n-k}](1+ty)^{nr}(1+y)^n \\ &= \frac{k}{n}[y^{n-k}]\sum_{i,j}\binom{nr}{i}t^i y^i \binom{n}{n-j} y^j \\ &= \frac{k}{n}[y^{n-k}]\sum_{\substack{i,j \\ i+j=n-k}}\binom{nr}{i}\binom{n}{n-j}t^i y^{i+j} \\ &= \frac{k}{n}\sum_{i}\binom{nr}{i}\binom{n}{i+k}t^i. \end{aligned}$$

So, we get

$$H^k = \sum_{n,i} \frac{k}{n}\binom{nr}{i}\binom{n}{i+k} t^i z^n$$
$$= \sum_{n,i} \frac{k}{n}\binom{nr}{i}\binom{n}{i+k} x^{nr} t^i.$$

For $k = 1$, we have

$$H(x,t) = \sum_{n,i} \frac{1}{n}\binom{nr}{i}\binom{n}{i+1} x^{nr} t^i.$$

Then we have

$$h(x,t) = 1 + tH$$
$$= 1 + \sum_{n\geq 1}\sum_{i\geq 0} \frac{1}{n}\binom{nr}{i}\binom{n}{i+1} x^{nr} t^{i+1}.$$

For $r = 2$, these numbers are A108767 or A120986 in the Online Encyclopedia of Integer Sequences [**18**]

4.2. Occurrences of UU and UDD

Count r-Dyck paths by occurrences of UU (weighted s) and UDD (weighted t), since we got an interesting result for $r = 1$ in section 2.9. We may use the same approach as section 2.9 to find the generating function. However, we can find a similar equation to equation (20) for the generating function by using Deutsch's decomposition. See Figure 4.1.

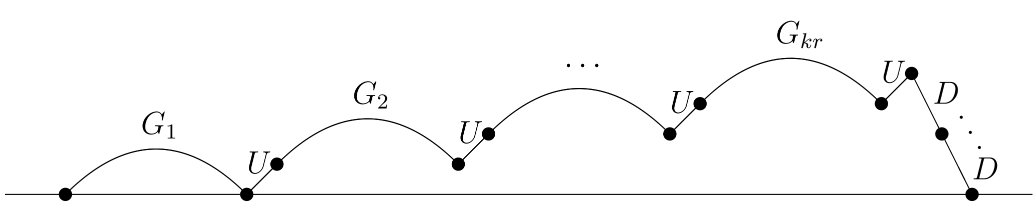


FIGURE 4.1. Deutsch's decomposition for r-Dyck paths

Let G be a nonempty r-Dyck path. Suppose there are exactly k consecutive down steps after the last up step in G. Then G can be factored uniquely as $G_1UG_2U\cdots G_{kr}UD^k$, where each G_j is a r-Dyck path. Let h be the generating function for counting all r-Dyck paths. Then this decomposition shows that h satisfies

$$h = 1 + \sum_{k=1}^{\infty}(hu)^{kr}d^k.$$

Now we assign weights s for occurrences of UDD and t for occurrences of UDD. If $k = 1$, the occurrences of UU in G are the same as those in G_1 to G_r, but for the occurrences of UU in G, the U between G_j and G_{j+1} is followed by another U for $1 \leq j \leq r$, giving additional $r-1$ occurrences of UU. If $k \geq 2$, then every UU or UDD in each G_j occurs in G. Moreover, for $1 \leq j \leq kr-1$, the U between G_j and G_{j+1} is followed by another U, giving additional $kr-1$ occurrences of UU. There is also one extra UDD from the last up step followed by at least two down steps. So the generating function h with weights s for UU and t for UDD satisfies

