THE COMBINATORICS OF FUNCTIONAL COMPOSITION AND INVERSION

A Dissertation

Presented to

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Professor Ira Gessel, Advisor

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by

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ABSTRACT

The Combinatorics of Functional Composition and Inversion

A Dissertation Presented to the Faculty of the
Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Susan Field Parker

This thesis is a generalization of a result of Carlitz, Scoville and Vaughan on sequence enumeration [**CSV**]: Let $\mathcal{A}$ be an alphabet and let L be a subset of $\mathcal{A}\times\mathcal{A}$; the elements of L and its complement $\bar{L}$ are called *links*. Then if $\sum_{n=0}^{\infty} f_n x^n$ is the generating function for sequences $a_1 a_2 \cdots a_n$ with a_i in $\mathcal{A}$ all of whose links are in L (that is, with each $a_i a_{i+1}$ in L), then $\left(\sum_{n=0}^{\infty}(-1)^n f_n x^n\right)^{-1}$ is the generating function for sequences $a_1 a_2 \cdots a_n$ all of whose links are in $\bar{L}$. In this work, the objects of study are ordered trees rather than sequences and the reciprocal of a generating function is replaced by the compositional inverse. We label the internal vertices of an ordered tree with elements of some alphabet $\mathcal{A}$ and its leaves with some indeterminate x. In this setting, a set L of links is defined to be a set of ordered triples (α,β,l); (α,β,l) is a link of a tree T if there is a vertex v in T such that v is labeled with α and the lth child (counted from left to right) of v is labeled with β. The basic result is that the generating function for trees all of whose links are in some set L is the compositional inverse (as a power series in x) of the generating function, with alternating signs, for trees all of whose links are in $\bar{L}$.

The primary application of this inversion theorem involves the study, for $n \geq 0$, of the nth iteration polynomial $p_n(k)$ of certain power series $f(x)$ which arise in counting trees with restricted links. When $p_n(k)$ is a polynomial in k of degree at most n, there is a polynomial $A_n(t)$ of degree at most n defined by

$$\sum_{k=0}^{\infty} p_n(k) t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

We find a combinatorial interpretation of the coefficients of $A_n(t)$ for the formal power series $f(x) = x + x^m$ and $f(x) = \dfrac{x}{1-x^{m-1}}$, where $m \geq 2$. In the latter case, when $m = 2$, $p_n(k) = k^n$ and $A_n(t)$ is the classical Eulerian polynomial, whose coefficients count permutations of $[n]$ according to descents (occurrences of $\pi(i) > \pi(i+1)$).

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CHAPTER 0

INTRODUCTION

0.1 Background

The theory of permutation and sequence enumeration plays an important role in the study of enumerative combinatorics. An important contribution to this theory was made by Carlitz, Scoville and Vaughan in [**CSV**] in 1976: Let $\mathcal{A}$ be an alphabet and let L be a subset of $\mathcal{A}\times\mathcal{A}$. Let $\bar{L}$ be the complement of L in $\mathcal{A}\times\mathcal{A}$; the elements of L and $\bar{L}$ are called *links*. If $\sum_{n=0}^{\infty} f_n x^n$ is the generating function for sequences $a_1 a_2 \cdots a_n$ with a_i in $\mathcal{A}$ all of whose links are in L (that is, with each $a_i a_{i+1}$ in L), then $\left(\sum_{n=0}^{\infty}(-1)^n f_n x^n\right)^{-1}$ is the generating function for sequences $a_1 a_2 \cdots a_n$ all of whose links are in $\bar{L}$. In his 1977 Ph.D. thesis, Gessel [**G**] studied the theorem of Carlitz, Scoville and Vaughan at length, and showed that much of the theory of permutation and sequence enumeration can be derived from it.

This thesis is a generalization of the theorem of Carlitz, Scoville and Vaughan, where the objects of study are ordered trees rather than sequences and where the reciprocal of a generating function is replaced by the compositional inverse. The result is an inversion theorem which reduces to the theorem of Carlitz, Scoville and Vaughan when the trees are unary.

After proving the inversion theorem, this work goes on to explore some of its applications. The primary one involves a problem which has historically been of considerable interest to combinatorialists. It is well-known (see, e.g., [**S2**], p. 204) that if $p_n(k)$ is a polynomial in k of degree at most n, then there is a polynomial $A_n(t)$ of degree at most n defined by

$$\sum_{k=0}^{\infty} p_n(k) t^k = \frac{A_n(t)}{(1-t)^{n+1}}.$$

The classical example of this occurs when $p_n(k) = k^n$ and $A_n(t)$ is the Eulerian polynomial, whose coefficients count permutations of $[n]$ according to descents (occurrences of $\pi(i) > \pi(i+1)$). The question often arises whether, given some polynomial $p_n(k)$, one can find an analogous combinatorial interpretation of the coefficients of the polynomial $A_n(t)$. We address this question when $p_n(k)$ is an "iteration polynomial", that is, when $p_n(k)$ is the coefficient of x^n in the kth iterate of some formal power series $f(x)$.

0.2 Summary of results

In Chapter 1, we give a generalization of the theorem of Carlitz, Scoville and Vaughan: Let $\mathcal{A}$ be an alphabet and to each α in $\mathcal{A}$ associate a positive integer $\delta(\alpha)$ called the *degree of* α. Let T be an ordered tree with vertex set V, and let X be an indeterminate not in $\mathcal{A}$. We attach letters to the vertices of T by means of a map $\lambda : V \to \mathcal{A}$ such that

(1) the degree of $\lambda(v)$ equals the out-degree of v, for each internal vertex v of T,
(2) $\lambda(v) = X$ if and only if v is a leaf of T.

We say that λ *letters* the vertices of T. Each tree T in some set $\mathbf{T}$ of ordered trees is then weighted by x^k times the product of the letters attached to its internal vertices, where k is the number of leaves of T. In this setting, a set L of links is defined to be a set of ordered triples (α, β, l), where $1 \le l \le \delta(\alpha)$; (α, β, l) is a link of a tree T if there is a vertex v in T such that v is lettered with α and the lth child of v is lettered with β.

The fundamental result, called the Inversion Theorem, is as follows. Let $\mathbf{T}$ be a set of ordered trees. Then the generating function for trees in $\mathbf{T}$ all of whose links are in a given set L is the compositional inverse (as a power series in x) of the generating function, with alternating signs, for trees in $\mathbf{T}$ all of whose links are in the complement of L.

In Chapter 2, we take an alphabet $\mathcal{A} = \{\, \alpha_1, \alpha_2, \ldots, \}$ such that $\delta(\alpha_i) = m$ for each i, and we put a linear ordering on $\mathcal{A}$ by $\alpha_i < \alpha_j$ if and only if $i < j$, for each i, j in $\mathbf{N}$. For each $k \ge 0$, we let $\mathbf{L}_k$ be the set of all links formed from the set $\{\, \alpha_1, \alpha_2, \ldots, \alpha_k \,\}$. We consider several examples of a subset L_k of $\mathbf{L}_k$ with the property that the generating function for trees with links restricted to the set L_k is the kth iterate of some formal power series $f(x)$. Now if

$$f(x) = x + \sum_{i=1}^{\infty} c_i x^{(m-1)i+1}$$

is a formal power series, where $r \ge 1$, then

$$f^{<k>}(x) = \sum_{n=0}^{\infty} p_n(k) x^{(m-1)n+1},$$

where $p_n(k)$ is a polynomial in k of degree at most n. Since $p_n(k)$ is a polynomial in k of degree at most n, we can define a polynomial $P_n(t)$ in t of degree at most n by

$$\sum_{k=0}^{\infty} p_n(k) t^k = \frac{P_n(t)}{(1-t)^{n+1}},$$

and we can ask if there is a combinatorial interpretation for the coefficient $P_{n,j}$ of t^j in $P_n(t)$.

For the formal power series $f(x) = x + x^m$ and $f(x) = \dfrac{x}{1 - x^{m-1}}$, where $m \ge 2$, we find such an interpretation as follows. We start with a set Ω_n of positive integers of length n, and if $\gamma = c_1 c_1 \cdots c_n$ belongs to Ω_n we define the *spaces* of γ to be the integers 0, 1, ..., n. We think of space i as lying between c_i and c_{i+1}, for $1 \leq i \leq n-1$, with space 0 lying to the left of c_1 and space n lying to the right of c_n. We call the spaces 1, 2, ..., n the *proper spaces* of γ. We then define a map $s : \Omega \to 2^{\mathbf{N}}$ to be a function which attaches to each sequence γ in Ω_n a subset of the set of proper spaces of γ which includes n; we call $s(\gamma)$ the *s-descent set of* γ and call an element of $s(\gamma)$ an *s-descent of* γ. Moreover, we define a *barred sequence on* γ to be a sequence of positive integers and bars formed from γ by inserting bars in some of the spaces of γ. Let $b_n(k)$ be the number of barred sequences on elements of Ω_n with at least one bar in each s-descent and with k bars. We show that $p_n(k) = b_n(k)$, and this gives us a combinatorial interpretation for the coefficient $P_{n,j}$ of t^j in $P_n(t)$: $P_{n,j}$ is the number of sequences in Ω_n with exactly j s-descents.

Suppose that $m = 3$ and $n = 2$ and that for every $k \geq 0$ we let $L_k = \{ (\alpha_i, \alpha_j, 1) : 1 \leq i, j \leq k \}$; then the generating function $f(x)$ for trees with links restricted to L_1 is

$$f(x) = \frac{x}{1-x},$$

and the generating function for trees with links restricted to L_k is

$$f^{<k>}(x) = \frac{x}{1-kx} = \sum_{n=0}^{\infty} k^n x^{n+1},$$

so $p_n(k) = k^n$, and in this case $P_n(t)$ is the classical Eulerian polynomial.

Let $\mathbf{T}_n^{(L_k)}$ be the number of trees with n internal vertices and links restricted to L_k; then $p_n(k)$ is the number of trees in $\mathbf{T}_n^{(L_k)}$. Since $p_n(k) = b_n(k)$, we are able to define a bijection between $\mathbf{T}_n^{(L_k)}$ and the set of barred sequences on elements of Ω_n with at least one bar in each s-descent and with k bars. Moreover, for each T in $\mathbf{T}_n^{(L_k)}$, we define a set called the *t-descent set of* T. We then find a subset of $\mathbf{T}_n^{(L_k)}$, whose elements are called *reduced trees*, and we define a descent-preserving bijection between this subset and the set Ω_n.

In Chapter 3, we look at q-analogues of some of the iteration polynomials studied in Chapter 2. We begin with a formal power series

$$f(x) = qx + \sum_{i=1}^{\infty} c_i x^{(m-1)i+1};$$

as in Chapter 2 we can write

$$f^{<k>}(x) = \sum_{n=0}^{\infty} p_n(k,q) x^{(m-1)n+1},$$

where $p_n(k,q)$ is a q-analogue of a polynomial of degree at most n. We then show that $p_n(k,q)$ defines a polynomial $P_n(t,q)$ in t of degree at most n by

$$\sum_{k=0}^{\infty} p_n(k,q) t^k = \frac{P_n(t,q)}{\prod_{i=0}^{n} (1 - q^{(m-1)i+1} t)}.$$

We find a combinatorial interpretation of $P_n(t,q)$ by again counting barred sequences in Ω_n, but this time we weight each sequence by a certain power of q. Moreover, if γ belongs to Ω_n, we define the *s-index of γ* to be the sum of the s-descents of γ. We find that $P_n(t,q)$ counts sequences in Ω_n according to the number of s-descents and the s-index.

If

$$f(x) = \frac{qx}{1-x},$$

then the polynomial $P_n(t)$ counts permutations of $[n]$ according to descents and the major index, so $P_n(t)$ is a q-analogue of the Eulerian polynomial.

CHAPTER 1

THE INVERSION THEOREM

1.1 Definitions and notation

A *directed graph* (or *digraph*) G is a pair (V, E), where V is a finite set whose elements are called *vertices* and E is a finite set of ordered pairs of elements of V called *edges*. If (u, v) is an edge of G, we say that it *leaves* u and *enters* v; we also say that u and v are *adjacent* vertices and that each of them is *incident with* (u, v). If $G = (V, E)$ and $G' = (V', E')$ are two directed graphs, then G and G' are *isomorphic* if there is a bijection $\psi : V \to V'$ such that for all u, v in V, (u, v) is in E if and only if $(\psi(u), \psi(v))$ is in E'.

A *path of length* n from v_0 to v_n in a digraph G is a sequence $v_0, v_1, \ldots, v_n$ of $n+1$ vertices such that (v_{i-1}, v_i) is an edge of G for each i in $[n]$. A path is *simple* if no vertex is repeated. If v is any vertex of G, the *in-degree* of v is the number of edges entering v and the *out-degree* of v is the number of edges leaving v; the *degree* of v, denoted by $\deg v$, is the sum of its in-degree and its out-degree.

A *rooted tree* T is a directed graph with a distinguished vertex, called the *root* of T, such that

(1) the root of T has in-degree zero,
(2) every other vertex of T has in-degree one,
(3) if v is any vertex of T, there is a unique simple path from the root of T to v.

Figure 1 gives an example of a tree. An arrow from one vertex to another indicates an edge leaving the first vertex and entering the second.

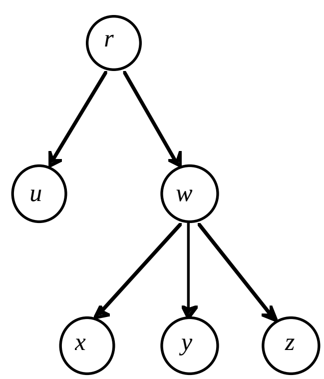


Figure 1

Henceforth, we shall follow the convention of drawing trees with the root at the top and the edges directed downward, so we will omit the arrows from our drawings.

If (u, v) is an edge of a tree T, then u is called the *parent* of v and v a *child* of u. If there is a path from u to v, then u is said to be an *ancestor* of v and

v a *descendent* of u. The vertices of T with at least one child are called *internal vertices*; those with no children are called *leaves*. The *height of* T, denoted by $\operatorname{ht} T$, is the length of the longest simple path in T which begins at the root; note that if T consists of a single vertex, then $\operatorname{ht} T$ equals zero.

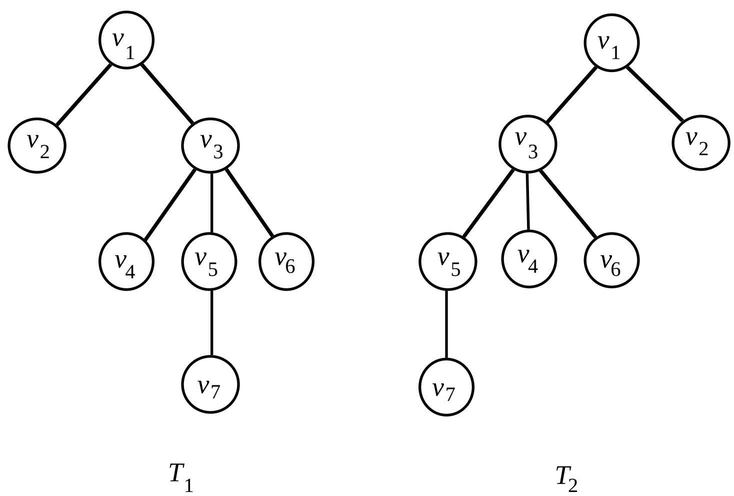
A *subtree* of a tree T is a tree T' such that

(1) each vertex of T' is a vertex of T,
(2) each edge of T' is an edge of T,
(3) each child of a vertex of T' is also a vertex of T'.

A *principal subtree* of a tree T is a subtree T' of T whose root is a child of the root of T.

We define a *rooted ordered tree* inductively to be a rooted tree whose principal subtrees have been linearly ordered and are themselves rooted ordered trees. Let T and T' be two rooted ordered trees with vertex sets V and V', respectively. Then T and T' are isomorphic as ordered trees if they are isomorphic as unordered trees via a bijection $\psi : V \to V'$ which is also order-preserving.

Unless otherwise stated, we will assume, in a drawing of an ordered tree T, that the children of any vertex of T are linearly ordered from left to right. (See Figure 2.)



T_1 and T_2 are isomorphic as unordered trees but not as ordered trees.

Figure 2

Let $\{T_1, T_2, \dots, T_n\}$ be a set of ordered trees; we call such a set a *forest of ordered trees*. An ordered n-tuple $(T_1, \dots, T_n)$ of ordered trees is called an *ordered forest of ordered trees*. If $n = 0$, we call the resulting forest the *empty forest*. Notice that we do not require the vertex sets of the trees in a forest (or an ordered forest) to be disjoint; in other words, if $F = (T_1, \dots, T_n)$ is an ordered forest with $T_i = (V_i, E_i)$ for i in $[n]$, then $V_i \cap V_j$ is not necessarily empty when $i \neq j$.

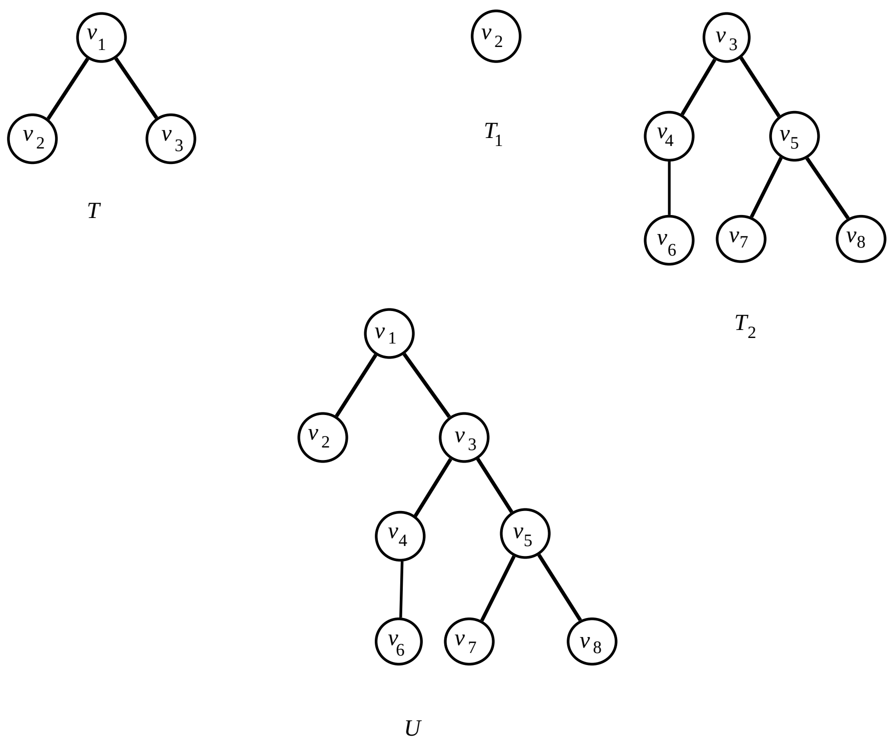
1.2 The Operation of substitution

Let $T = (V, E)$ be an ordered tree with m leaves, $v_1, v_2, \dots, v_m$. Suppose that $F = (T_1, \dots, T_n)$ is an ordered forest of ordered trees such that for each i in

$[n]$, $T_i = (V_i, E_i)$. We define an operation, called the *substitution of F into T* and denoted by $T * F$, as follows:

(1) if $m = n$ and if for each i, j in $[m]$, T_i has root v_i, $V \cap V_i = \{v_i\}$, and $V_i \cap V_j = \emptyset$ whenever $i \neq j$, then $T * F$ is the ordered tree $U = (V', E')$, where $V' = V \cup \bigcup_{i=1}^{m} V_i$ and $E' = E \cup \bigcup_{i=1}^{m} E_i$,
(2) otherwise, $T * F$ is undefined.

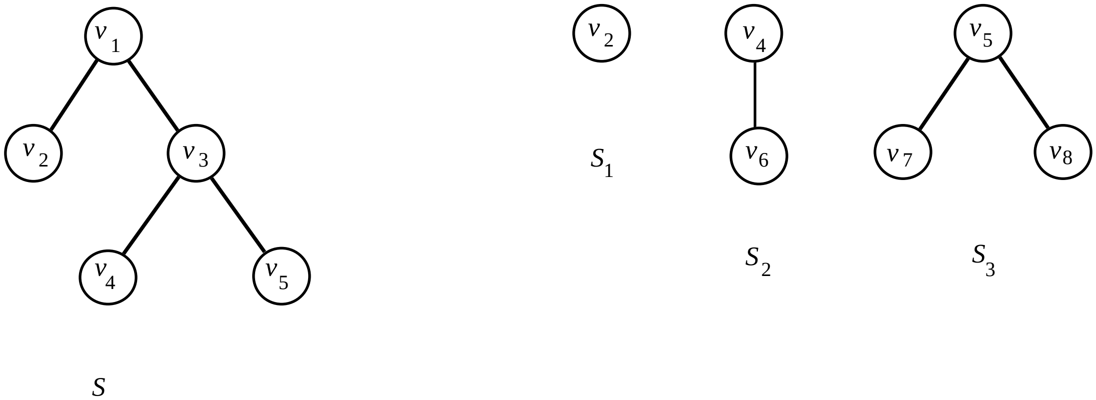
Figure 3a gives an example of the substitution of a forest into a tree.



When the forest $F = (T_1, T_2)$ is substituted into the tree T, the result is the tree U. So $U = T * F$.

Figure 3a

If $U = T * F$, then we call the ordered pair (T, F) a *factorization of U*. Clearly, an ordered tree may have more than one factorization. Figure 3b shows an another factorization of the tree U showed in Figure 3a.



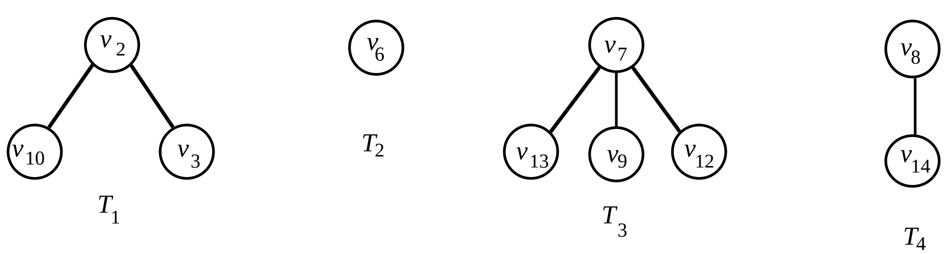
The forest G contains the trees S_1, S_2, and S_3; (S, G) is another factorization of U.

Figure 3b

We can use the definition of the operation of substitution of a forest into a tree to define a similar operation on two ordered forests. Let $F = (T_1, \ldots, T_n)$ and $G = (S_1, \ldots, S_m)$ be two ordered forests of ordered trees, with $T_i = (V_i, E_i)$ for each i in $[n]$ and $S_j = (V'_j, E'_j)$ for each j in $[m]$. Suppose that each T_i has r_i leaves, which we denote by v_{i1}, v_{i2}, $\ldots$, v_{ir_i}. Intuitively speaking, the *substitution of G into F*, denoted by $F * G$, is, when it is defined, the ordered forest that results from substituting the first r_1 trees of G into T_1, then substituting the next r_2 trees of G into T_2, etc. We define $F * G$ more formally as follows:

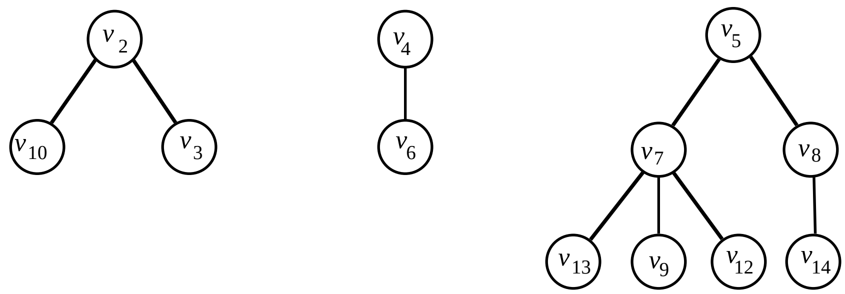
(1) if $r_1 + r_2 + \cdots + r_n = m$ and if for each i in $[n]$ and j, k in $[r_i]$, $S_{r_1+\cdots+r_{i-1}+j}$ has root v_{ij}, $V_i \cap V'_{r_1+\cdots+r_{i-1}+j} = \{v_{ij}\}$, and $V'_{r_1+\cdots+r_{i-1}+j} \cap V'_{r_1+\cdots+r_{i-1}+k} = \emptyset$ whenever $j \neq k$, then $F * G = (U_1, \ldots, U_n)$, where, for each i in $[n]$, $U_i = T_i * (S_{r_1+\cdots+r_{i-1}+1}, S_{r_1+\cdots+r_{i-1}+2}, \ldots, S_{r_1+r_2+\cdots+r_i})$,
(2) otherwise, $F * G$ is undefined.

Note that if F consists of a single tree T, then $F*G$ is just $T*G$. Let $F = (S_1, S_2, S_3)$ be the ordered forest shown in Figure 3b. Let $G = (T_1, T_2, T_3, T_4)$ be the ordered forest pictured in Figure 4a. Then $F * G$ is the ordered forest shown in Figure 4b.



The forest $G = (T_1, T_2, T_3, T_4)$

Figure 4a



The forest $F * G$

Figure 4b

1.3 Lettering of a tree

We will often want to attach a letter from some alphabet $\mathcal{A}$ to each vertex of an ordered tree T. If $T = (V, E)$, let $\lambda : V \to \mathcal{A}$ be a map from the vertex set of T into the alphabet $\mathcal{A}$; then we say that λ is a *lettering* of T and that T is a *lettered ordered tree*.

Let $\mathcal{A} = \{\alpha_1, \alpha_2, \dots\}$ be an alphabet and let $\delta : \mathcal{A} \to \mathbf{Z}^+$ be a map. Then for each α in $\mathcal{A}$, we call $\delta(\alpha)$ the *degree of* α. Let $T = (V, E)$ be an ordered tree and let X be an indeterminate not in $\mathcal{A}$. We will define a special kind of lettering of T by elements of $\mathcal{A} \cup \{X\}$ which we call a *degree-matching lettering*. Let $\lambda : V \to \mathcal{A} \cup \{X\}$ be a lettering of T. Then λ is degree-matching if

(1) the degree of $\lambda(v)$ equals the out-degree of v, for each internal vertex v of T,
(2) $\lambda(v) = X$ if and only if v is a leaf of T.

We call the pair (T, λ) a *matched* $\mathcal{A}$*-tree*. As an example, let T be the ordered tree shown in Figure 5.

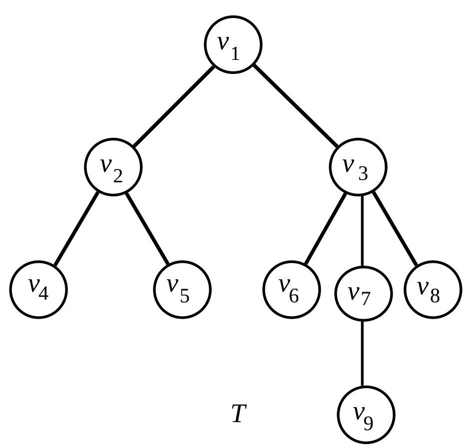


Figure 5

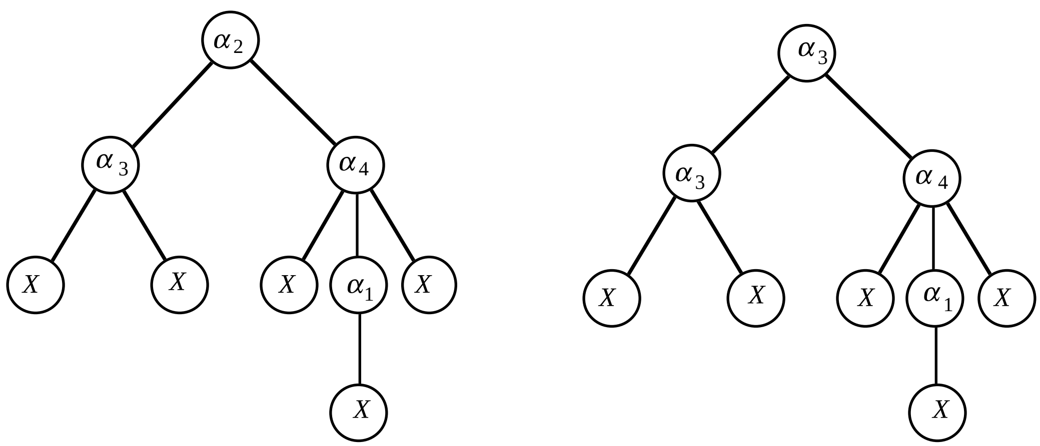
Suppose that $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\delta(\alpha_1) = 1$, $\delta(\alpha_2) = \delta(\alpha_3) = 2$, $\delta(\alpha_4) = 3$. Then (T, λ) is a matched $\mathcal{A}$-tree, where λ is given by

$$
\begin{array}{ll}
v_1 \mapsto \alpha_2 & v_6 \mapsto X \\
v_2 \mapsto \alpha_3 & v_7 \mapsto \alpha_1 \\
v_3 \mapsto \alpha_4 & v_8 \mapsto X \\
v_4 \mapsto X & v_9 \mapsto X \\
v_5 \mapsto X &
\end{array}
$$

Note that (T, λ') is also a matched $\mathcal{A}$-tree, where λ' is given by

$$
\begin{array}{ll}
v_1 \mapsto \alpha_3 & v_6 \mapsto X \\
v_2 \mapsto \alpha_3 & v_7 \mapsto \alpha_1 \\
v_3 \mapsto \alpha_4 & v_8 \mapsto X \\
v_4 \mapsto X & v_9 \mapsto X \\
v_5 \mapsto X &
\end{array}
$$

When we wish to emphasize in a diagram that an ordered tree (T, λ) is a matched $\mathcal{A}$-tree, we will draw it so that each vertex v is replaced by $\lambda(v)$. (See Figure 6.)



The trees (T, λ) (left) and (T, λ')

Figure 6

1.4 The R-algebra A of formal sums of ordered forests

Let $\mathcal{A} = \{\, \alpha_1, \alpha_2, \dots, \,\}$ be an alphabet each of whose elements has a degree assigned to it, and let $\mathcal{V}$ be a set of vertices. Let $\mathbf{T}$ be the set of all matched $\mathcal{A}$-trees trees with vertices in $\mathcal{V}$ and let $\mathbf{F}$ be the set of all ordered forests of trees in $\mathbf{T}$. Note that $\mathbf{F}$ contains the empty forest, which we denote by F_0. We can define multiplication in $\mathbf{F}$ by juxtaposition: if $F = (T_1, T_2, \dots, T_n)$ and $G = (S_1, S_2, \dots, S_m)$ belong to $\mathbf{F}$, then we define FG to be the ordered forest $(T_1, \dots, T_n, S_1, \dots, S_m)$. Since $FF_0 = F_0F = F$ for any F in $\mathbf{F}$, $\mathbf{F}$ is a free monoid.

Let $\bar{\mathbf{T}}$ be the set of all isomorphism classes of lettered ordered trees in $\mathbf{T}$ and let $\bar{\mathbf{F}}$ be the set of all isomorphism classes of forests in $\mathbf{F}$. The multiplication that we defined on $\mathbf{F}$ is in fact well-defined on the isomorphism classes of elements of $\mathbf{F}$, that is, if $\bar{F}$ and $\bar{G}$ belong to $\bar{\mathbf{F}}$, then $\bar{F}\bar{G}$ equals $\overline{FG}$. Since $\bar{F}\bar{F}_0 = \bar{F}_0\bar{F} = \bar{F}$ for any $\bar{F}$ in $\bar{\mathbf{F}}$, $\bar{\mathbf{F}}$ too is a free monoid.

Let R be a commutative ring and let A be the set of all formal sums $\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\bar{F}$, where $c_{\bar{F}}$ is in R. Multiplication in A is defined by

$$\Big(\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\bar{F}\Big)\Big(\sum_{\bar{G}\in\bar{\mathbf{F}}} d_{\bar{G}}\bar{G}\Big) = \sum_{\bar{F},\bar{G}\in\bar{\mathbf{F}}} c_{\bar{F}}d_{\bar{G}}\,\bar{F}\bar{G}.$$

Note that this operation is associative. If we define scalar multiplication by elements of R in the obvious way, then A becomes an associative R-algebra.

Recall that if F and G belong to $\mathbf{F}$, we can define $F*G$ if the number of leaves in F equals the number of trees in G and if the vertex sets of the trees in F and G intersect in a certain way. If we are considering isomorphism classes of elements of $\mathbf{F}$, only the former condition matters to us. So if $\bar{F}$ and $\bar{G}$ belong to $\bar{\mathbf{F}}$, we can define the substitution of $\bar{G}$ into $\bar{F}$, which we denote by $\bar{F}*\bar{G}$, as follows:

(1) if there is some F in $\bar{F}$ and some G in $\bar{G}$ such that $F*G$ is defined, then $\bar{F}*\bar{G}=\overline{F*G}$,
(2) otherwise, $\bar{F}*\bar{G}$ is undefined.

In other words, if $\bar{F}=(\bar{T}_1,\bar{T}_2,\ldots,\bar{T}_n)$ and $\bar{G}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_m)$, and if each T_i in $\bar{\mathbf{T}}_i$ has r_i leaves, then $\bar{F}*\bar{G}$ is defined if $\sum_{i=1}^{n} r_i = m$.

We can use this definition to define another operation on A. If $\phi=\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\bar{F}$ and $\gamma=\sum_{\bar{G}\in\bar{\mathbf{F}}} d_{\bar{G}}\bar{G}$ are any two elements of A, then we define the *substitution of* γ *into* ϕ, denoted by $\phi*\gamma$, to be

$$\phi*\gamma = \sum_{\substack{\bar{F},\bar{G}\in\bar{\mathbf{F}}\\ \bar{F}*\bar{G}\text{ defined}}} c_{\bar{F}}d_{\bar{G}}\,\bar{F}*\bar{G}.$$

Recall that an ordered tree T can be considered as the ordered forest containing the single tree T. Let $\tau=\sum_{\bar{F}\in\bar{\mathbf{F}}} a_{\bar{F}}\bar{F}$ be an element of A such that $a_{\bar{F}}$ equals zero unless each forest in $\bar{F}$ contains a single tree. We may rewrite τ as $\tau=\sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}}\bar{T}$. Let B be the subset of A containing all elements of this form; then we define an operation on B as follows. If $\tau=\sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}}\bar{T}$ and $\sigma=\sum_{\bar{S}\in\bar{\mathbf{T}}} b_{\bar{S}}\bar{S}$ are in B, then the *composition of* τ *and* σ, denoted by $\tau[\sigma]$, is defined to be

$$\tau[\sigma]=\tau*(1-\sigma)^{-1}.$$

Since

$$(1-\sigma)^{-1}=\sum_{n\ge 0}\sigma^n=\sum_{n\ge 0}\sum_{\substack{\bar{F}\in\bar{\mathbf{F}}\\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n)}} b_{\bar{S}_1}b_{\bar{S}_2}\cdots b_{\bar{S}_n}\,\bar{F},$$

we have

$$\tau[\sigma]=\tau*\sum_{n\ge 0}\sum_{\substack{\bar{F}\in\bar{\mathbf{F}}\\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n)}} b_{\bar{S}_1}b_{\bar{S}_2}\cdots b_{\bar{S}_n}\,\bar{F}.$$

1.5 The image of A in $R[[\alpha_1, \alpha_2, \ldots, X]]$

Let T be a tree in $\mathbf{T}$ with n vertices, where $n \geq 1$. We wish to associate with T a word ω_T in $(\mathcal{A} \cup \{X\})^*$ of length n. To do this, we first define a total ordering on the vertices of T called *postorder*. It is defined recursively as follows. Let r be the root of T and let τ_1, τ_2, ... , τ_k be the principal subtrees of T, listed in the order defining T as an ordered tree. Let $\mathrm{ord}(T)$ be a listing of the vertices of T in postorder; then $\mathrm{ord}(T) = \mathrm{ord}(\tau_1)$, $\mathrm{ord}(\tau_2)$, ... , $\mathrm{ord}(\tau_k)$, r. For example, if T is the tree shown in Figure 5, then $\mathrm{ord}(T) = v_4$, v_5, v_2, v_6, v_9, v_7, v_8, v_3, v_1.

Since T is a matched $\mathcal{A}$-tree, each vertex of T has an element of $\mathcal{A} \cup \{X\}$ attached to it by a degree-matching lettering λ. If $\mathrm{ord}(T) = v_1$, v_2, ... , v_{n-1}, r, then we define ω_T to be the word $\lambda(v_1)\lambda(v_2)\cdots\lambda(v_{n-1})\lambda(r)$. For example, if we take the tree T shown in Figure 5, lettered with the lettering λ as shown on the left in Figure 6, then $\omega_T = XX\alpha_3XX\alpha_1X\alpha_4\alpha_2$.

Note that any tree T in $\mathbf{T}$ is uniquely determined by ω_T, up to isomorphism. Hence we can associate with any element $\bar{T}$ in $\bar{\mathbf{T}}$ a unique word $\omega_{\bar{T}}$ in $(\mathcal{A} \cup \{X\})^*$: $\omega_{\bar{T}}$ simply equals ω_T, where T is any element of $\bar{T}$.

If $F = (T_1, T_2, \ldots, T_m)$ belongs to $\mathbf{F}$, then we can define a total ordering of the vertices of F by $\mathrm{ord}(F) = \mathrm{ord}(T_1)$, $\mathrm{ord}(T_2)$, ... , $\mathrm{ord}(T_m)$. If for each i in $[m]$, T_i has n_i vertices, then we can associate with F a word ω_F in $(\mathcal{A} \cup \{X\})^*$ of length $n_1 + n_2 + \cdots + n_m$ by $\omega_F = \omega_{T_1}\omega_{T_2}\cdots\omega_{T_m}$. (See Figure 7a.) Clearly, if F and G are two elements of $\mathbf{F}$, then $\omega_{FG} = \omega_F\omega_G$. Since F is uniquely determined by ω_F, up to isomorphism, we can associate with any element $\bar{F}$ in $\bar{\mathbf{F}}$ a unique word $\omega_{\bar{F}}$ in $(\mathcal{A} \cup \{X\})^*$; as before, $\omega_{\bar{F}}$ equals ω_F, where F is any element of $\bar{F}$. Moreover, $\omega_{\bar{F}\bar{G}} = \omega_{\overline{FG}} = \omega_{\bar{F}}\omega_{\bar{G}}$.

Let $G = (S_1, S_2, \ldots, S_r)$ be in $\mathbf{F}$. If $F * G$ is defined, then ω_{F*G} is obtained by replacing the first X in ω_F by ω_{S_1}, the second X in ω_F by ω_{S_2}, and so on. (See Figure 7b.)

Let $\hat{A}$ be the R-algebra of all formal sums $\sum_{\bar{F} \in \bar{\mathbf{F}}} c_{\bar{F}}\omega_{\bar{F}}$. Since each $\bar{F}$ in $\bar{\mathbf{F}}$ corresponds uniquely to a word $\omega_{\bar{F}}$ in $(\mathcal{A} \cup \{X\})^*$ and since $\omega_{\bar{F}}\omega_{\bar{G}} = \omega_{\bar{F}\bar{G}}$, for each $\bar{F}$, $\bar{G}$ in $\bar{\mathbf{F}}$, the R-algebra A is isomorphic to $\hat{A}$, with the isomorphism $\rho : A \to \hat{A}$ given by

$$\sum_{\bar{F} \in \bar{\mathbf{F}}} c_{\bar{F}}\bar{F} \overset{\rho}{\mapsto} \sum_{\bar{F} \in \bar{\mathbf{F}}} c_{\bar{F}}\omega_{\bar{F}}.$$

Let $\hat{\phi} = \sum_{\bar{F} \in \bar{\mathbf{F}}} c_{\bar{F}}\omega_{\bar{F}}$ and $\hat{\gamma} = \sum_{\bar{G} \in \bar{\mathbf{F}}} d_{\bar{G}}\omega_{\bar{G}}$ be two elements of $\hat{A}$. We define the *substitution of* $\hat{\gamma}$ *into* $\hat{\phi}$, denoted by $\hat{\phi} * \hat{\gamma}$, to be

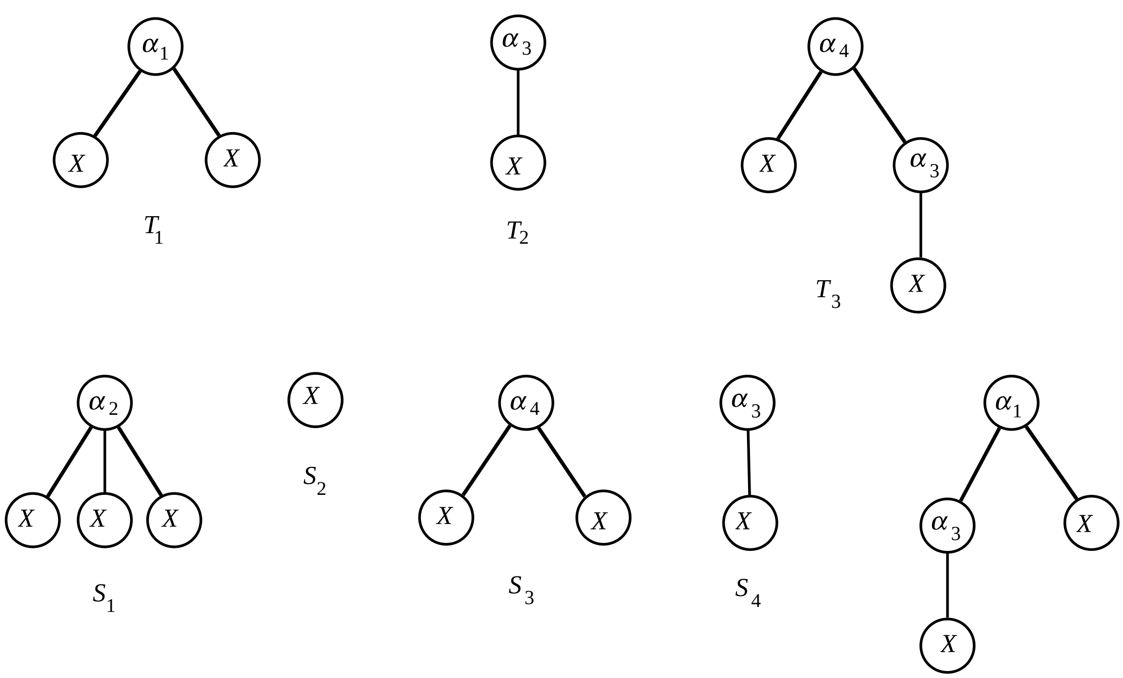
$$\hat{\phi} * \hat{\gamma} = \sum_{\substack{\bar{F}, \bar{G} \in \bar{\mathbf{F}} \\ \bar{F} * \bar{G} \text{ defined}}} c_{\bar{F}} d_{\bar{G}}\, \omega_{\bar{F} * \bar{G}}.$$

Since

$$\rho\left(\sum_{\bar{F} \in \bar{\mathbf{F}}} c_{\bar{F}}\bar{F} * \sum_{\bar{G} \in \bar{\mathbf{F}}} d_{\bar{G}}\bar{G}\right) = \rho\left(\sum_{\substack{\bar{F}, \bar{G} \in \bar{\mathbf{F}} \\ \bar{F} * \bar{G} \text{ defined}}} c_{\bar{F}} d_{\bar{G}}\, \bar{F} * \bar{G}\right)$$

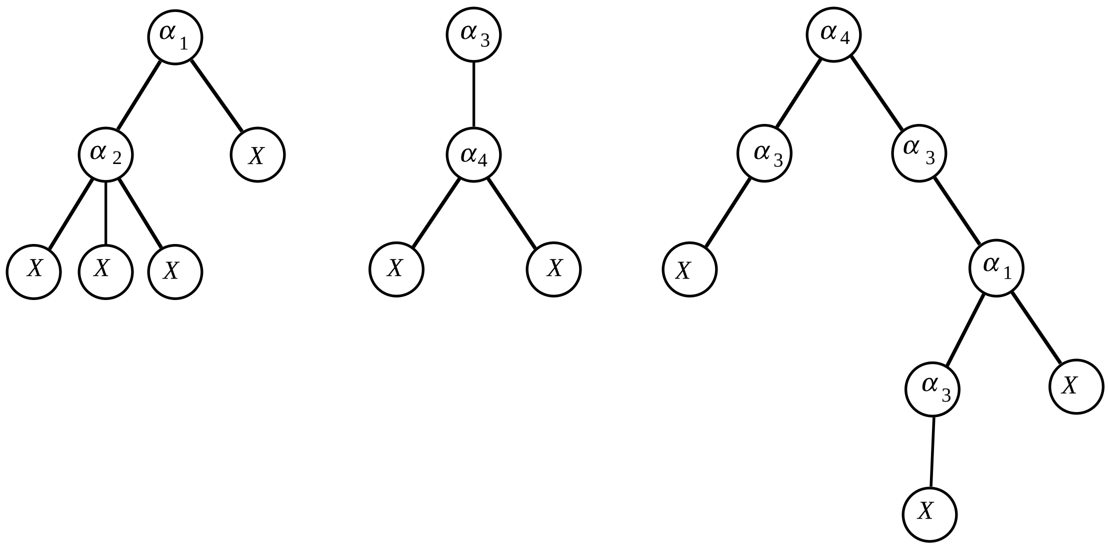
$$
\begin{aligned}
&= \sum_{\substack{\bar F,\bar G\in\bar{\mathbf{F}}\\ \bar F*\bar G \text{ defined}}} c_{\bar F} d_{\bar G}\, \omega_{\bar F*\bar G} \\
&= \sum_{\bar F\in\bar{\mathbf{F}}} c_{\bar F}\omega_{\bar F} * \sum_{\bar G\in\bar{\mathbf{F}}} d_{\bar G}\omega_{\bar G} \\
&= \rho\left(\sum_{\bar F\in\bar{\mathbf{F}}} c_{\bar F}\bar F\right) * \rho\left(\sum_{\bar G\in\bar{\mathbf{F}}} d_{\bar G}\bar G\right),
\end{aligned}
$$

ρ preserves the operation of substitution.