$$h = 1 + s^{r-1}(hu)^r d + \sum_{k=2}^{\infty} s^{kr-1}t(hu)^{kr}d^k.$$

Replacing u by x and d by 1, we get

$$h = 1 + s^{r-1}(xh)^r + \frac{s^{2r-1}t(xh)^{2r}}{1 - s^r(xh)^r}. \tag{37}$$

Let

$$H = sxh.$$

We multiply both sides of equation (37) by sx and substitute H for sxh

$$H = x\left(s + H^r + t\frac{H^{2r}}{1-H^r}\right).$$

By Lagrange inversion [**20**, Ch. 5, Page. 38], we have

$$\begin{aligned}
[x^n]H^k &= \frac{k}{n}[y^{n-k}]\left(s + y^r + t\frac{y^{2r}}{1-y^r}\right)^n \\
&= \frac{k}{n}[y^{n-k}]\sum_{\substack{i,j,l\\ n=i+j+l}}\binom{n}{i,j,l}s^i\left(t\frac{y^{2r}}{1-y^r}\right)^j(y^r)^l \\
&= \frac{k}{n}[y^{n-k}]\sum_{\substack{i,j,l,m\\ n=i+j+l}}\binom{n}{i,j,l}s^it^jy^{lr}y^{2jr}\binom{j+m-1}{m}(y^r)^m \\
&= \frac{k}{n}[y^{n-k}]\sum_{\substack{i,j,l,m\\ n=i+j+l}}\binom{n}{i,j,l}\binom{j+m-1}{m}s^it^jy^{lr+2jr+mr}.
\end{aligned}$$

For this to be nonzero, $n-k$ must be a multiple of r. Let $n-k=pr$. Thus

$$\begin{aligned}
[x^n]H^k &= \frac{k}{pr+k}[y^{pr}]\sum_{\substack{i,j,l,m\\ pr+k=i+j+l}}\binom{pr+k}{i,j,l}\binom{j+m-1}{m}s^it^jy^{lr+2jr+mr} \\
&= \frac{k}{pr+k}[y^{p}]\sum_{\substack{i,j,l,m\\ pr+k=i+j+l}}\binom{pr+k}{i,j,l}\binom{j+m-1}{m}s^it^jy^{l+2j+m} \\
&= \frac{k}{pr+k}\sum_{\substack{i,j,l,m\\ pr+k=i+j+l\\ p=l+2j+m}}\binom{pr+k}{i,j,l}\binom{j+m-1}{m}s^it^j \\
&= \frac{k}{pr+k}\sum_{\substack{i,j,l\\ pr+k=i+j+l}}\binom{pr+k}{i,j,l}\binom{p-j-l-1}{p-l-2j}s^it^j \\
&= \frac{k}{pr+k}\sum_{i,j}\binom{pr+k}{i,j,pr+k-i-j}\binom{p-pr-k+i-1}{p-pr-k+i-j}s^it^j.
\end{aligned}$$

So

$$H^k = \sum_{\substack{i,j,p\\ n=pr+k}}\frac{k}{pr+k}\binom{pr+k}{i,j,pr+k-i-j}\binom{p-pr-k+i-1}{p-pr-k+i-j}x^ns^it^j.$$

For k=1, we have

$$H = \sum_{\substack{i,j,p\\ n=pr+1}} \frac{1}{pr+1}\binom{pr+1}{i,j,pr+1-i-j}\binom{p-pr+i-2}{p-pr+i-j-1}x^{pr+1}s^i t^j.$$

Therefore, the generating function for r-Dyck paths counting by occurrences of UU (weighted s) and UDD (weighted t) is

$$\begin{aligned} h(x,s,t) &= \sum_{i,j,p} \frac{1}{pr+1}\binom{pr+1}{i,j,pr+1-i-j}\binom{p-pr+i-2}{p-pr+i-j-1}x^{pr}s^{i-1}t^j \\ &= \sum_{i,j,p} \frac{1}{pr+1}\binom{pr+1}{i+1,j,pr-i-j}\binom{p-pr+i-1}{p-pr+i-j}x^{pr}s^i t^j \qquad (38) \end{aligned}$$

Here, for all nonnegative integers p, i, and j, the coefficient of $x^{pr}s^it^j$ is nonzero and equal to

$$\frac{1}{pr+1}\binom{pr+1}{i+1,j,pr-i-j}\binom{p-pr+i-1}{p-pr+i-j}$$

for $pr \geq i+j$ and $i \geq j$ and is 0 otherwise.

In particular for $r = 1$, equation (38) reduces to equation (21) which counts for Dyck paths by occurrences of UU and UDD.

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