The forests $F = (T_1, T_2, T_3)$ (top) and $G = (S_1, S_2, S_3, S_4, S_5)$; $\omega_F = XX\alpha_1X\alpha_3XX\alpha_3\alpha_4$ and $\omega_G = XXX\alpha_2XXX\alpha_4X\alpha_3X\alpha_3X\alpha_1$.

Figure 7a



The forest $F * G$; $\omega_{F*G} = XXX\alpha_2 X\alpha_1 XX\alpha_4\alpha_3 X\alpha_3 X\alpha_3 X\alpha_1\alpha_3\alpha_4$.

Figure 7b

1.6 The image of A in $R[[a_1, a_2, \ldots, x]]$

Notice that the algebra $\hat{A}$ is a power series algebra in the non-commuting variables α_1, α_2, $\ldots$, X. Let $\widetilde{\mathcal{A}} = \{a_1, a_2, \ldots, x\}$ be a set of commuting variables and let $\widetilde{A} = R[[a_1, a_2, \ldots, x]]$ be the power series algebra in these variables. Then we can define a homomorphism $\eta : \hat{A} \to \widetilde{A}$ by $X \overset{\eta}{\mapsto} x$ and $\alpha_i \overset{\eta}{\mapsto} a_i$, for each i, such that η preserve infinite sums. Then, for each $\bar{T}$ in $\bar{\mathbf{T}}$, $\eta(\omega_{\bar{T}})$ is a monomial in the commuting variables a_1, a_2, $\ldots$, x. If $\bar{T}$ has $r_{\bar{T}}$ leaves, we will write $\eta(\omega_{\bar{T}})$ as $m_{\bar{T}} x^{r_{\bar{T}}}$, where $m_{\bar{T}}$ is a monomial in the variables a_1, a_2, $\ldots$. Hence $\sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}}\omega_{\bar{T}} \overset{\eta}{\mapsto} \sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}} m_{\bar{T}} x^{r_{\bar{T}}}$. If $\bar{F} = (\bar{T}_1, \bar{T}_2, \ldots, \bar{T}_n)$, then $\omega_{\bar{F}} = \omega_{\bar{T}_1}\omega_{\bar{T}_2}\cdots\omega_{\bar{T}_n}$, so $\omega_{\bar{F}} \overset{\eta}{\mapsto} m_{\bar{T}_1} m_{\bar{T}_2}\cdots m_{\bar{T}_n}\, x^{r_{\bar{T}_1}+r_{\bar{T}_2}+\cdots+r_{\bar{T}_n}}$. We denote $m_{\bar{T}_1} m_{\bar{T}_2}\cdots m_{\bar{T}_n}\, x^{r_{\bar{T}_1}+r_{\bar{T}_2}+\ldots r_{\bar{T}_n}}$ by $m_{\bar{F}} x^{r_{\bar{F}}}$; then $\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\omega_{\bar{F}} \overset{\eta}{\mapsto} \sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}} m_{\bar{F}} x^{r_{\bar{F}}}$.

If $\bar{G} = (\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_m)$ and if $\bar{F} * \bar{G}$ is defined, then (since the elements of of the algebra $\widetilde{A}$ commute) $\omega_{\bar{F}*\bar{G}} \overset{\eta}{\mapsto} m_{\bar{T}_1}\cdots m_{\bar{T}_n} m_{\bar{S}_1}\cdots m_{\bar{S}_m}\, x^{r_{\bar{S}_1}+\cdots+r_{\bar{S}_m}}$, which equals $m_{\bar{F}} m_{\bar{G}} x^{r_{\bar{G}}}$. So

$$\eta\left(\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\omega_{\bar{F}} \;*\; \sum_{\bar{G}\in\bar{\mathbf{F}}} d_{\bar{G}}\omega_{\bar{G}}\right) = \eta\left(\sum_{\substack{\bar{F},\bar{G}\in\bar{\mathbf{F}} \\ \bar{F}*\bar{G} \text{ defined}}} c_{\bar{F}} d_{\bar{G}}\,\omega_{\bar{F}*\bar{G}}\right)$$

$$= \sum_{\substack{\bar{F},\bar{G}\in\bar{\mathbf{F}} \\ \bar{F}*\bar{G} \text{ defined}}} c_{\bar{F}} d_{\bar{G}}\, m_{\bar{F}} m_{\bar{G}} x^{r_{\bar{G}}}.$$

Let $\tilde{\rho} = \eta \circ \rho$. Then $\tilde{\rho} : A \to \widetilde{A}$ is a homomorphism that preserves infinite sums and $\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\bar{F} \overset{\tilde{\rho}}{\mapsto} \sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}} m_{\bar{F}} x^{r_{\bar{F}}}$. Moreover,

$$\tilde{\rho}\left(\sum_{\bar{F}\in\bar{\mathbf{F}}} c_{\bar{F}}\bar{F} * \sum_{\bar{G}\in\bar{\mathbf{F}}} d_{\bar{G}}\bar{G}\right) = \sum_{\substack{\bar{F},\bar{G}\in\bar{\mathbf{F}} \\ \bar{F}*\bar{G} \text{ defined}}} c_{\bar{F}} d_{\bar{G}}\, m_{\bar{F}} m_{\bar{G}} x^{r_{\bar{G}}}.$$

Recall that B is the subset of A consisting of all elements of the form $\sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}}\bar{T}$. So $\tilde{\rho}(B)$ is the subset of $\widetilde{A}$ consisting of all elements of the form $\sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}} m_{\bar{T}} x^{r_{\bar{T}}}$. We can think of elements of $\tilde{\rho}(B)$ as power series in x, so the usual operation of composition is defined in $\tilde{\rho}(B)$.

Lemma 1.6. *Let $\tau = \sum_{\bar{T}\in\bar{\mathbf{T}}} a_{\bar{T}}\bar{T}$ and $\sigma = \sum_{\bar{S}\in\bar{\mathbf{T}}} b_{\bar{S}}\bar{S}$ be elements of B. Then $\tilde{\rho}\big(\tau[\sigma]\big) = \tilde{\rho}(\tau) \circ \tilde{\rho}(\sigma)$.*

Proof. Recall that

$$\begin{aligned}
\tau[\sigma] &= \tau * \sum_{n\geq 0} \sigma^n \\
&= \tau * \sum_{n\geq 0} \sum_{\substack{\bar{F}\in\bar{\mathbf{F}} \\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n)}} b_{\bar{S}_1} b_{\bar{S}_2} \cdots b_{\bar{S}_n}\, \bar{F}.
\end{aligned}$$

Denote $b_{\bar{S}_1} b_{\bar{S}_2} \cdots b_{\bar{S}_n}$ by $b_{\bar{F}}$. Then

$$\tau[\sigma] = \sum_{n\geq 0} \sum_{\substack{\bar{T}\in\bar{\mathbf{T}},\, \bar{F}\in\bar{\mathbf{F}} \\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n) \\ \bar{T}*\bar{F} \text{ defined}}} a_{\bar{T}} b_{\bar{F}}\, \bar{T} * \bar{F}.$$

For each $n \geq 1$, let $\bar{\mathbf{T}}_n$ be the set of trees in $\bar{\mathbf{T}}$ with n leaves. Since the substitution of the empty forest into any $\bar{T}$ in $\bar{\mathbf{T}}$ is undefined, we can write

$$\tau[\sigma] = \sum_{n\geq 1} \sum_{\substack{\bar{T}\in\bar{\mathbf{T}}_n \\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n) \\ \bar{T}*\bar{F} \text{ defined}}} a_{\bar{T}} b_{\bar{F}}\, \bar{T} * \bar{F}.$$

So

$$\begin{aligned}
\tilde{\rho}\big(\tau[\sigma]\big) &= \tilde{\rho}\left(\sum_{n\geq 1} \sum_{\substack{\bar{T}\in\bar{\mathbf{T}}_n \\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n) \\ \bar{T}*\bar{F} \text{ defined}}} a_{\bar{T}} b_{\bar{F}}\, \bar{T} * \bar{F}\right) \\
&= \sum_{n\geq 1} \sum_{\substack{\bar{T}\in\bar{\mathbf{T}}_n \\ \bar{F}=(\bar{S}_1,\bar{S}_2,\ldots,\bar{S}_n) \\ \bar{T}*\bar{F} \text{ defined}}} a_{\bar{T}} b_{\bar{F}}\, m_{\bar{T}} m_{\bar{F}} x^{r_{\bar{F}}}.
\end{aligned}$$

On the other hand,

$$\tilde{\rho}(\tau) \circ \tilde{\rho}(\sigma) = \sum_{\bar{T} \in \bar{\mathbf{T}}} a_{\bar{T}} m_{\bar{T}} x^{r_{\bar{T}}} \circ \sum_{\bar{S} \in \bar{\mathbf{T}}} b_{\bar{S}} m_{\bar{S}} x^{r_{\bar{S}}}.$$

Since $\tilde{\rho}(\sigma)(0) = 0$,

$$\begin{aligned}\sum_{\bar{T} \in \bar{\mathbf{T}}} a_{\bar{T}} m_{\bar{T}} x^{r_{\bar{T}}} \circ \sum_{\bar{S} \in \bar{\mathbf{T}}} b_{\bar{S}} m_{\bar{S}} x^{r_{\bar{S}}} &= \sum_{\bar{T} \in \bar{\mathbf{T}}} a_{\bar{T}} m_{\bar{T}} \big(\sum_{\bar{S} \in \bar{\mathbf{T}}} b_{\bar{S}} m_{\bar{S}} x^{r_{\bar{S}}} \big)^{r_{\bar{T}}} \\ &= \sum_{\bar{T} \in \bar{\mathbf{T}}} \sum_{\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_{r_{\bar{T}}}} a_{\bar{T}} m_{\bar{T}} b_{\bar{S}_1} \cdots b_{\bar{S}_{r_{\bar{T}}}} m_{\bar{S}_1} \cdots m_{\bar{S}_{r_{\bar{T}}}} x^{r_{\bar{S}_1} + \cdots + r_{\bar{S}_{r_{\bar{T}}}}}.\end{aligned}$$

But $r_{\bar{T}}$ is the number of leaves of $\bar{T}$, so this last expression may be written as

$$\sum_{n \geq 1} \sum_{\substack{\bar{T} \in \bar{\mathbf{T}}_n \\ \bar{F} = (\bar{S}_1, \ldots, \bar{S}_n)}} a_{\bar{T}} m_{\bar{T}} b_{\bar{S}_1} \cdots b_{\bar{S}_n} m_{\bar{S}_1} \cdots m_{\bar{S}_n} x^{r_{\bar{S}_1} + \cdots + r_{\bar{S}_n}},$$

which equals

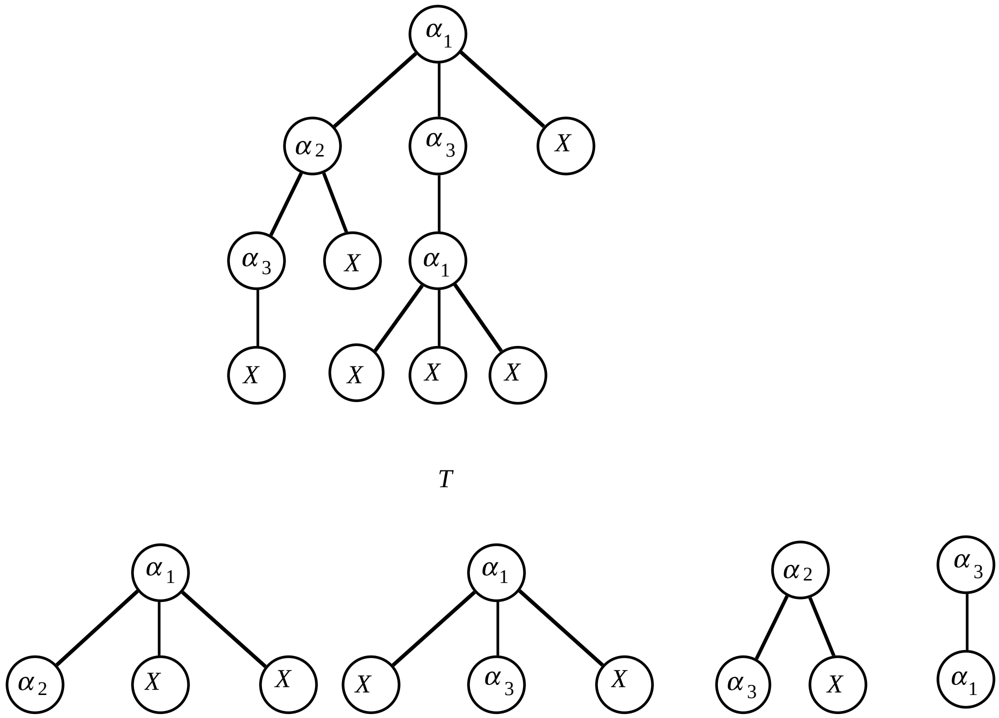
$$\sum_{n \geq 1} \sum_{\substack{\bar{T} \in \bar{\mathbf{T}}_n \\ \bar{F} = (\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_n) \\ \bar{T} * \bar{F} \text{ defined}}} a_{\bar{T}} b_{\bar{F}} \, m_{\bar{T}} m_{\bar{F}} x^{r_{\bar{F}}}.$$

So $\tilde{\rho}\big(\tau[\sigma]\big) = \tilde{\rho}(\tau) \circ \tilde{\rho}(\sigma)$.

1.7 The links of an ordered tree

From now on, we will primarily be interested in the letters attached to the vertices of a tree and in the relationship among these letters. These do not vary among the trees in an isomorphism class $\bar{T}$ in $\bar{\mathbf{T}}$. Therefore, in order to simplify our notation we will simply refer an element $\bar{T}$ of $\bar{\mathbf{T}}$ as T, and we will write $\mathbf{T}$ for $\bar{\mathbf{T}}$. Similarly, we will write F for $\bar{F}$ and $\mathbf{F}$ for $\bar{\mathbf{F}}$.

Let $\mathbf{L} = \{(\alpha, \beta, n) : \alpha, \beta \in \mathcal{A} \text{ and } n \in \mathbf{Z}^+ \text{ with } 1 \leq n \leq \delta(\alpha)\}$. We call the elements of $\mathbf{L}$ *links*. If (T, λ) is a matched $\mathcal{A}$-tree in $\mathbf{T}$ and T contains an edge (u, v) such that $\lambda(u) = \alpha$, $\lambda(v) = \beta$ and v in the nth child of u (the children of any vertex in T being counted from left to right), then we say that (α, β, n) is a *link of* T and that (α, β, n) is the link of T *associated with the edge* (u, v). (See Figure 8.) Note that if (u, v) is an edge of T such that v is a leaf of T, then there is no link associated with (u, v). Moreover, if $\operatorname{ht} T = 0$ or $\operatorname{ht} T = 1$, T has no links.



The links of the tree T are $(\alpha_1, \alpha_2, 1)$, $(\alpha_1, \alpha_3, 2)$, $(\alpha_2, \alpha_3, 1)$ and $(\alpha_3, \alpha_1, 1)$. They can be pictured as the trees of height one showed below T.

Figure 8

If L is any subset of $\mathbf{L}$, we define $\mathbf{T}^{(L)}$ to be the set of elements of $\mathbf{T}$ all of whose links are in L and $\mathbf{F}^{(L)}$ to be the set of all ordered forests of trees in $\mathbf{T}^{(L)}$. Let $\overline{L} = \mathbf{L} - L$; then we define $\mathbf{T}^{(\overline{L})}$ and $\mathbf{F}^{(\overline{L})}$ analogously. Note that $\mathbf{T}^{(L)}$ and $\mathbf{T}^{(\overline{L})}$ both contain all trees in $\mathbf{T}$ of height zero or one.

For any tree T in $\mathbf{T}$, we define the *size of* T, denoted by $|T|$, to be the number of internal vertices of T. If $S = (S_1, S_2, \ldots, S_n)$ is an ordered forest of trees in $\mathbf{T}$, then the size of F is $|F| = |S_1| + |S_2| + \cdots + |S_n|$. Recall that B is the subset of the algebra A consisting of all elements of the form $\sum_{T \in \mathbf{T}} a_T T$. Let $\tau = \sum_{T \in \mathbf{T}} a_T T$ be an element of B such that

$$a_T = \begin{cases} 1 \text{ if } T \in \mathbf{T}^{(L)} \\ 0 \text{ if } T \notin \mathbf{T}^{(L)}, \end{cases}$$

and let $\sigma = \sum_{S \in \mathbf{T}} b_S S$ be an element of B such that

$$b_S = \begin{cases} (-1)^{|S|} \text{ if } S \in \mathbf{T}^{(\overline{L})} \\ 0 \text{ if } S \notin \mathbf{T}^{(\overline{L})}. \end{cases}$$

Note that we can write τ as $\sum_{T \in \mathbf{T}^{(L)}} T$ and σ as $\sum_{S \in \mathbf{T}^{(\overline{L})}} (-1)^{|S|} S$.

1.8 The Inversion Theorem

Theorem 1.8 (The Inversion Theorem). *Let L be a subset of $\mathbf{L}$. Let $\tau = \sum_{T \in \mathbf{T}^{(L)}} T$ and $\sigma = \sum_{S \in \mathbf{T}^{(\overline{L})}} (-1)^{|S|} S$. Then $\tau[\sigma] = t_1$, where t_1 is the tree consisting of a single vertex.*

Proof. Let $\Theta = \{\, \theta : \theta = T * F$ for some T in $\mathbf{T}^{(L)}$ and some F in $\mathbf{F}^{(\overline{L})}\,\}$. Let $\varrho(\theta) = \{(T, F) : T * F = \theta,\ T \in \mathbf{T}^{(L)},\ F \in \mathbf{F}^{(\overline{L})}\,\}$. We call the elements of $\varrho(\theta)$ *L-$\overline{L}$ factorizations of θ*. Then

$$\begin{aligned} \tau[\sigma] &= \sum_{\substack{T \in \mathbf{T}^{(L)},\, F \in \mathbf{F}^{(\overline{L})} \\ T*F \text{ defined}}} (-1)^{|F|}\, T * F \\ &= \sum_{\theta \in \Theta} \theta \sum_{(T,F) \in \varrho(\theta)} (-1)^{|F|}. \end{aligned}$$

We will show that $\sum_{(T,F) \in \varrho(\theta)} (-1)^{|F|}$ equals 0 if $\theta \neq t_1$ and equals 1 if $\theta = t_1$.

Let θ be an element of Θ with vertex set V. If u, v are any two internal vertices of θ such that (u, v) is an edge of θ, then there is a link in L associated with (u, v) which belongs to one of L or $\overline{L}$. Therefore we may think of each edge of θ which is not incident with a leaf as being labeled with an L or an $\overline{L}$.

Note that θ may be considered as the Hasse diagram of a poset. With this in mind, we define a *cut of* θ to be a subset C of V such that C is an antichain of θ and every minimal element of θ is less than or equal to some element of C. If C is a cut of θ, let $I_C = \{v \in V : v \geq u$ for some u in $C\}$ and let $J_C = \{v \in V : v \leq w$ for some w in $C\}$. If v is in I_C, we say that v lies *above* C and if it is in $I_C - C$, we say that it lies *strictly above* C. Similarly, if v is in J_C, it lies *below* C and if it is in $J_C - C$, it lies *strictly below* C. If (v, v') is an edge of θ and v, v' both belong to one of I_C, $I_C - C$, J_C, or $J_C - C$, then we also say that (v, v') lies above, strictly above, below or strictly below C, respectively. Note that $(I_C - C) \cap (J_C - C) = \emptyset$.

Lemma 1.8. *Any vertex of θ that is not in a cut C of θ lies strictly above C or strictly below C.*

Proof. Let C be a cut of θ and let v be a vertex of θ that is not in C. Suppose that v is not in $I_C - C$. Then there is no u in C such that $v > u$. Now there is some minimal element x of θ such that $x \leq v$. Since C is a cut, there is some y in C such that $x \leq y$. Since v and y are both ancestors of x and since the ancestors of a vertex in a tree form a chain, y and v must be comparable. So either $y > v$, $y < v$, or $y = v$. The last two are impossible by assumption, so $v < y$. Hence v belongs to $J_C - C$.

Note that it follows from Lemma 1.2 that any edge of θ lies above C or below C.

We define a *good cut of* θ to be a cut C such that every edge labeled $\overline{L}$ lying above C is incident with an element of C and such that every labeled edge lying below C is labeled with an $\overline{L}$. We also define the *order of a cut* C, denoted by $o(C)$, to be the number of vertices lying strictly above C.

To prove the theorem, it suffices to show that if θ has more than one vertex, then

(1) there is a bijection between L-$\overline{L}$ factorizations of θ and good cuts of θ,
(2) there is an involution on good cuts of θ that changes the order by ± 1.

For if these conditions hold, then each L-$\overline{L}$ factorization (T, F) of θ corresponds uniquely to another L-$\overline{L}$ factorization $(\widetilde{T}, \widetilde{F})$, with $|F| - |\widetilde{F}| = \pm 1$. So if $\theta \neq t_1$,

$$\sum_{\theta \in \Theta} \theta \sum_{(T,F) \in \varrho(\theta)} (-1)^{|F|} = 0.$$

If $\theta = t_1$, then the only factorization of θ is (t_1, t_1) and $|t_1| = 0$, so

$$\sum t_1 \sum_{\substack{(T,F) \in \varrho(\theta) \\ T * F = t_1}} (-1)^{|F|} = t_1 \cdot (-1)^0 = t_1.$$

Let $\mathbf{C}$ be the set of good cuts of θ. We define a bijection $f : \varrho(\theta) \to \mathbf{C}$ as follows. If (T, F) is in $\varrho(\theta)$, let $f\big((T, F)\big)$ be the set of leaves of T; call this set C. Clearly C is a cut of θ. Suppose that (u, v) is an edge of θ lying above C; then (u, v) is an edge of T. Since T is in $\mathbf{T}^{(L)}$, (u, v) can be labeled with an $\overline{L}$ only if v is a leaf of T, that is, only if v is in C. Now suppose that (u, v) is an edge of θ lying below C; then (u, v) is an edge of some tree in F. Since F is in $\mathbf{F}^{(L)}$, each such edge is labeled with a $\overline{L}$ or is incident with a leaf of θ. So if (u, v) is labeled, it must be labeled with an $\overline{L}$. Hence, C is a good cut.

Conversely, let C' be a good cut of θ. Then every vertex of θ not in C' lies strictly above C' or strictly below C'. Those lying strictly above C' are the internal vertices of a tree T' whose leaves are the elements of C'. Those lying strictly below C' are the non-root vertices of an ordered forest F' of trees whose roots are the elements of C'. Hence, (T', F') is a factorization of θ. Since C' is a good cut, it is clear that T' is in $\mathbf{T}^{(L)}$ and that F' is in $\mathbf{F}^{(\overline{L})}$, so (T', F') belongs to $\varrho(\theta)$. It is easy to see that the correspondence $C' \mapsto (T', F')$ defines f^{-1}.

Next we define an involution $g : \mathbf{C} \to \mathbf{C}$ such that if $g(C) = \widetilde{C}$, then $o(C) - o(\widetilde{C}) = \pm 1$. Let C be in $\mathbf{C}$. Descend from the root of θ, always taking the left-most edge labeled with L, until there are no more edges labeled with L. Suppose we end at vertex v. Since the edge entering v is labeled with an L, v cannot be a leaf of θ. So v has children $w_1, w_2, \ldots, w_n$, where $n \in \mathbf{Z}^+$. For each i in $[n]$, there is a minimal element x of θ such that $x \leq w_i < v$. Since C is a cut of θ, there is some y in C such that $x \leq y$. Hence, since θ is a tree, y belongs to the unique chain containing w_i and v. Since C is a good cut, we cannot have $y < w_i$, for then (v, w_i) would be an edge lying above C labeled with an $\overline{L}$ but not incident with an element of C. If v is the root of θ, clearly we cannot have $y > v$. If v has parent u, we still cannot have $y > v$, for then (u, v) would be an edge lying below C labeled with an L. So $y = v$ or $y = w_i$. Since C is an antichain, for each i in $[n]$ we have that either v belongs to C or w_i belongs to C. Hence, C contains v or C contains all of $w_1, w_2, \ldots, w_n$. If C contains v, let $g(C) = C'$, where $C' = C - \{v\} \cup \{w_1, w_2, \ldots, w_n\}$. If C contains $w_1, w_2, \ldots, w_n$, let $g(C) = C''$, where $C'' = C - \{w_1, w_2, \ldots, w_n\} \cup \{v\}$. Clearly, $g^2(C) = C$, so g is an involution. Moreover, $o(C') = o(C) + 1$ and $o(C'') = o(C) - 1$.

If (T,F) belongs to $\varrho(\theta)$ and $f\big((T,F)\big)=C$, then $o(C)=|T|$ and $|F|=|\theta|-|T|$, so $|F|=|\theta|-o(C)$. Suppose that $g(C)=\widetilde{C}$ and that $\widetilde{C}=f\big((T,F)\big)$. Then $|F|-|\widetilde{F}|=\big(|\theta|-o(C)\big)-\big(|\theta|-o(\widetilde{C})\big)=o(\widetilde{C})-o(C)=\pm 1$. This proves the theorem.

Recall that r_T is the number of leaves of a tree T.

Corollary 1.8. *Let* $\beta(x)=\sum_{T\in\mathbf{T}^{(L)}} m_T x^{r_T}$ *and* $\overline{\beta}=\sum_{S\in\mathbf{T}^{(\overline{L})}}(-1)^{|S|}m_S x^{r_S}$*. Then* β *and* $\overline{\beta}$ *are functional inverses, that is,* $\big(\beta\circ\overline{\beta}\big)(x)=x$*.*

Proof. Both $\beta(x)$ and $\overline{\beta}(x)$ belong the the R-algebra $\widetilde{A}$. Recall that the homomorphism $\tilde{\rho}:A\to\widetilde{A}$ is given by

$$\sum_{F\in\mathbf{F}} c_F F \overset{\tilde{\rho}}{\mapsto} \sum_{F\in\mathbf{F}} c_F m_F x^{r_F}.$$

Let $\tau=\sum_{T\in\mathbf{T}^{(L)}} T$ and $\sigma=\sum_{S\in\mathbf{T}^{(\overline{L})}}(-1)^{|S|}S$ belong to the subset B of A. Then $\beta(x)=\tilde{\rho}(\tau)$ and $\overline{\beta}(x)=\tilde{\rho}(\sigma)$, so $\big(\beta\circ\overline{\beta}\big)(x)=\tilde{\rho}(\tau)\circ\tilde{\rho}(\sigma)$. By Lemma 1.1, $\tilde{\rho}(\tau)\circ\tilde{\rho}(\sigma)=\tilde{\rho}\big(\tau[\sigma]\big)$. The Inversion Theorem tells us that $\tau[\sigma]=t_1$, where t_1 is the tree consisting of a single vertex. Since $\tilde{\rho}(t_1)=x$, $\big(\beta\circ\overline{\beta}\big)(x)=x$.

1.9 Some simple applications of the Inversion Theorem

In this section, we give some simple applications of the Inversion Theorem. In each case, we take a commutative ring R and an alphabet $\mathcal{A}=\{\,\alpha_1,\alpha_2,\ldots,\,\}$ such that each α_i has a degree $\delta(\alpha_i)$ attached to it, and we consider a particular subset L of the set $\mathbf{L}$ of all links formed from elements of $\mathcal{A}$. As in Section 1.7, we let $\mathbf{T}^{(L)}$ denote the set of all matched $\mathcal{A}$-trees whose links are restricted to the set L and we let $\mathbf{T}_n^{(L)}$ be the subset of trees in $\mathbf{T}^{(L)}$ with n internal vertices. We let $\widetilde{\mathcal{A}}=\{\,a_1,a_2,\ldots,\,\}$ be a set of commuting variables and we let $\beta(x)$ be the image of $\sum_{T\in\mathbf{T}^{(L)}} T$ in $R[[a_1,a_2,\ldots,x]]$ under the homomorphism $\tilde{\rho}$ defined in Section 1.6. Therefore

$$\beta(x)=\sum_{T\in\mathbf{T}^{(L)}} m_T x^{r_T},$$

where m_T is a monomial in $R[[a_1,a_2,\ldots,]]$ such that m_T contains j_i factors of a_i if and only if T has j_i vertices lettered with α_i, and where r_T is the number of leaves of T. Similarly, $\overline{\beta}(x)$ is the image of $\sum_{S\in\mathbf{T}^{(\overline{L})}}(-1)^{|S|}S$ under $\tilde{\rho}$, so

$$\overline{\beta}(x)=\sum_{S\in\mathbf{T}^{(\overline{L})}}(-1)^{|S|}m_S x^{r_S}.$$

For our first application, let $\mathcal{A}=\{\,\alpha\,\}$, where $\delta(\alpha)=1$. Then the set $\mathbf{L}$ of all links formed from elements of $\mathcal{A}$ contains the single link $(\alpha,\alpha,1)$. Let $L=\mathbf{L}$ and let $\overline{L} = \emptyset$. All trees in $\mathbf{T}^{(L)}$ have the form shown on the left in Figure 9. There are only two trees in $\mathbf{T}^{(\overline{L})}$, and they are shown on the right in Figure 9.

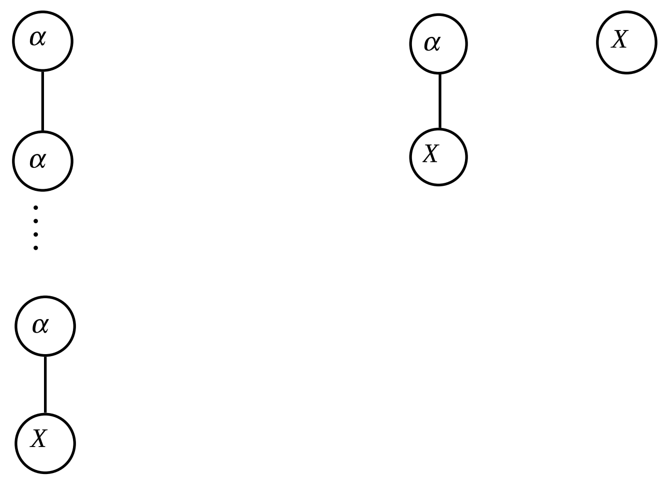


Figure 9

Note that $\widetilde{\mathcal{A}} = \{\, a \,\}$. Clearly

$$\beta(x) = x + ax + a^2x + a^3x + \cdots = \sum_{n=0}^{\infty} a^n x = \frac{x}{1-a},$$

and

$$\overline{\beta}(x) = x - ax,$$

and it is obvious that β and $\overline{\beta}$ are compositional inverses.

Notice that this is a special case of the theorem of Carlitz, Scoville and Vaughan, in which $\mathcal{A} = \{\, \alpha \,\}$, $\mathbf{L} = \mathcal{A} \times \mathcal{A} = \{\, (\alpha,\alpha) : \alpha \in \mathcal{A} \,\}$, $L = \mathbf{L}$ and $\overline{L} = \emptyset$. The generating function for sequences of elements of $\mathcal{A}$ all of whose links are in L is

$$f(x) = \sum_{n=0}^{\infty} \alpha^n x^n,$$

the generating function, with alternating signs, for sequences all of whose links are in $\overline{L}$ is

$$g(x) = 1 - \alpha x,$$

and $f(x) = (g(x))^{-1}$. If we set $x = 1$, $\beta(x) = f(x)$ and $\overline{\beta}(x) = g(x)$. So, as mentioned earlier, the Inversion Theorem reduces to the Reciprocity Theorem of Carlitz, Scoville and Vaughan when the trees involved are unary.

Next, let $\mathcal{A} = \{\, \alpha \,\}$ as in the previous example, but let $\delta(\alpha) = 2$. Then $\mathbf{L} = \{\, (\alpha,\alpha,1), (\alpha,\alpha,2) \,\}$. Once again, let $L = \mathbf{L}$. Then $\overline{L} = \emptyset$, and the only two trees in $\mathbf{T}^{(\overline{L})}$ are shown in Figure 10.

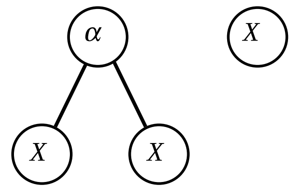


Figure 10

Hence

$$\overline{\beta}(x) = x - ax^2,$$

and we can use Lagrange inversion to find that

$$\begin{aligned}\beta(x) &= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} a^{n-1} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} a^n x^{n+1}.\end{aligned}$$

This is what we would expect for $\beta(x)$, since the number of trees in $\mathbf{T}_n^{(L)}$ is the number of ordered binary trees on n internal vertices. This number is the nth Catalan number, $\frac{1}{n+1}\binom{2n}{n}$.

Finally, we generalize by taking $\mathcal{A} = \{\,\alpha\,\}$, where $\delta(\alpha) = m$ for some $m \geq 1$. Then $\mathbf{L} = \{\,(\alpha,\alpha,1),\ldots,(\alpha,\alpha,m)\,\}$, so $\mathbf{L}$ contains m links. Suppose that L contains k of those links, for $0 \leq k \leq m$. Then counting trees in $\mathbf{T}_n^{(L)}$ is the same as counting k-ary ordered trees on n internal vertices. There are $\frac{1}{(k-1)n+1}\binom{kn}{n} = \frac{1}{n}\binom{kn}{n-1}$ such trees (see, e.g., [**GJ**], p. 112), so

$$\beta(x) = x + \sum_{n=1}^{\infty} \frac{1}{n}\binom{kn}{n-1} a^n x^{(m-1)n+1}.$$

Similarly,

$$\overline{\beta}(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}\binom{(m-k)n}{n-1} a^n x^{(m-1)n+1},$$

and β and $\overline{\beta}$ are compositional inverses.

CHAPTER 2

ITERATION POLYNOMIALS

2.1 Definitions and Notation

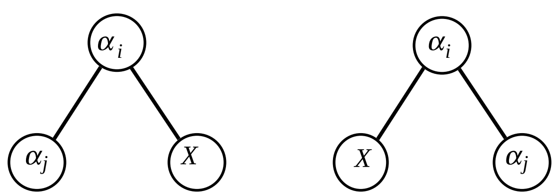
If $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation of $[n]$, we define the *spaces* of π to be the integers 0, 1, … , n. We think of space i as lying between π_i and π_{i+1}, for $1 \le i \le n-1$, with space 0 lying to the left of π_1 and space n lying to the right of π_n. We call the spaces 1, 2, … , n the *proper spaces* of π. (The empty permutation has one space, which is not proper.) A *descent of* π is a proper space i such that $\pi_i > \pi_{i+1}$ or the space n for $n > 0$. The *Eulerian number* $A_{n,j}$ is the number of permutations of $[n]$ with j descents; for each $n \ge 0$, $A_{n,j}$ is the coefficient of t^j in the classical Eulerian polynomial $A_n(t)$. The latter may be defined by

$$\sum_{k=0}^{\infty} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}} = \frac{\sum_{j=0}^{n} A_{n,j}\, t^j}{(1-t)^{n+1}}.$$

In this chapter, we define the links of our trees in a slightly different way from Chapter One. Let $\mathcal{A} = \{\, \alpha_1, \alpha_2, \dots \}$ be an alphabet such that $\deg \alpha_i = m$ for each i and let $\mathbf{L}$ be the set of all possible links formed from elements of $\mathcal{A}$. For each $k \ge 0$, let $\mathbf{L}_k$ be the set of links in $\mathbf{L}$ formed from the set $\{\, \alpha_1, \alpha_2, \dots, \alpha_k \,\}$.

2.2 The kth iterate of a formal power series

We start by considering a certain sequence of subsets of $\mathbf{L}$. Let $m = 2$. For each $k \ge 0$, let $L_k = \{\, (\alpha_i, \alpha_j, 1) : 1 \le i, j \} \cap \mathbf{L}_k$; then $L_k = \{\, (\alpha_i, \alpha_j, 1) : 1 \le i, j \le k \,\}$ and its complement in $\mathbf{L}_k$ is the set $\overline{L}_k = \{\, (\alpha_i, \alpha_j, 2) : 1 \le i, j \le k \,\}$. The links in L_k and $\overline{L}_k$ may be pictured as in Figure 1.



The links in L_k may be pictured as on the left and those in $\overline{L}_k$ as on the right.

Figure 1

The images of $\sum_{T \in \mathbf{T}^{(L_k)}} T$ and $\sum_{S \in \mathbf{T}^{(\overline{L}_k)}} (-1)^{|S|} S$ in $R[[a_1, a_2, \dots, a_k, x]]$ will be denoted by $\beta_k(x)$ and $\overline{\beta}_k(x)$. Note that L_0 is empty; $T^{(L_0)}$ and $T^{(\overline{L}_0)}$ each contain one tree, that consisting of a single vertex. If we set $a_1 = a_2 = \cdots = a_k = 1$, it is easy to see that for $k \geq 1$,

$$\beta_k(x) = \sum_{n=0}^{\infty} k^n x^{n+1} = \frac{x}{1-kx}$$

and

$$\overline{\beta}_k(x) = \sum_{n=0}^{\infty} (-1)^n k^n x^{n+1} = \frac{x}{1+kx},$$

while $\beta_0(x) = \overline{\beta}_0(x) = x$. The Inversion Theorem confirms the (obvious) fact that $\beta_k(x)$ and $\overline{\beta}_k(x)$ are compositional inverses for all $k \geq 0$.

Note that $\beta_0(x) = x$, $\beta_1(x) = \dfrac{x}{1-x}$, and for each $k \geq 1$, $\beta_1(x) \circ \beta_{k-1}(x) = \beta_k(x)$. Now if $f(x)$ is any formal power series, the *iterate of order k of f*, denoted by $f^{<k>}$, is defined as follows:

$$f^{<0>} = x,\ f^{<1>} = f,\ f^{<2>} = f \circ f, \ldots,\ f^{<k>} = f \circ f^{<k-1>}.$$

So

$$\beta_k(x) = \left(\frac{x}{1-x} \right)^{<k>}.$$

Let $O(x^j)$ denote some unspecified formal power series in which every term is divisible by x^j. Then it is straightforward (see, e.g., [**P**], pp.42–44) to show that if $f(x)$ is a formal power series and $f(x) = x + O(x^{r+1})$, for $r \geq 1$, then $[x^d]f^{<k>}$ is a polynomial in k of degree at most $\lfloor \frac{d-1}{r} \rfloor$. In the previous example, for instance, $f(x) = \dfrac{x}{1-x}$, $r = 1$ and the coefficient of x^{n+1} in $f^{<k>}$ is k^n. As mentioned earlier, $\sum\limits_{n=0}^{\infty} k^n t^k = \dfrac{A_n(t)}{(1-t)^{n+1}}$, and the coefficient of t^j in $A_n(t)$ is the Eulerian number $A_{n,j}$.

2.3 The case $(x + x^2)^{<k>}$

2.3.1 Introduction

Next we consider a different sequence of subsets of $\mathbf{L}$. Again, let $m = 2$. Let the alphabet $\mathcal{A}$ be linearly ordered by $\alpha_i < \alpha_j$ if and only if $i < j$, for each i, j in $\mathbf{N}$, and for each $k \geq 0$ let $L_k = \{ (\alpha_i, \alpha_j, q) : q \in [2],\ i > j \geq 1 \} \cap \mathbf{L}_k$. Then $L_k = \{ (\alpha_i, \alpha_j, q) : q \in [2],\ k \geq i > j \geq 1 \}$ and its complement in $\mathbf{L}_k$ is $\overline{L}_k = \{ (\alpha_i, \alpha_j, q) : q \in [2],\ k \geq j \geq i \geq 1 \}$. For example, the two links on the left in Figure 2 belong to L_k, while the two on the right belong to $\overline{L}_k$.

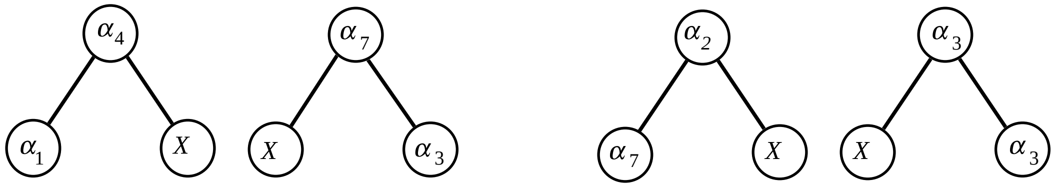


Figure 2

Any tree in $\mathbf{T}^{(L_k)}$ either has root α_i, where $i < k$, or has root α_k. Hence we get the recurrence

$$\sum_{T \in \mathbf{T}^{(L_k)}} T = \sum_{T \in \mathbf{T}^{(L_{k-1})}} T + \alpha_k \Bigg(\sum_{T \in \mathbf{T}^{(L_{k-1})}} T \Bigg)^2 \qquad \text{for } k \geq 1.$$

So if we set $a_1 = a_2 = \cdots = a_k = 1$, we get that $\beta_k(x) = \beta_{k-1}(x) + (\beta_{k-1}(x))^2$ for $k \geq 1$. Since $\beta_0(x) = x$ and $\beta_1(x) = x + x^2$, it follows that

$$\beta_k(x) = (x + x^2)^{<k>}.$$

Therefore, $\beta_k(x)$ is again the kth iterate of a formal power series. In this case, $\beta_1(x) = x + x^2 = x + O(x^{r+1})$, where r equals 1. So for each $k \geq 0$ we can write

$$\beta_k(x) = \sum_{n=0}^{\infty} p_n(k) x^{n+1},$$

where $p_n(k)$ is a polynomial in k of degree at most n; we call $p_n(k)$ the *nth iteration polynomial of $x + x^2$*. It is well known (see, e.g., [**S2**], p. 204) that if $p_n(k)$ is a polynomial in k of degree at most n, then we can define a polynomial $P_n(t)$ in t of degree at most n by

$$\sum_{k=0}^{\infty} p_n(k) t^k = \frac{P_n(t)}{(1-t)^{n+1}}.$$

We would like to know whether the coefficient $P_{n,j}$ of t^j in $P_n(t)$ has a combinatorial interpretation analogous to that of the Eulerian number $A_{n,j}$.

2.3.2 A recurrence for the iteration polynomial $p_n(k)$

Note that $p_n(k)$ is the number of trees in $\mathbf{T}^{(L_k)}$ with $n+1$ leaves (and hence n internal vertices). For each $k \geq 0$, $p_0(k)$ counts the tree consisting of a single vertex, lettered with an X, so $p_0(k) = 1$. Clearly $p_n(0) = 0$ for all $n > 0$. The following lemma gives a recurrence for $p_n(k)$ for $n \geq 0$, $k > 0$.

Lemma 2.3.2. $p_n(k) = \sum_{i=\lceil \frac{n-1}{2} \rceil}^{n} \binom{i+1}{n-i} p_i(k-1)$ *for* $n \geq 0$, $k > 0$.

Proof. Since $\beta_k(x) = (x + x^2)^{<k>}$, $\beta_k(x) = \beta_{k-1}(x) \circ (x + x^2)$, so

$$\begin{aligned}
\beta_k(x) &= \sum_{i=0}^{\infty} p_i(k-1)(x+x^2)^{i+1} \\
&= \sum_{i=0}^{\infty} p_i(k-1) \sum_{j=0}^{i+1} \binom{i+1}{j} x^{j+i+1} \\
&= \sum_{i=0}^{\infty} \sum_{n=i}^{2i+1} p_i(k-1) \binom{i+1}{n-i} x^{n+1} \qquad \text{(by setting } n = j+i\text{)} \\
&= \sum_{n=0}^{\infty} \sum_{i=\lceil \frac{n-1}{2} \rceil}^{n} p_i(k-1) \binom{i+1}{n-i} x^{n+1} \text{ (by changing the order of summation).}
\end{aligned}$$

Since $\beta_k(x) = \sum_{n=0}^{\infty} p_n(k)x^{n+1}$, equating coeffcients gives the desired result.

2.3.3 Counting sequences by s-descents

We need some additional definitions. Let Ω be a set of finite sequences of positive integers, and for each $n \geq 0$, let Ω_n be the set of sequences in Ω of length n. Let $\gamma = c_1c_2\cdots c_n$ belong to Ω_n. We define the *spaces* and *proper spaces* of γ exactly as we did for the permutation π. Now let $s : \Omega \to 2^{\mathbf{N}}$ be a function which attaches to each sequence γ of length n a subset of the set of proper spaces of γ which includes n; we call $s(\gamma)$ the *s-descent set of* γ and call an element of $s(\gamma)$ an *s-descent of* γ.

Let $n \geq 0$. Suppose we wish to count the elements of Ω_n according to s-descents; one of the easiest ways to do so is as follows. Let $\gamma = c_1c_2\cdots c_n$ be in Ω_n. We define a *barred sequence on* γ to be a sequence of positive integers and bars formed from γ by inserting bars in some of the spaces of γ. For example, if $\gamma = 14173$, then $||1|41|7|||3|$ is a barred sequence on γ. Let B_n be the set of barred sequences on elements of Ω_n with at least one bar in each s-descent. Let $B_{n,k}$ be the set of elements of B_n with k bars, and let $b_n(k) = |B_{n,k}|$.

Lemma 2.3.3. $\sum_{k=0}^{\infty} b_n(k)t^k = \dfrac{D_n(t)}{(1-t)^{n+1}}$*, where* $D_n(t) = \sum_{j=0}^{n} D_{n,j}\, t^j$ *and* $D_{n,j}$ *is the number of sequences in* Ω_n *with* j *s-descents.*

Proof. If γ is in Ω_n, let $d(\gamma)$ be the number of s-descents of γ. Note that there are $n+1$ spaces in γ into which bars may be inserted. Since every element of $B_{n,k}$ can be obtained uniquely from some γ in Ω_n by first inserting a bar into each of the $d(\gamma)$ s-descents of γ and then inserting $k - d(\gamma)$ bars arbitrarily into the $n+1$ spaces of γ,

$$\begin{aligned}
\sum_{k=0}^{\infty} b_n(k)t^k &= \sum_{\gamma\in\Omega_n} t^{d(\gamma)}\ (1+t+t^2+\cdots)^{n+1} \\
&= \frac{\sum_{j=0}^{n} D_{n,j}\, t^j}{(1-t)^{n+1}}.
\end{aligned}$$

The basic idea of this proof goes back to MacMahon, in his investigations of the "Lattice Function" and "Permutation Functions" [**M2**].

2.3.4 A combinatorial interpretation for $P_n(t)$

Let Ω be the set of all finite sequences $\gamma = c_1c_2\cdots c_r$ of positive integers such that $1 \le c_i \le i$ for each i. For each $n \ge 0$, let Ω_n be the set of elements of Ω of length n. If $\gamma = c_1c_2\cdots c_n$ belongs to Ω_n, we define the s-descent set of γ to be $s(\gamma) = \{\, i \in [n] : c_i \le c_{i+1} \text{ or } i = n\,\}$. Let B_n be the set of barred sequences on elements of Ω_n with at least one bar in each s-descent. Let $B_{n,k}$ be the set of elements of B_n with k bars and let $b_n(k) = |B_{n,k}|$.

Theorem 2.3.4. *$b_n(k) = p_n(k)$, for all $n \ge 0$, $k \ge 0$.*

Proof. For each $k \ge 0$, $b_0(k)$ counts the sequence consisting only of k bars, so $b_0(k) = 1$. Since every barred sequence in $B_{n,k}$ must have a bar in its final space, $b_n(0) = 0$ for all $n > 0$. Now suppose that $\bar\gamma$ belongs to $B_{i,k-1}$ for some i such that $0 \le i \le n$; then $\bar\gamma$ is a barred sequence on some sequence $\gamma = c_1c_2\cdots c_i$ with $k-1$ bars and with at least one bar in each s-descent of γ. We can construct an element of $B_{n,k}$ from $\bar\gamma$ by choosing $n-i$ integers $c_{i+1}, c_{i+2}, \ldots, c_n$ such that $1 \le c_j \le j$ for each j that satisfies $i+1 \le j \le n$ and such that $c_j > c_{j+1}$ for each j that satisfies $i+1 \le j \le n-1$. If we choose an $(n-i)$-element subset of $[i+1]$ and arrange it in decreasing order we will have a set of the desired form. There are $\binom{i+1}{n-i}$ ways to choose such a set. If we insert the sequence $c_{i+1}c_{i+2}\cdots c_n$ after the final bar of $\bar\gamma$ and put a bar in space n of the resulting sequence, we have an element of $B_{n,k}$. Since each sequence in $B_{n,k}$ can be constructed in this way from a unique sequence $\bar\gamma$ in $B_{i,k-1}$ for some i such that $0 \le i \le n$, we have

$$b_n(k) = \sum_{i=0}^{n} \binom{i+1}{n-i} b_i(k-1).$$

Since $\binom{i+1}{n-i} = 0$ if $i < \lceil \frac{n-1}{2} \rceil$, we can rewrite this as

$$b_n(k) = \sum_{i=\lceil \frac{n-1}{2} \rceil}^{n} \binom{i+1}{n-i} b_i(k-1).$$

Therefore $b_n(k)$ and $p_n(k)$ satisfy the same initial conditions and the same recurrence, so $b_n(k) = p_n(k)$.

Theorem 2.3.4 and Lemma 2.3.3 together give us a combinatorial interpretation for the numbers $P_{n,j}$. Since

$$\sum_{k=0}^{\infty} p_n(k)t^k = \frac{\sum_{j=0}^{n} P_{n,j}\, t^j}{(1-t)^{n+1}}$$

and since $p_n(k)$ is the number of sequences in $B_{n,k}$, $P_{n,j}$ is the number of sequences in Ω_n with j s-descents. So for $n > 0$, $P_{n,j}$ is the number of sequences $c_1c_2\cdots c_n$ such that $1 \le c_i \le i$ for each i in $[n]$ with exactly $j-1$ values of i such that $c_i \le c_{i+1}$.

Table 1 gives values for $P_{n,j}$ for $0 \le n, j \le 8$.

$n \backslash j$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	0	2	0	0	0	0	0	0
3	0	0	1	5	0	0	0	0	0
4	0	0	0	10	14	0	0	0	0
5	0	0	0	8	70	42	0	0	0
6	0	0	0	4	160	424	132	0	0
7	0	0	0	1	250	1978	2382	429	0
8	0	0	0	0	302	6276	19508	12804	1430

The numbers $P_{n,j}$

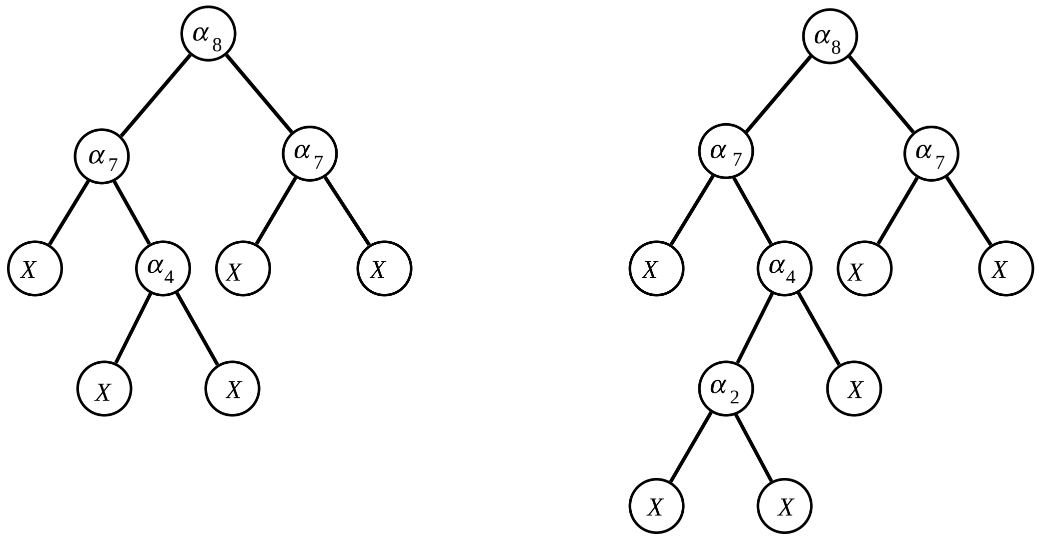
Table 1

2.3.5 A bijection between linked trees and barred sequences Let $\mathbf{T}_n^{(L_k)}$ be the set of trees in $\mathbf{T}^{(L_k)}$ with n internal vertices; then Theorem 2.3.4 implies that for each $n \ge 0$ and $k \ge 0$, there is a bijection $\phi_{n,k} : B_{n,k} \to \mathbf{T}_n^{(L_k)}$. We now give an explicit description of such a bijection. First we make a definition: if a leaf of tree T is the ith leaf to appear when $\operatorname{ord}(T)$ is read from *right to left*, then we call it the ith *leaf of* T.

Let $\bar{\gamma}$ be in $B_{n,k}$; then $\bar{\gamma}$ is a barred sequence on some $\gamma = c_1 c_2 \cdots c_n$ with k bars and at least one bar in each s-descent. We define $\phi_{n,k}(\bar{\gamma})$ inductively. If $n = 0$, we define $\phi_{n,k}(\bar{\gamma})$ to be the tree consisting of a single vertex, lettered with an X. If $n > 0$, let $\bar{\gamma}'$ be the element of $B_{n-1,k}$ obtained by deleting c_n from $\bar{\gamma}$. Let $T' = \phi_{n-1,k}(\bar{\gamma}')$. Note that T' has n leaves and recall that $1 \le c_n \le n$. Let l be the number of bars to the left of c_n in $\bar{\gamma}$. We construct $\phi_{n,k}(\bar{\gamma})$ from T' by first replacing the c_nth leaf of T' with a vertex v lettered with α_{k-l} and then adding two children to v, each lettered with an X.

For example, let $\bar{\gamma} = 1|21|||3||4||$. Then $n = 5$, $k = 8$, $l = 6$ and $\bar{\gamma}' = 1|21|||3||||$. The tree T' on the left in Figure 3 is $\phi_{4,8}(\bar{\gamma}')$; the tree T on the right is $\phi_{5,8}(\bar{\gamma})$.

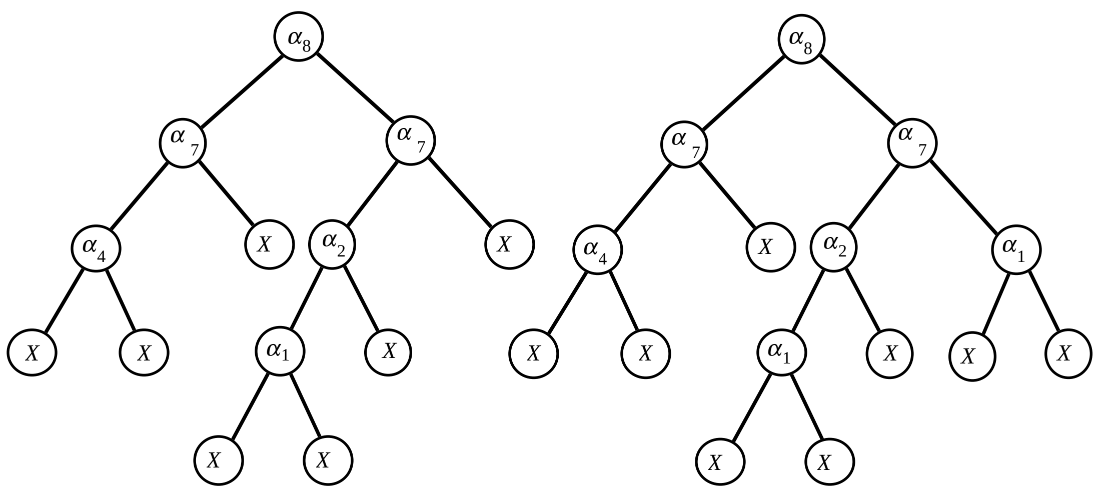
Next we define a map $\psi_{n,k} : \mathbf{T}_n^{(L_k)} \to B_{n,k}$; we leave it to the reader to show that $\psi_{n,k} = \phi_{n,k}^{-1}$ for each $n \ge 0$, $k \ge 0$. Let T be in $\mathbf{T}_n^{(L_k)}$; then T has n internal vertices, each lettered with an element of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. Let α_m be the smallest letter to appear in T. We define $\psi_{n,k}(T)$ inductively. If $n = 0$, we define $\psi_{n,k}(T)$ to be the barred sequence consisting of k bars. If $n > 0$, let v be the last vertex lettered with α_m to appear in $\operatorname{ord}(T)$ (read in the usual way, from left to right). Let T' be the tree obtained from T by deleting the children of v (which must be leaves of T) and replacing the letter α_m attached to v with an X. Then $\psi_{n-1,k}(T')$ is a barred sequence $\bar{\gamma}'$ on some $c_1 c_2 \cdots c_{n-1}$ with k bars. Now the vertex v is the ith leaf of T', for some i in $[n]$. We construct $\psi_{n,k}(T)$ from $\bar{\gamma}'$ by letting $c_n = i$ and inserting it into $\bar{\gamma}'$ so that there are $k - m$ bars to its left. Since $1 \leq m \leq k$, it is always possible to do this and still have a final bar in the resulting sequence.



The trees T' (left) and T

Figure 3

For example, suppose T is the tree shown on the right in Figure 4. Then $n = 7$, $k = 8$ and $m = 1$, and T' is the tree shown on the left in Figure 4. Moreover, $\psi_{6,8}(T') = \bar{\gamma}' = 1|21|||4||2|3|$, while $\psi_{7,8}(T) = \bar{\gamma} = 1|21|||4||2|31|$.



The trees T' (left) and T

Figure 4

2.3.6 The t-descent set of a tree and reduced trees

The existence of the bijection $\psi_{n,k}$ between $\mathbf{T}_n^{(L_k)}$ and $B_{n,k}$ suggests that it may be possible to associate to each tree T in $\mathbf{T}_n^{(L_k)}$ a "descent set" S in such a way that S equals the s-descent set of the sequence corresponding to T under $\psi_{n,k}$. We define such a set S as follows:

Suppose that T is in $\mathbf{T}_n^{(L_k)}$ and that for each j in $[k]$ there are q_j vertices of T lettered with α_{k-j+1}; we will denote these vertices by $v_{q_1+\cdots+q_{j-1}+1}$, $v_{q_1+\cdots+q_{j-1}+2}$, $\ldots$, $v_{q_1+\cdots+q_j}$, in the order in which they appear in $\mathrm{ord}(T)$. Then we define the *t-descent set of* T to be $\{\, i \in [n] : v_{i+1}$ precedes v_i in $\mathrm{ord}(T)$ or $i = n \,\}$.

For example, suppose that T is the tree shown on the left in in Figure 5. If we label the internal vertices of T in the manner described above, the result is the tree shown on the right in Figure 5.

Using this labeling of the vertices of T, and ignoring the leaves of T, we get that $\mathrm{ord}(T) = v_4v_2v_7v_6v_5v_3v_1$, so the t-descent set of T is $\{\,1,3,5,6,7\,\}$. Notice that $\psi_{7,8}(T) = 1|21|||4|2|2|4|$ and that the s-descent set of 1214224 is $\{\,1,3,5,6,7\,\}$. In fact, if T belongs to $\mathbf{T}_n^{(L_k)}$ and the internal vertices of T are labeled in the above manner, and if $\psi_{n,k}(T) = \bar{\gamma}$, where $\bar{\gamma}$ is a barred sequence on $\gamma = c_1c_2\cdots c_n$ in Ω_n, then v_{i+1} precedes v_i in $\mathrm{ord}(T)$ if and only if $c_i \le c_{i+1}$. So i is a t-descent of T if and only if it is an s-descent of γ.

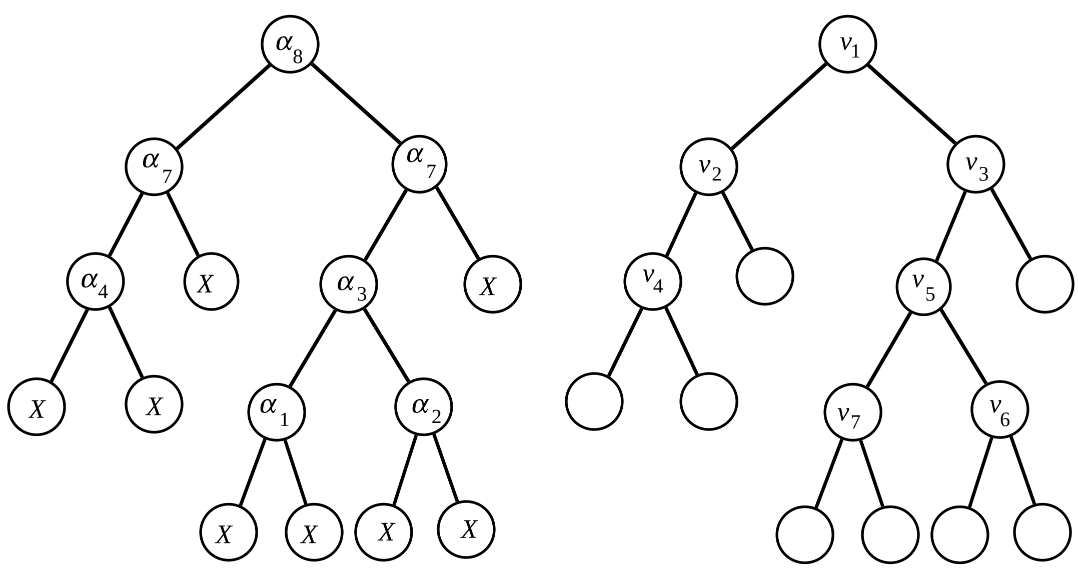


Figure 5

We now ask if we can find a descent-preserving bijection between matched $\mathcal{A}$-trees with n internal vertices and (unbarred) sequences $\gamma = c_1c_2\cdots c_n$ in Ω_n. The natural way to proceed would appear to be to use the bijection $\psi_{n,k} : \mathbf{T}_n^{(L_k)} \to B_{n,k}$ defined in Section 2.3.4 and then remove the bars from $\psi_{n,k}(T)$, for each T in $\mathbf{T}_n^{(L_k)}$. The difficulty with this, however, is that given any $k \ge |s(\gamma)|$, we can find at least one tree in $\mathbf{T}_n^{(L_k)}$ which will correspond to γ. For example, Figure 6 shows two trees in $\mathbf{T}_3^{(L_8)}$ and one in $\mathbf{T}_3^{(L_2)}$, all of which correspond to the sequence 121. Suppose, however, that given any sequence γ in Ω_n, we can find the unique tree T which corresponds under $\psi_{n,|s(\gamma)|}$ to the barred sequence on γ which has a bar in space i if and only if i is an s-descent of γ. Then we can define a bijection, in the manner described above, between this subset of the set of matched $\mathcal{A}$-trees and Ω_n.

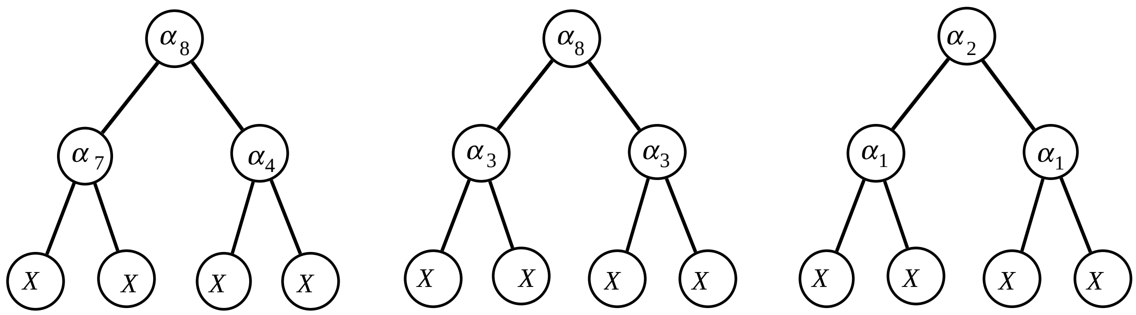


Figure 6

Let T be a matched $\mathcal{A}$-tree, and let $d(T)$ be the cardinality of the t-descent set of T. We will say that T is a *reduced tree* if

(1) $\lambda(v_1) = \alpha_{d(T)}$,
(2) $\lambda(v_{i+1}) = \lambda(v_i) - 1$ whenever i is a t-descent of T,
(3) $\lambda(v_{i+1}) = \lambda(v_i)$ whenever i is not a t-descent of T.

Note that the first two trees (from left to right) in Figure 6 are not reduced, whereas the third is reduced.

It is clear from the definition of a reduced tree that given any $n \geq 0$ and any subset S of $[n]$ containing n, there is a unique reduced matched $\mathcal{A}$-tree with n internal vertices and with t-descent set S. Moreover, T belongs to $\mathbf{T}_n^{(L_{|S|})}$ and it follows from the definition of $\psi_{n,|S|}$ that there is a bar in space i of $\psi_{n,|S|}(T)$ if and only if i is a t-descent of T. Recall that this implies that there is a bar in space i of $\psi_{n,|S|}(T)$ if and only if i is an s-descent of the underlying sequence. Therefore, given any sequence $\gamma = c_1c_2\cdots c_n$ in Ω_n with s-descent set $s(\gamma)$, there is a unique reduced tree T in $\mathbf{T}_n^{(L_{|s(\gamma)|})}$ such that $\psi_{n,|s(\gamma)|}(T)$ is the barred sequence on γ having a bar in space i if and only if i is in $s(\gamma)$. Hence we can remove the bars from our sequences and we have a descent-preserving bijection between the set of reduced matched $\mathcal{A}$-trees with n internal vertices and the set Ω_n.

2.3.7 Generalization to $(x + x^m)^{<k>}$

Now let m be an arbitrary positive integer; we can generalize the above discussion to m-ary trees. For each $k \geq 0$, let $L_k^{(m)} = \{\, (\alpha_i, \alpha_j, q) : q \in [m],\ i > j \geq 1 \,\} \cap \mathbf{L}_k$; then $L_k^{(m)} = \{\, (\alpha_i, \alpha_j, q) : q \in [m],\ k \geq i > j \geq 1 \,\}$. We denote the image of

$\sum\limits_{T\in\mathbf{T}^{(L_k^{(m)})}} T$ in $R[[a_1,a_2,\ldots,a_k,x]]$ by $\beta_k^{(m)}(x)$. If we set $a_1=a_2=\cdots=a_k=1$, then an argument similar to that used in the case $m=2$ gives us the recurrence $\beta_k^{(m)}(x)=\beta_{k-1}^{(m)}(x)+(\beta_{k-1}^{(m)}(x))^m$, for $k\geq 1$. Since $\beta_0^{(m)}(x)=x$ and $\beta_1^{(m)}(x)=x+x^m$,

$$\beta_k^{(m)}(x)=(x+x^m)^{<k>}.$$

In this case, $x+x^m=x+O(x^{r+1})$, where $r=m-1$, so we can write

$$\beta_k^{(m)}(x)=\sum_{n=0}^{\infty}p_n^{(m)}(k)x^{(m-1)n+1},$$

where $p_n^{(m)}(k)$ is a polynomial in k of degree at most n; for each $k\geq 0$, $n\geq 0$, $p_n^{(m)}(k)$ is the number of trees with links in $L_k^{(m)}$ with n internal vertices and $(m-1)n+1$ leaves. As in the binary case, $p_0^{(m)}(k)=1$ for each $k\geq 0$ and $p_n^{(m)}(0)=0$ for all $n>0$. Moreover, we have the recurrence

$$p_n^{(m)}(k)=\sum_{i=\lceil\frac{n-1}{m}\rceil}^{n}\binom{(m-1)i+1}{n-i}p_i^{(m)}(k-1)\quad\text{for } n\geq 0, k>0.$$

Let $\Omega^{(m)}$ be the set of all finite sequences $\gamma=c_1c_2\cdots c_r$ of positive integers such that $1\leq c_i\leq(m-1)i-(m-2)$ for each i; for each $n\geq 0$, let $\Omega_n^{(m)}$ be the set of elements of length n. As before, we define the s-descent set of $\gamma=c_1c_2\cdots c_n$ to be $s(\gamma)=\{\,i\in[n]: c_i\leq c_{i+1}\text{ or } i=n\,\}$. Let $B_n^{(m)}$ be the set of barred sequences on elements of $\Omega_n^{(m)}$ with at least one bar in each s-descent and let $B_{n,k}^{(m)}$ be the set of elements of $B_n^{(m)}$ with k bars. If we let $b_n^{(m)}(k)=|B_{n,k}^{(m)}|$, then a proof similar to that of Theorem 2.3.4 establishes that $p_n^{(m)}(k)=b_n^{(m)}(k)$. Therefore, if we write

$$\sum_{k=0}^{\infty}p_n^{(m)}(k)t^k=\frac{P_n^{(m)}(t)}{(1-t)^{n+1}},$$

where $P_n^{(m)}(t)=\sum\limits_{j=0}^{n}P_{n,j}^{(m)}t^j$, then $P_{n,j}^{(m)}$ is the number of sequences in $\Omega^{(m)}$ with j s-descents. So for $n>0$, $P_{n,j}^{(m)}$ is the number of sequences $c_1c_2\cdots c_n$ such that $1\leq c_i\leq(m-1)i-(m-2)$ for each i in $[n]$ with exactly $j-1$ values of i such that $c_i\leq c_{i+1}$.

Since $p_n^{(m)}(k)=b_n^{(m)}(k)$, there is a bijection $\phi_{n,k}^{(m)}: B_{n,k}^{(m)}\to\mathbf{T}_n^{(L_k^{(m)})}$ for each $n\geq 0$, $k\geq 0$. The definition of $\phi_{n,k}$ given earlier can easily be extended to a definition of $\phi_{n,k}^{(m)}$ for any $m\geq 2$. So for each $m\geq 2$, we get a descent-preserving bijection between reduced trees in $\mathbf{T}_n^{(L_k^{(m)})}$ and (unbarred) sequences of integers $c_1c_2\cdots c_n$ with $1\leq c_i\leq(m-1)i-(m-2)$.

2.4 The case $\left(\dfrac{x}{1-x^{m-1}}\right)^{<k>}$

2.4.1 Introduction

In this section we consider another sequence of subsets of $\mathbf{L}$. As before, let $\mathcal{A} = \{\, \alpha_1, \alpha_2, \ldots, \,\}$ be an alphabet such that $\deg \alpha_i = m$, for each i, and let $\mathcal{A}$ be linearly ordered as in Section 2.3.

For each $k \geq 0$, let

$$L_k^{(m)} = \big(\{\, (\alpha_i, \alpha_j, q) : q \in [m-1], i > j \geq 1 \,\} \cup \{\, (\alpha_i, \alpha_j, m) : i \geq j \geq 1 \,\}\big) \cap \mathbf{L}_k.$$

So

$$L_k^{(m)} = \{\, (\alpha_i, \alpha_j, q) : q \in [m-1], k \geq i > j \geq 1 \,\} \cup \{\, (\alpha_i, \alpha_j, m) : k \geq i \geq j \geq 1 \,\}.$$

A typical tree in $\mathbf{T}_n^{(L_k^{(3)})}$ is shown in Figure 7.

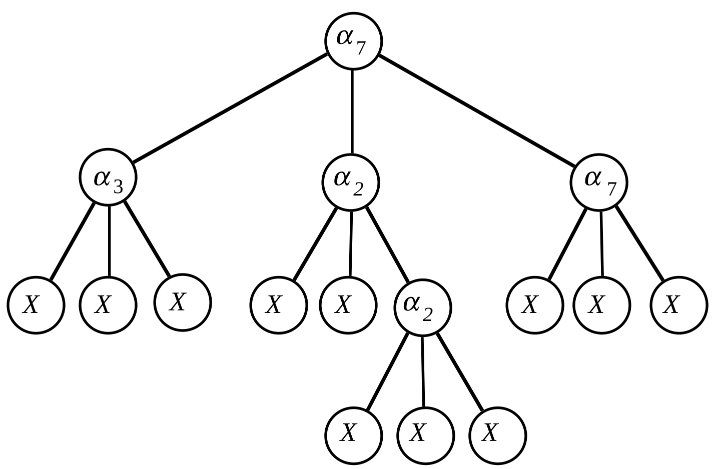


Figure 7

Let $\beta_k^{(m)}(x)$ be the image of $\sum_{T \in \mathbf{T}^{(L_k^{(m)})}} T$ in $R[[a_1, a_2, \ldots, a_k, x]]$. It is easily seen that

$$\beta_k^{(m)}(x) = \beta_{k-1}^{(m)}(x) + \big(\beta_{k-1}^{(m)}(x)\big)^{m-1} \cdot \beta_k^{(m)}(x).$$

Now each tree in $\mathbf{T_n}^{(L_1^{(m)})}$ has the form shown in Figure 8 (where there are n vertices lettered with α_1), so

$$\beta_1^{(m)}(x) = x + x^m + x^{2m-1} + \cdots = \frac{x}{1-x^{m-1}}.$$

Since $\beta_0(x) = x$,

$$\beta_k^{(m)}(x) = \left(\frac{x}{1-x^{m-1}}\right)^{<k>}.$$

In this case, $\dfrac{x}{1-x^{m-1}} = x + O(x^{r+1})$, where $r = m-1$, so we can write

$$\beta_k^{(m)}(x) = \sum_{n=0}^{\infty} r_n^{(m)}(k) x^{(m-1)n+1},$$

where $r_n^{(m)}(k)$ is a polynomial in k of degree at most n and counts trees with links in $L_k^{(m)}$ with n internal vertices and $(m-1)n+1$ leaves. Since $r_n^{(m)}(k)$ is a polynomial in k of degree at most n, we can define a polynomial $R_n^{(m)}(t)$ of degree at most n by

$$\sum_{k=0}^{\infty} r_n^{(m)}(k) t^k = \frac{R_n^{(m)}(t)}{(1-t)^{n+1}}.$$

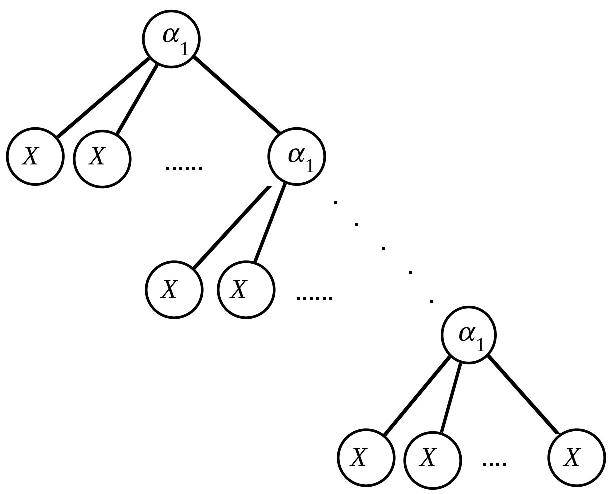


Figure 8

2.4.2 A reccurrence for $r_n^{(m)}(k)$

Clearly, $r_0^{(m)}(k) = 1$ for all $k \geq 0$ and $r_n^{(m)}(0) = 0$ for all $n > 0$. For $k > 0$, $n \geq 0$ we have the following reccurence for $r_n^{(m)}(k)$.

Lemma 2.4.2. $r_n^{(m)}(k) = \sum\limits_{i=0}^{n} \dbinom{n+(m-2)i}{(m-1)i} r_i^{(m)}(k-1)$.

Proof. Since $\beta_k^{(m)}(x) = \left(\dfrac{x}{1-x^{m-1}}\right)^{<k>}$, $\beta_k^{(m)}(x) = \beta_{k-1}^{(m)}(x) \circ \dfrac{x}{1-x^{m-1}}$, so

$$\begin{aligned}
\beta_k^{(m)}(x) &= \sum_{i=0}^{\infty} r_i^{(m)}(k-1) \left(\frac{x}{1-x^{m-1}}\right)^{(m-1)i+1} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{j+(m-1)i}{(m-1)i} r_i^{(m)}(k-1)x^{(m-1)(i+j)+1} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \binom{n+(m-2)i}{(m-1)i} r_i^{(m)}(k-1)x^{(m-1)n+1} \qquad \text{(by setting } n=i+j\text{)}.
\end{aligned}$$

Since $\beta_k^{(m)}(x) = \sum_{n=0}^{\infty} r_n^{(m)}(k)x^{(m-1)n+1}$, equating coefficients gives the desired result.

2.4.3 A combinatorial interpretation for $R_n^{(m)}(t)$

Let $\Omega^{(m)}$ be the set of all finite sequences $\gamma = c_1c_2\cdots c_r$ of positive integers such that $1 \le c_i \le (m-1)i-(m-2)$ for each i; for each $n \ge 0$, let $\Omega_n^{(m)}$ be the set of elements of length n. If $\gamma = c_1c_2\cdots c_n$ belongs to $\Omega_n^{(m)}$, we define the s-descent set of γ to be $s(\gamma) = \{\, i \in [n] : c_i \ge c_{i+1}-(m-2) \text{ or } i=n \,\}$. Let $B_n^{(m)}$ be the set of barred sequences on elements of $\Omega_n^{(m)}$ with at least one bar in each s-descent. Let $B_{n,k}^{(m)}$ be the set of elements of $B_n^{(m)}$ with k bars and let $b_n^{(m)}(k) = |B_{n,k}^{(m)}|$.

Theorem 2.4.3. $r_n^{(m)}(k) = b_n^{(m)}(k)$, *for all* $n \ge 0$, $k \ge 0$.

Proof. For each $k \ge 0$, $b_0^{(m)}(k)$ counts the sequence consisting only of k bars, so $b_0^{(m)}(k) = 1$. Since every barred sequence in $B_{n,k}^{(m)}$ must have a bar in its final space, $b_n^{(m)}(0) = 0$ for all $n > 0$. Now suppose that $\bar\gamma$ belongs to $B_{i,k-1}^{(m)}$ for some i in $[n]$; then $\bar\gamma$ is a barred sequence on some $\gamma = c_1c_2\cdots c_i$ with $k-1$ bars and at least one bar in each s-descent of γ. We can construct an element of $B_{n,k}^{(m)}$ from γ by choosing $n-i$ positive integers $c_{i+1}, c_{i+2}, \ldots, c_n$ such that $1 \le c_j \le (m-1)j-(m-2)$ for each j that satisfies $i+1 \le j \le n$ and such that $c_j < c_{j+1}-(m-2)$ for each j that satisfies $i+1 \le j \le n-1$. The number of ways to choose such a set of integers is the number of subsets $\{\, x_1, x_2, \ldots, x_{n-i} \,\}$ of $[(m-1)n-(m-2)]$ such that $x_j < x_{j+1}-(m-2)$ for each j in $[n-i-1]$, which, if we let $z_j = x_j-(j-1)(m-2)$, is the same as the number of subsets $\{\, z_1, z_2, \ldots, z_{n-i} \,\}$ of $[n+(m-2)i]$. So there are $\binom{n+(m-2)i}{n-i} = \binom{n+(m-2)i}{(m-1)i}$ ways to choose the integers $c_{i+1}, c_{i+2}, \ldots, c_n$. If we insert the sequence $c_{i+1}c_{i+2}\cdots c_n$ after the final bar of $\bar\gamma$ and put a bar in space n of the resulting sequence, we have an element of $B_{n,k}^{(m)}$. Since each sequence in $B_{n,k}^{(m)}$ can be constructed in this way from a unique sequence $\bar\gamma$ in $B_{i,k-1}^{(m)}$ for some

i such that $0 \le i \le n$, we have

$$b_n^{(m)}(k) = \sum_{i=0}^{n} \binom{n+(m-2)i}{(m-1)i} b_i^{(m)}(k-1).$$

Since $r_n^{(m)}(k)$ and $b_n^{(m)}(k)$ satisfy the same initial conditions and the same recurrence, they are equal.

So it follows from Theorem 2.4.3 and Lemma 2.3.3 that if $R_n^{(m)}(t) = \sum_{j=0}^{n} R_{n,j}^{(m)} t^j$, then $R_{n,j}^{(m)}$ is the number of sequences in $\Omega_n^{(m)}$ with j s-descents. So for $n > 0$, $R_{n,j}^{(m)}$ is the number of sequences $c_1 c_2 \cdots c_n$ such that $1 \le c_i \le (m-1)i-(m-2)$ for each i in $[n]$ with exactly $j-1$ values of i such that $c_i \ge c_{i+1}-(m-2)$.

Table 2 gives values for $R_{n,j}^{(3)}$ for $0 \le n, j \le 6$.

$n\backslash j$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	3	0	0	0	0
3	0	1	15	12	0	0	0
4	0	1	55	169	55	0	0
5	0	1	183	1470	1713	273	0
6	0	1	588	10488	29220	16515	1428

The numbers $R_{n,j}^{(3)}$

Table 2

2.4.4 A bijection between linked trees and barred sequences

Since $r_n^{(m)}(k) = b_n^{(m)}(k)$, there is a bijection $\rho_{n,k}^{(m)}$ between $B_{n,k}^{(m)}$ and $\mathbf{T_n}^{(L_k^{(m)})}$ for all $k \ge 0$, $n \ge 0$. Let T belong to $\mathbf{T_n}^{(L_k^{(m)})}$. We define another total ordering on the vertices of T; it is defined recursively as follows. Let r be the root of T and let $\tau_1, \tau_2, \ldots, \tau_m$ be the principal subtrees of T, listed in the order defining T as an ordered tree. Let $\widetilde{\mathrm{ord}}(T)$ be a listing of the vertices of T in this total order; then $\widetilde{\mathrm{ord}}(T) = \widetilde{\mathrm{ord}}(\tau_1), \widetilde{\mathrm{ord}}(\tau_2), \ldots, \widetilde{\mathrm{ord}}(\tau_{m-1}), v_0, \widetilde{\mathrm{ord}}(\tau_m)$. For example, if T is the tree shown in Figure 9, then $\widetilde{\mathrm{ord}}(T) = v_4, v_6, v_2, v_7, v_3, v_1, v_5$.

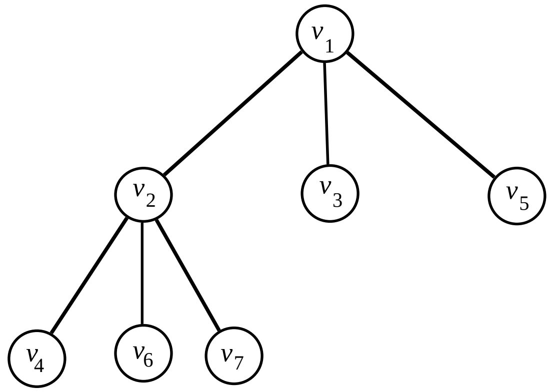


Figure 9

Here we define the ith leaf of T to be the ith leaf to appear in $\widetilde{\text{ord}}(T)$ (read in the usual way, from left to right). If we substitute this for the definition of the ith leaf of a tree T used in the construction of $\phi_{n,k}^{(m)}$ in the previous section, the result defines the bijection $\rho_{n,k}^{(m)}$.

For example, if we take $m = 3$, the tree in Figure 10 corresponds under $\rho_{9,7}^{(3)}$ to the barred sequence $13||138||6|9|2|1|$.

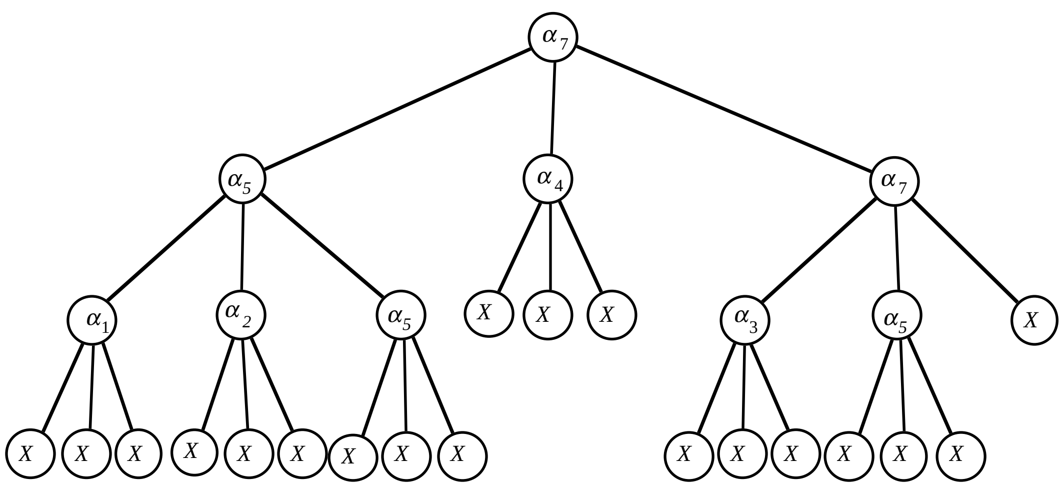


Figure 10

2.4.5 Stirling permutations and inverse descents

The case $m = 3$ provides an interesting combinatorial interpretation for these sequences. A *Stirling permutation* is a permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that if $i < j < k$ and $\pi_i = \pi_k$, then $\pi_j \geq \pi_i$. Stirling permutations were first studied by Gessel and Stanley in [**GS**]. The permutation 24421331, for example, is a Stirling permutation of the multiset $\{1^2, \ldots, 4^2\}$. We say that i is an *inverse descent* of a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$ if some $i+1$ occurs to the left of some i in π. For example, if $\pi = 14422133$ its inverse descents are 1 and 3.

Let Q be the set of all Stirling permutations, and for each $n \geq 0$ let Q_n be the set of all Stirling permutations of the multiset $\{1^2, \ldots, n^2\}$. If π is in Q_n, we define the spaces 0, 1, ... , $2n$ of π just as we did for ordinary permutations. Every permutation π in Q_n can be obtained uniquely by choosing a permutation π' in Q_{n-1} and inserting two consecutive n's into π'. There are $2n-1$ spaces into which the n's may be inserted. Hence each Stirling permutation π in Q_n corresponds uniquely to a sequence $\theta(\pi) = c_1c_2\cdots c_n$ of integers such that for each i in $[n]$, $1 \leq c_i \leq 2i-1$, that is, $1 \leq c_i \leq (m-1)i-(m-2)$ with $m=3$; the integer c_i in $\theta(\pi)$ signifies that two consecutive i's were inserted into space $c_i - 1$ during the construction of π. For example, if $\pi = 14422133$, $\theta(\pi) = 1252$. It is easily seen that i is an inverse descent of π if and only if two $i+1$'s are inserted into π to the left of some i, so i is an inverse descent of π if and only if $c_i \geq c_{i+1} - 1$ (that is, if $c_i \geq c_{i+1} - (m-2)$) in $\theta(\pi)$. Therefore, counting Stirling permutations of $\{1^2, \ldots, n^2\}$ by inverse descents is equivalent to counting sequences $c_1c_2\cdots c_n$ such that $1 \leq c_i \leq 2i-1$ according to s-descents, where we define the s-descent set of $c_1c_2\cdots c_n$ to be $\{i \in [n] : c_i \geq c_{i+1} - 1 \text{ or } i = n\}$. So for each j in $[n]$, there are $R^{(3)}_{n,j}$ Stirling permutations with exactly $j-1$ inverse descents.

We can generalize this interpretation. Let $r \geq 1$; we define a *multipermutation* of the multiset $\{1^r, 2^r, \ldots, n^r\}$ to be a permutation $\pi = \pi_1\pi_2\cdots\pi_{rn}$ such that if $i < j < k$ and $\pi_i = \pi_k$, then $\pi_j \geq \pi_i$ (see [**P**], p. 3). As before, i is an inverse descent of π if some $i+1$ occurs to the left of some i in π. An argument analogous to that used in the case $r=2$ shows that we can count multipermutations according to inverse descents by counting sequences $\theta(\pi) = c_1c_2\cdots c_n$ with $1 \leq c_i \leq ri-(r-1)$ according to s-descents, where we define the s-descent set of $\theta(\pi)$ to be $\{i \in [n] : c_i \geq c_{i+1} - (r-1) \text{ or } i = n\}$. Then for each j in $[n]$, there are $R^{(r+1)}_{n,j}$ multipermutations of $\{1^r, 2^r, \ldots, n^r\}$ with exactly $j-1$ inverse descents.

Note that if $r=1$, a multipermutation of $\{1^r, 2^r, \ldots, n^r\}$ is just a permutation of $[n]$. In this case,

$$\beta^{(r+1)}_k(k) = \beta^{(2)}_k(k) = \left(\frac{x}{1-x}\right)^{<k>} = \frac{x}{1-kx},$$

so $r^{(2)}_n(k) = k^n$. Since $\sum\limits_{k=0}^{\infty} k^n t^k = \dfrac{\sum_{j=0}^{n} A_{n,j}\, t^j}{(1-t)^{n+1}}$, the polynomial $R^{(2)}_n(t)$ is the Eulerian polynomial. Therefore descents and inverse descents have the same distribution on permutations of $[n]$. We note that if $r=1$, then i is an inverse descent of a permutation π if and only if it is a descent of π^{-1}. This explains the origin of the term *inverse descent*.

2.4.6 Reduced trees and Stirling permutations

Suppose that for each j in $[n]$ there are q_j vertices of T lettered with α_{k-j+1}; we will denote these vertices by $v_{q_1+\cdots+q_{j-1}+1}$, $v_{q_1+\cdots+q_{j-1}+2}$, ... , $v_{q_1+\cdots+q_j}$ in the order in which they appear in $\widetilde{\mathrm{ord}}(T)$. As we did before, we define the t-descent set of T to be $\{\, i \in [n] : v_{i+1}$ precedes v_i in $\widetilde{\mathrm{ord}}(T)$ or $i = n \,\}$.

If we label the internal vertices of the tree in Figure 10 in the manner described above, we get the tree shown in Figure 11. Since $\widetilde{\mathrm{ord}}(T) = v_9 v_8 v_3 v_4 v_6 v_1 v_7 v_5 v_2$, the t-descent set of T is $\{\, 2, 5, 7, 8, 9 \,\}$, which is the s-descent set of the sequence 131386921. It is easy to see that, as before, if $\bar{\gamma}$ belongs to $B^{(m)}_{n,k}$, then i is an s-descent of γ if and only if it is a t-descent of $\rho^{(m)}_{n,k}(\bar{\gamma})$.

If we define a reduced tree exactly as we did earlier, we again have a descent-preserving bijection between the set of reduced matched $\mathcal{A}$-trees with n internal vertices and (unbarred) sequences in $\Omega^{(m)}_n$. Consequently, there is a bijection between the set of reduced matched $\mathcal{A}$-trees with n internal vertices and multipermutations of the multiset $\{\, 1^{m-1}, 2^{m-1}, \ldots, n^{m-1} \,\}$ with the property that i in $[n]$ is a t-descent of a reduced tree if and only if it is an inverse descent of the corresponding multipermutation.

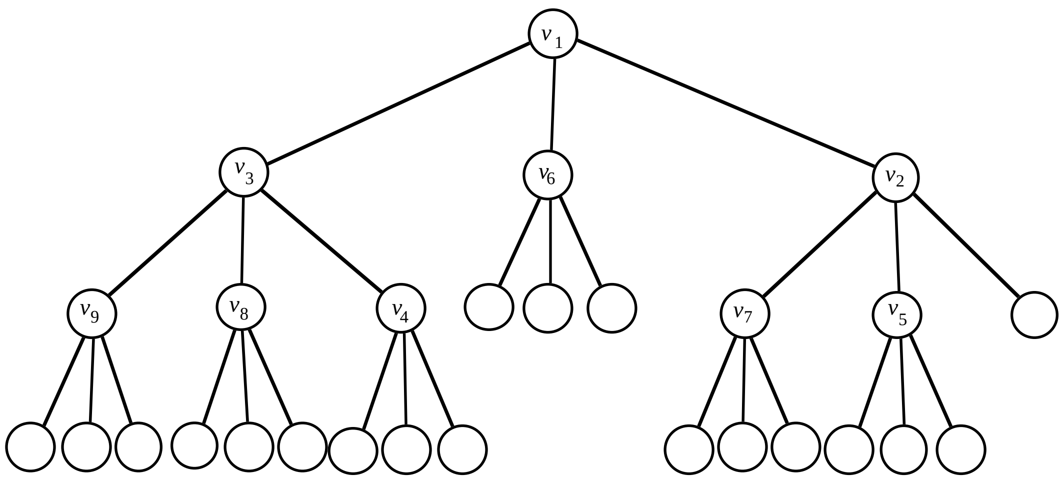


Figure 11

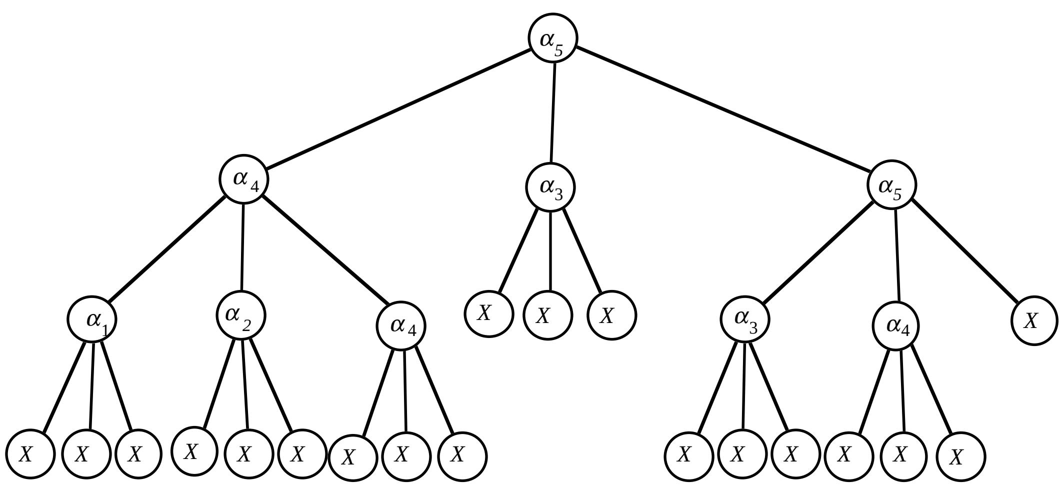


Figure 12

For example, if we reduce the tree shown in Figure 10, we get the tree shown in Figure 12, which corresponds to the sequence 131386921. The latter corresponds to the multipermutation $\pi = 993983441661772552$. As expected, the inverse descent set of π is $\{2, 5, 7, 8, 9\}$, which is the t-descent set of tree.

2.5 The general case

2.5.1 Introduction

Next we look at a generalization of the example we've been considering. Let p be an integer such that $1 \le p \le m$. We define a set of links $L_k^{(m,p)}$ by

$$L_k^{(m,p)} = \big(\{ (\alpha_i, \alpha_j, q) : 1 \le q \le p,\ i > j \ge 1 \} \\ \cup \{ (\alpha_i, \alpha_j, q) : p+1 \le q \le m,\ i \ge j \ge 1 \}\big) \cap \mathbf{L}_k.$$

Then

$$L_k^{(m,p)} = \{ (\alpha_i, \alpha_j, q) : 1 \le q \le p,\ k \ge i > j \ge 1 \} \\ \cup \{ (\alpha_i, \alpha_j, q) : p+1 \le q \le m,\ k \ge i \ge j \ge 1 \}.$$

(Note that in the previous example, we had $p = m - 1$.) If we let $\beta_k^{(m,p)}(x)$ denote the image of $\sum_{T \in \mathbf{T}^{(L_k^{(m,p)})}} T$ in $R[[a_1, a_2, \ldots, a_k, x]]$, then

$$\beta_k^{(m,p)}(x) = \beta_{k-1}^{(m,p)}(x) + (\beta_{k-1}^{(m,p)}(x))^p\, (\beta_k^{(m,p)}(x))^{m-p}.$$

Since $\beta_0^{(m,p)}(x) = x$,

$$\beta_1^{(m,p)}(x) = x + x^p\, (\beta_1^{(m,p)}(x))^{m-p}.$$

Moreover,

$$\beta_1^{(m,p)}(\beta_{k-1}^{(m,p)}) = \beta_{k-1}^{(m,p)} + (\beta_{k-1}^{(m,p)})^p\, (\beta_1^{(m,p)}(\beta_{k-1}^{(m,p)}))^{m-p} = \beta_k^{(m,p)}(x),$$

so

$$\beta_k^{(m,p)}(x) = (\beta_1^{(m,p)}(x))^{<k>}.$$

Note that if T belongs to $\mathbf{T}^{(L_1^{(m,p)})}$, then any vertex in T has q children, where $0 \le q \le m-p$. So for each $n \ge 0$ there is a bijection between $\mathbf{T_n}^{(L_1^{(m,p)})}$ and the set of $(m-p)$-ary trees with n internal vertices. Hence (see, e.g., [**GJ**], p. 112)

$$\beta_1^{(m,p)}(x) = \sum_{n=0}^{\infty} \frac{1}{(m-p)n+1} \binom{(m-p)n+1}{n} x^{(m-1)n+1}.$$

So

$$\beta_k^{(m,p)}(x) = \left(\sum_{n=0}^{\infty} \frac{1}{(m-p)n+1} \binom{(m-p)n+1}{n} x^{(m-1)n+1} \right)^{<k>}.$$

As before, for each $k \geq 0$ we can write

$$\beta_k^{(m,p)}(x) = \sum_{n=0}^{\infty} r_n^{(m,p)}(k) x^{(m-1)n+1},$$

where $r_n^{(m,p)}(k)$ is a polynomial in k of degree at most n and, moreover, where $r_n^{(m,p)}(k)$ is the number of trees with links in $L_k^{(m,p)}$ with n internal vertices and $(m-1)n+1$ leaves. Since $r_n^{(m,p)}(k)$ is a polynomial in k of degree at most n, we can define a polynomial $R_n^{(m,p)}(t)$ of degree at most n by

$$\sum_{k=0}^{\infty} r_n^{(m,p)}(k) t^k = \frac{R_n^{(m,p)}(t)}{(1-t)^{n+1}}.$$

2.5.2 A recurrence for $r_n^{(m,p)}(k)$

It is clear that for each $k \geq 0$, $r_0^{(m,p)}(k) = 1$, and for each $n > 0$, $r_n^{(m,p)}(0) = 0$.

Lemma 2.5.2. *We have the following recurrence for $n \geq 0$, $k > 0$:*

$$r_n^{(m,p)}(k) = \sum_{i=0}^{n} \frac{(m-1)i+1}{(m-p)n+(p-1)i+1} \binom{(m-p)n+(p-1)i+1}{n-i} r_i^{(m,p)}(k-1).$$

Proof. Since

$$\beta_k^{(m,p)}(x) = \beta_{k-1}^{(m,p)}(x) \circ \left(\sum_{j=0}^{\infty} \frac{1}{(m-p)j+1} \binom{(m-p)j+1}{j} x^{(m-1)j+1} \right),$$

we have that

$$\begin{aligned} &\beta_k^{(m,p)}(x) \\ &= \sum_{i=0}^{\infty} r_i^{(m,p)}(k-1) \left(\sum_{j=0}^{\infty} \frac{1}{(m-p)j+1} \binom{(m-p)j+1}{j} x^{(m-1)j+1} \right)^{(m-1)i+1}. \qquad (*) \end{aligned}$$

If we apply Lagrange inversion (see, e.g., [**CO**], p. 153) to the functional equation

$$\beta_1^{(m,p)}(x) = x + x^p (\beta_1^{(m,p)}(x))^{m-p},$$

we get that for any s in $\mathbf{Z}^+$

$$(\beta_1^{(m,p)}(x))^s = \sum_{l=0}^{\infty} \frac{s}{s+(m-p)l} \binom{s+(m-p)l}{l} x^{s+(m-1)l}.$$

Hence

$$\begin{aligned}
&(\beta_1^{(m,p)}(x))^{(m-1)i+1} \\
&\quad = \sum_{l=0}^{\infty} \frac{(m-1)i+1}{(m-1)i+(m-p)l+1} \binom{(m-1)i+(m-p)l+1}{l} x^{(m-1)i+(m-1)l+1}.
\end{aligned}$$

So

$$\begin{aligned}
(*) = \sum_{i=0}^{\infty} r_i^{(m,p)}(k-1) \sum_{l=0}^{\infty} &\frac{(m-1)i+1}{(m-1)i+(m-p)l+1} \\
&\times \binom{(m-1)i+(m-p)l+1}{l} x^{(m-1)i+(m-1)l+1}. \quad (**)
\end{aligned}$$

Setting $n = i + l$ gives us

$$\begin{aligned}
(**) = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} r_i^{(m,p)}(k-1) &\frac{(m-1)i+1}{(m-1)i+(m-p)(n-i)+1} \\
&\times \binom{(m-1)i+(m-p)(n-i)+1}{n-i} x^{(m-1)n+1}.
\end{aligned}$$

If we change the order of summation and equate coefficients, we get

$$r_n^{(m,p)}(k) = \sum_{i=0}^{n} \frac{(m-1)i+1}{(m-p)n+(p-1)i+1} \binom{(m-p)n+(p-1)i+1}{n-i} r_i^{(m,p)}(k-1).$$

2.5.3 A combinatorial interpretation for $R_n^{(m,p)}(t)$

For each $n \geq 0$, let $\Omega_n^{(m,p)}$ be the set of sequences $\gamma = c_1 c_2 \cdots c_n$ such that $1 \leq c_i \leq (m-1)i - (m-2)$ for each i in $[n]$. We define the s-descent set of γ to be $s(\gamma) = \{\, i \in [n] : c_i \geq c_{i+1} - (p-1) \text{ or } i = n \,\}$. Let $B_n^{(m,p)}$ be the set of barred sequences on elements of $\Omega_n^{(m,p)}$ having at least one bar in each s-descent, and let $B_{n,k}^{(m,p)}$ be the set of elements of $B_n^{(m,p)}$ with k bars. Let $b_n^{(m,p)}(k)=|B_{n,k}^{(m,p)}|$.

Theorem 2.5.3. *$r_n^{(m,p)}(k) = b_n^{(m,p)}(k)$ for all $n \geq 0$, $k > 0$.*

Proof. We prove the theorem by defining a bijection $\rho_{n,k}^{(m,p)}$ between $B_{n,k}^{(m,p)}$ and $\mathbf{T_n}^{(L_k^{(m,p)})}$. Let T be a tree in $\mathbf{T_n}^{(L_k^{(m,p)})}$; we define a total ordering on the vertices of T as follows. Let r be the root of T and the $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the principal subtrees of T, listed in the order defining T as an ordered tree. Let $\widetilde{\mathrm{ord}}_{(p)}(T)$ be a listing of the vertices of T in this total order; then $\widetilde{\mathrm{ord}}_{(p)}(T)$=$\widetilde{\mathrm{ord}}_{(p)}(\lambda_1), \ldots, \widetilde{\mathrm{ord}}_{(p)}(\lambda_p)$, r, $\widetilde{\mathrm{ord}}_{(p)}(\lambda_{p+1}), \ldots, \widetilde{\mathrm{ord}}_{(p)}(\lambda_m)$. In the previous example, we defined a bijection $\rho_{n,k}^{(m)}$ between $B_{n,k}^{(m)}$and $\mathbf{T_n}^{(L_k^{(m)})}$; if we replace the total ordering on the vertices of trees in $\mathbf{T_n}^{(L_k^{(m)})}$ used in that definition with the total ordering just defined, we get a bijection $\rho_{n,k}^{(m,p)}$ between $B_{n,k}^{(m,p)}$ and $\mathbf{T_n}^{(L_k^{(m,p)})}$.

Once again, it follows from Theorem 2.5.3 and Lemma 2.3.3. that if $R_n^{(m,p)}(t) = \sum_{j=0}^{n} R_{n,j}^{(m,p)} t^j$, then $R_{n,j}^{(m,p)}$ is the number of sequences in $\Omega_n^{(m,p)}$ with j s-descents. So for $n > 0$, $R_{n,j}^{(m,p)}$ is the number of sequences $\gamma = c_1 c_2 \cdots c_n$ such that $1 \leq c_i \leq (m-1)i - (m-2)$ for each i in $[n]$ with exactly $j-1$ values of i such that $c_i \geq c_{i+1} - (p-1)$.

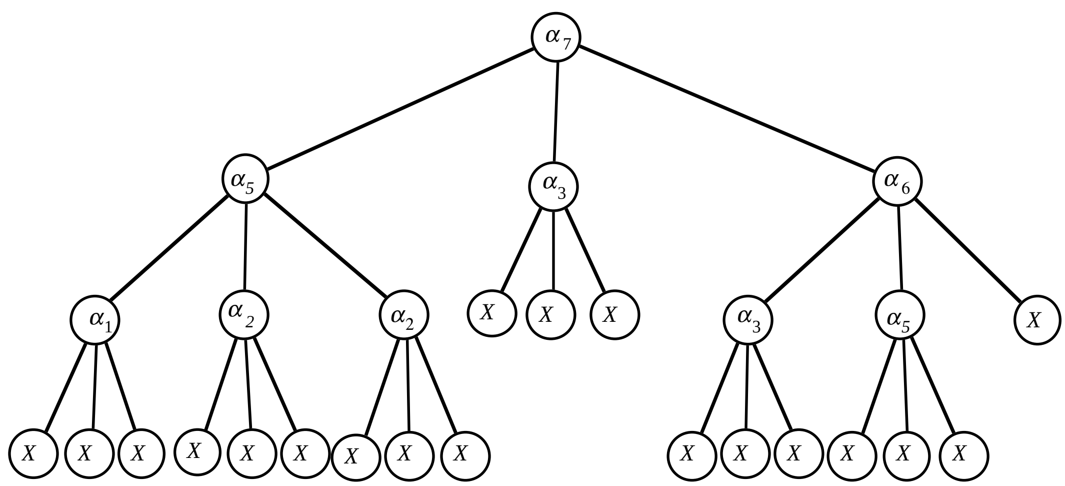


Figure 13

Notice that if $p = m$, then the set of links $L_k^{(m,m)}$ is the set we considered in section 2.3.7, namely $L_k^{(m,m)} = \{\, (\alpha_i, \alpha_j, q) : 1 \leq q \leq m,\, k \geq i > j \geq 1 \,\}$, and $\beta_1^{(m,p)}(x)$ is simply the polynomial $x + x^m$. In this earlier example, we defined a bijection $\phi_{n,k}^{(m)}$ between trees with links in $L_k^{(m,m)}$ and barred sequences on sequences $\gamma = c_1 c_2 \cdots c_n$ with $1 \leq c_i \leq (m-1)i - (m-2)$ and with at least one bar in each s-descent, where the s-descent of $\gamma = c_1 c_2 \cdots c_n$ is $\{\, i : c_i \leq c_{i+1} \text{ or } i = n \,\}$. In this section, on the other hand, we defined the bijection $\rho_{n,k}^{(m,p)}$ between the same set of trees and the the set of barred sequences on sequences $\gamma = c_1c_2\cdots c_n$ with $1 \le c_i \le (m-1)i-(m-2)$ and with at least one bar in each s-descent, where the s-descent set of $\gamma = c_1c_2\cdots c_n$ is $\{\, i : c_i \ge c_{i+1}-(m-1) \text{ or } i = n \,\}$. So there is a descent-preserving bijection between these two sets of barred sequences.

Let $m = 3$. Then the tree in Figure 13 corresponds under $\phi_{9,7}^{(3)}$ to the sequence $1\,|\,1\,|\,5\,2\,||\,6\,5\,|\,12\,11\,|\,17\,|$, whereas it corresponds under $\rho_{6,5}^{(3,3)}$ to the sequence $1|3|16||47|25|1|$; both sequences have s-descent set $\{\,1,2,4,6,8,9\,\}$.

2.5.4 Complementary links and the s-ascent set

If $L_k^{(m,p)}$ is the set of links defined above, then

$$\begin{aligned}\overline{L}_k^{(m,p)} &= \{\,(\alpha_i,\alpha_j,q) : 1\le q\le p,\ 1\le i\le j\le k\,\}\\ &\qquad\qquad\qquad \cup \{\,(\alpha_i,\alpha_j,q) : p+1\le q\le m,\ 1\le i< j\le k\,\}.\end{aligned}$$

We let $\overline{\beta}_k^{(m,p)}(x)$ denote the image of $\sum_{S\in\mathbf{T}(\overline{L}_k^{(m,p)})} S$ in $R[[a_1,a_2,\ldots,a_k,x]]$. Since $\overline{\beta}_k^{(m,p)}(x)$ equals $(\beta_k^{(m,p)}(x))^{<-1>}$ for each $k>0$, we have in particular that

$$\overline{\beta}_1^{(m,p)}(x) = (\beta_1^{(m,p)}(x))^{<-1>},$$

and we can use Lagrange inversion to get that

$$\overline{\beta}_1^{(m,p)}(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{pn+1}\binom{pn+1}{n}x^{(m-1)n+1}.$$

So

$$\overline{\beta}_k^{(m,p)}(x) = \left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{pn+1}\binom{pn+1}{n}x^{(m-1)n+1}\right)^{<k>}.$$

Notice that if we ignore the alternating signs in $\overline{\beta}_1^{(m,p)}(x)$ we see that it is just $\beta_1^{(m,m-p)}(x)$. This makes sense, since clearly there is a bijection between the set of trees with links in $\overline{L}_k^{(m,p)}$ and the set of trees with links in $\widetilde{L}_k^{(m,p)} = \{\,(\alpha_i,\alpha_j,q) : 1\le q\le p,\ k\ge i\ge j\ge 1\,\}\cup\{\,(\alpha_i,\alpha_j,q) : p+1\le q\le m,\ k\ge i> j\ge 1\,\}$.

As we did before, we may write

$$\overline{\beta}_k^{(m,p)}(x) = \sum_{n=0}^{\infty}(-1)^n s_n^{(m,p)}(k)x^{(m-1)n+1},$$

where $s_n^{(m,p)}(k)$ is a polynomial in k of degree at most n and where $s_n^{(m,p)}(k)$ is the number of trees with links in $\overline{L}_k^{(m,p)}$ (or $\widetilde{L}_k^{(m,p)}$) with n internal vertices. For each $n \geq 0$, let $\Omega_n^{(m,p)}$ be the set of sequences $\gamma = c_1c_2\cdots c_n$ such that $1 \leq c_i \leq (m-1)i-(m-2)$ for each i in $[n]$, and define the s-descent set of γ to be $s(\gamma) = \{\, i : c_i < c_{i+1}-(p-1) \text{ or } i = n \,\}$. It is easy to see that $s_n^{(m,p)}(k)$ is the number of barred sequences on elements of $\Omega_n^{(m,p)}$ with at least one bar in each s-descent. The map $\rho_{n,k}^{(m,p)}$ defined above gives a bijection between the set of trees with links in $\widetilde{L}_k^{(m,p)}$ and the set of such barred sequences with k bars.

Since $s_n^{(m,p)}(k)$ is a polynomial in k of degree at most n, we can define a polynomial $S_n^{(m,p)}(t)$ of degree at most n by

$$\sum_{k=0}^{\infty} s_n^{(m,p)}(k)t^k = \frac{S_n^{(m,p)}(t)}{(1-t)^{n+1}}.$$

As before, if $S_n^{(m,p)}(t) = \sum\limits_{j=0}^{\infty} S_{n,j}^{(m,p)} t^j$, then $S_{n,j}^{(m,p)}$ is the number of sequences $\gamma = c_1c_2\cdots c_n$ such that $1 \leq c_i \leq (m-1)i-(m-2)$ for each i in $[n]$ with exactly $j-1$ values of i such that $c_i < c_{i+1}-(p-1)$. This suggests Theorem 2.5.4 and its corollary. First, however, we make a definition: Let $n > 0$. If a sequence $\gamma = c_1c_2\cdots c_n$ has s-descent set S, then the *s-ascent set* of the sequence $\gamma = c_1c_2\cdots c_n$ is the set complement of S in $\{\,1,2,\ldots,n-1\,\} \cup \{\,n\,\}$.

Theorem 2.5.4. *Let $F(x) = x + \sum_{i+1}^{\infty} c_i x^{ri+1}$ be a formal power series, where $r \geq 1$. Then $F^{<k>}(x) = \sum_{n=0}^{\infty} f_n(k)x^{rn+1}$, where $f_n(k)$ is a polynomial in k of degree at most n. Let $G(x) = F^{<-1>}(x)$; then $G^{<k>}(x) = \sum_{n=0}^{\infty} g_n(k)x^{rn+1}$, where $g_n(k)$ is also a polynomial in k of degree at most n. Let $n > 0$. We can define a polynomial $A_n(t)$ of degree at most n by*

$$\sum_{k=0}^{\infty} f_n(k)t^k = \frac{\sum_{j=0}^{n} A_{n,j}\, t^j}{(1-t)^{n+1}}.$$

It follows that

$$\sum_{k=1}^{\infty} g_n(k)t^k = (-1)^n \frac{\sum_{j=0}^{n} A_{n,n-j+1}\, t^j}{(1-t)^{n+1}}.$$

To prove the theorem, we need the following lemma (see [**S1**], pp. 114–115).

Lemma 2.5.4. *Suppose that $\sum\limits_{k=0}^{\infty} p(k)t^k = \dfrac{P(t)}{(1-t)^{n+1}}$, where $P(t)$ is a polynomial of degree at most n. Then $\sum\limits_{k=1}^{\infty} p(-k)t^k = (-1)^n \dfrac{t^{n+1}\,P(1/t)}{(1-t)^{n+1}}$.*

Proof. Let $P(t) = t^d$ and use linearity.

Proof of Theorem 2.5.4. Since $g_n(k) = f_n(-k)$ for each $n \geq 0$ and each $k \geq 0$, $\sum_{k=0}^{\infty} g_n(k)t^k = \sum_{k=0}^{\infty} f_n(-k)t^k$, which equals $f_n(0) + (-1)^n \dfrac{t^{n+1}A_n(1/t)}{(1-t)^{n+1}}$. Since

$n > 0$, $f_n(0) = 0$, so we get

$$\sum_{k=0}^{\infty} f_n(-k)t^k = (-1)^n \frac{t^{n+1} A_n(1/t)}{(1-t)^{n+1}}$$

$$= (-1)^n \frac{t^{n+1} \sum\limits_{j=0}^{n} A_{n,j} t^{-j}}{(1-t)^{n+1}}$$

$$= (-1)^n \frac{\sum\limits_{j=0}^{n} A_{n,n-j+1} t^{j}}{(1-t)^{n+1}}.$$

Corollary 2.5.4. *Let $n > 0$ and let $f_n(k)$ and $g_n(k)$ be as above; $f_n(k)$ and $g_n(k)$ define polynomials $A_n(t)$ and $B_n(t)$ by*

$$\sum_{k=0}^{\infty} f_n(k)t^j = \frac{\sum\limits_{j=0}^{n} A_{n,j} t^{j}}{(1-t)^{n+1}}$$

and

$$\sum_{k=0}^{\infty} g_n(k)t^j = \frac{\sum\limits_{j=0}^{n} B_{n,j} t^{j}}{(1-t)^{n+1}}.$$

Then if $A_{n,j}$ is the number of sequences $\gamma = c_1 c_2 \cdots c_n$ with exactly j s-descents, $B_{n,j}$ is the number of such sequences with exactly j s-ascents.

Proof. Since $B_{n,j}$ equals $A_{n,n-j+1}$, $B_{n,j}$ is the number of sequences $\gamma = c_1 c_2 \cdots c_n$ with $n - j + 1$ s-descents and hence with j s-ascents (recall that n is both an s-descent and an s-ascent).

CHAPTER 3

q-ANALOGUES OF ITERATION POLYNOMIALS

3.1 Introduction

In the previous chapter, we considered several examples of formal power series $f(x)$ which arose in counting trees with restricted links. In each case we could write

$$f^{<k>}(x) = \sum_{n=0}^{\infty} d_n^{(m)}(k) x^{(m-1)n+1}$$

for some $m \geq 2$, where $d_n^{(m)}(k)$ was a polynomial in k of degree at most n called the nth iteration polynomial of f. We then focused our attention on finding a combinatorial interpretation for the polynomial $D_n^{(m)}(t)$ defined by

$$\sum_{k=0}^{\infty} d_n^{(m)}(k) t^k = \frac{D_n^{(m)}(t)}{(1-t)^{n+1}}.$$

In this chapter, we will look at q-analogues of some of the examples we studied in Chapter 2. We will see if the nth iteration q-polynomial $d_n^{(m)}(k,q)$ of certain formal power series $f(x)$ defines a polynomial $D_n^{(m)}(t,q)$; if so, we will ask if there is a combinatorial interpretation for the coefficient of t^j in $D_n^{(m)}(t,q)$.

3.2 The kth iterate of $f(x) = qx + O(x^m)$

Let $f(x) = qx + O(x^m)$, for some $m \geq 2$, where $O(x^m)$ denotes some unspecified formal power series of the form $ax^m + bx^{m+1} + cx^{m+2} + \ldots$. Then we can write

$$f^{<k>}(x) = \sum_{n=0}^{\infty} d_n^{(m)}(k,q) x^{(m-1)n+1},$$

where $d_n^{(m)}(k,q)$ is a q-analogue of a polynomial of degree less at most n. Note that for each $n > 0$, $d_n^{(m)}(0,q) = 0$ and for each $k \geq 0$, $d_0^{(m)}(k,q) = q^k$.

Let L be the linear transformation on formal power series defined by $L(g) = g \circ f$. Then L has the property that if $g(x) = a_i x^i + a_{i+1} x^{i+1} \ldots$, then $L(g)$ has no powers of x less than x^i. So, using the basis $\{ x, x^2, x^3, \ldots, \}$, L can be represented by an infinite dimensional, lower triangular matrix $(A_{i,j})_{i,j=1,2,\ldots,}$. We denote by $[x^j]$

the vector $(0,0,\ldots,0,1,0\ldots,)$ with one in the jth position and zeroes in all other positions; then by $\left[\sum_{j=0}^{\infty} c_j x^j\right]$ we mean $\sum_{j=0}^{\infty} c_j[x^j]$. Then

$$\left[L^k(x)\right] = A^k\cdot[x] = \left[f^{<k>}(x)\right].$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} A^k\cdot[x]\,t^k &= \sum_{k=0}^{\infty}\left[f^{<k>}(x)\right]t^k \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} d_n^{(m)}(k,q)[x^{(m-1)n+1}]t^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} d_n^{(m)}(k,q)t^k[x^{(m-1)n+1}].\end{aligned}$$

Now $\sum\limits_{k=0}^{\infty} A^k t^k = \dfrac{1}{I-tA}$, and $I-tA$ is also a lower triangular matrix, so we can use the following lemma, which follows easily from the formula for the inverse of a matrix, to find $(I-tA)^{-1}$.

Lemma 3.2. *Let $(B_{i,j})_{i,j=1,2,\ldots,}$ be an invertible lower triangular matrix, and let $(B_{i,j})^{-1} = (C_{ij})$. Then, for $1 \le l \le n$, we have*

$$C_{n,l} = \frac{(-1)^{n-l}}{B_{l,l}B_{l+1,l+1}\cdots B_{n,n}} \mid B_{l+i+1,l+j} \mid_{i,j=0,1,\ldots,n-l-1}$$

and for $l > n$ we have $C_{n,l} = 0$.

Let $I - tA = (B_{i,j})_{i,j=1,2,\ldots,}$ and let $(C_{i,j}) = (B_{i,j})^{-1}$. Notice that the matrix A has the property that $A_{i,i} = q^i$ for $i \ge 1$. Hence $B_{i,i} = 1 - q^i t$, and for $0 \le l \le n$,

$$C_{n,l} = \frac{(-1)^{n-l}}{\prod_{i=l}^{n}(1-q^i t)} \mid B_{l+i+1,l+j} \mid_{i,j=0,1,\ldots,n-l-1}.$$

Since $(I-tA)^{-1}\cdot[x] = \sum_{n=1}^{\infty} C_{n,1}[x^n]$, we have that

$$(I-tA)^{-1}\cdot[x] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\prod_{i=1}^{n}(1-q^i t)} \mid B_{2+i,1+j} \mid_{i,j=0,1,\ldots,n-2} [x^n].$$

Note that $A_{n,l}$ is the coefficient of x^n in $x^l \circ f(x)$, so $A_{n,l}$ is the coefficient of x^n in $(f(x))^l$. Since $f(x) = qx + O(x^m)$, the coefficient of x^n in $(f(x))^l$ is 0 unless $n = (m-1)k + l$ for some $k \ge 0$. Hence $B_{n,l} = 0$ unless $n = (m-1)k + l$ for some $k \ge 0$, so the matrix $(B_{2+i,1+j})_{i,j=0,1,\ldots,n-2}$ has the form

$$\begin{pmatrix} 0 & 1-q^2t & 0 & 0 & & & \cdots & 0 \\ 0 & 0 & 1-q^3t & 0 & 0 & & \cdots & 0 \\ \vdots & & & \ddots & 0 & & \cdots & 0 \\ -A_{m,1}t & 0 & & & 1-q^m t & 0 & \cdots & 0 \\ & -A_{m+1,2}t & & & & 1-q^{m+1}t & \cdots & 0 \\ \vdots & & & & & & \cdots & 1-q^{n-1}t \\ -A_{n,1}t & -A_{n,2}t & & & & & \cdots & 0 \end{pmatrix}.$$

It is easy to show that if $n=(m-1)k+1$ for some $k \geq 0$, then

$$(-1)^{n-1} \mid B_{2+i,1+j} \mid_{i,j=0,1,\ldots,n-2} = \prod_{r=0}^{k-1} \prod_{s=2}^{m-1} (1-q^{(m-1)r+s}t) \cdot D_k^{(m)}(t,q),$$

where $D_k^{(m)}(t,q)$ is a polynomial in t of degree at most k. On the other hand, if $n \neq (m-1)k+1$ for some k, then $\mid B_{2+i,1+j} \mid_{i,j=0,1,\ldots,n-2}$ equals 0. So

$$\begin{aligned} (I-tA)^{-1} \cdot [x] &= \sum_{k=0}^{\infty} A^k \cdot [x]\, t^k \\ &= \sum_{n=0}^{\infty} \frac{D_n^{(m)}(t,q)}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)} [x^{(m-1)n+1}], \end{aligned}$$

and therefore

$$\sum_{k=0}^{\infty} d_n^{(m)}(k,q) t^k = \frac{D_n^{(m)}(t,q)}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)},$$

where $D_n^{(m)}(t,q)$ is a polynomial in t of degree at most n.

3.3 A combinatorial interpretation for the coefficients of $D_n^{(m)}(t,q)$

In this section we look at several power series of the form $f(x)=qx+O(x^m)$ and we ask in each case if there is a combinatorial interpretation for the coefficient of t^j in $D_n^{(m)}(t,q)$. As in Chapter 2, we let Ω be a set of sequences of positive integers, and for each $n \geq 0$ we let Ω_n be the set of elements of Ω of length n. If γ is in Ω_n, we define the s-descent set $s(\gamma)$ to be a subset of $[n]$ containing n.

Recall that if $\pi = \pi_1\pi_2\cdots\pi_n$ is a permutation of n, then the descent set $D(\pi)$ of π is $\{\, i \in [n] : \pi_i > \pi_{i=1} \text{ or } i=n \,\}$; this permutation statistic motivated the definition of the s-descent set $s(\gamma)$ of a sequence γ which we gave in Chapter 2. Another important statistic of a permutation π is the *major index maj(π) of* π, first studied by MacMahon in [**M1**] and [**M2**], and defined by

$$\operatorname{maj}(\pi) = \sum_{j \in D(\pi)} j.$$

In a similar spirit, if γ is a sequence in Ω_n with s-descent set $s(\gamma)$, we define the *s-m-index $i(\gamma)$ of* γ to be

$$i(\gamma) = \sum_{j \in s(\gamma)} (m-1)j + 1.$$

Let $n \geq 0$. Suppose we want to count the elements of Ω_n according to the number of s-descents and the s-m-index. As in Chapter 2, we can do this by counting barred sequences; this time, however, we weight each sequence in a certain way. Let B_n be the set of barred sequences in Ω_n with at least one bar in each s-descent, and let $B_{n,k}$ be the set of elements of B_n with k bars. If $\bar{\gamma}$ belongs to $B_{n,k}$, we weight $\bar{\gamma}$ by $q^{\sigma_m(\bar{\gamma})}$, where

$$\sigma_m(\bar{\gamma}) = \sum_{\substack{i:\bar{\gamma} \text{ has a} \\ \text{bar in space } i}} (m-1)i + 1,$$

and we let $b_n^{(m)}(k,q) = \sum_{\bar{\gamma} \in B_{n,k}} q^{\sigma_m(\bar{\gamma})}$.

Lemma 3.3. $\sum_{k=0}^{\infty} b_n^{(m)}(k,q)t^k = \dfrac{G_n^{(m)}(t,q)}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)}$, *where $G_n^{(m)}(t,q)$ is a polynomial in t defined by $G_n^{(m)}(t,q) = \sum_{\gamma \in \Omega_n} q^{i(\gamma)}\, t^{d(\gamma)}$.*

Proof. Recall that $d(\gamma)$ is the number of s-descents of a sequence γ. Every element of $B_{n,k}$ along with its weighting can be obtained uniquely from some γ in Ω_n as follows: insert a bar into each of the $d(\gamma)$ s-descents of γ and weight the resulting barred sequence by $\omega = \prod_{j \in s(\gamma)} q^{(m-1)j+1}$. Then insert $k - d(\gamma)$ bars arbitrarily into the $n+1$ spaces of γ; for each such bar inserted into space j of γ, for $0 \leq l \leq n$, multiply ω by a factor of $q^{(m-1)l+1}$. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} b_n^{(m)}(k,q)t^k &= \sum_{\gamma \in \Omega_n} \left(q^{\sum_{j \in s(\gamma)} (m-1)j+1}\, t^{d(\gamma)}\right) \\
&\quad \times (1 + qt + q^2t^2 + \cdots)(1 + q^m t + q^{2m}t^2 + \cdots) \cdots (1 + q^{(m-1)n+1}t + \cdots) \\
&= \frac{\sum_{\gamma \in \Omega_n} q^{i(\gamma)}\, t^{d(\gamma)}}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)}.
\end{aligned}$$

3.4 The case $(qx + x^m)^{<k>}$

Recall that in Chapter 2 we looked at the kth iterate of $x + x^m$; here we look at a a q-analogue of this example. Let $f(x) = qx + x^m$, where $m \geq 1$. Then

$$f^{<k>}(x) = (qx + x^m)^{<k>} = \sum_{n=0}^{\infty} p_n^{(m)}(k,q)x^{(m-1)n+1}$$

for $k \geq 0$, where $p_n^{(m)}(k,q)$ is a the q-analogue of a polynomial of degree at most n, and we can define a polynomial in t of degree at most n by

$$\sum_{k=0}^{\infty} p_n^{(m)}(k,q) = \frac{P_n^{(m)}(t,q)}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)}.$$

From Section 3.1 we know that for each $n > 0$, $p_n^{(m)}(0,q) = 0$ and for each $k \geq 0$, $p_0^{(m)}(k,q) = q^k$. If we adopt the proof of Lemma 2.3.2 to the present situation, it is easy to show that we have the following recurrence for $n \geq 0$, $k > 0$:

$$p_n^{(m)}(k,q) = \sum_{i=\lceil \frac{n-1}{m} \rceil}^{n} \binom{(m-1)i+1}{n-i} p_i^{(m)}(k-1,q)\, q^{mi-n+1}.$$

Let $\Omega^{(m)}$ be the set of all finite sequences $\gamma = c_1 c_2 \cdots c_r$ of positive integers such that $1 \leq c_i \leq (m-1)i - (m-2)$ for each i. For each $n \geq 0$, let $\Omega_n^{(m)}$ be the set of elements of $\Omega^{(m)}$ of length n, and if γ is in $\Omega_n^{(m)}$ define the s-descent set $s(\gamma)$ to be $s(\gamma = \{\, i \in [n] : c_i \leq c_{i+1} \text{ or } i = n \,\}$. Let $B_n^{(m)}$ be the set of barred sequences in $\Omega_n^{(m)}$ with at least one bar in each s-descent and let $B_{n,k}^{(m)}$ be the set of elements of $B_n^{(m)}$ with k bars. Let $b_n^{(m)}(k,q) = \sum_{\bar{\gamma} \in B_{n,k}} q^{\sigma_m(\bar{\gamma})}$ and let $\tilde{b}_n^{(m)}(k,q) = q^{-mn} \cdot b_n^{(m)}(k,q)$, that is, let

$$\tilde{b}_n^{(m)}(k,q) = \sum_{\bar{\gamma} \in B_{n,k}^{(m)}} q^{-mn+\sigma(\bar{\gamma})}.$$

Theorem 3.4. $\tilde{b}_n^{(m)}(k,q) = p_n^{(m)}(k,q)$.

Proof. For each $k \geq 0$, $\tilde{b}_0^{(m)}(k,q)$ counts the sequence consisting only of k bars; since each of the k bars is considered to be in space 0 of the sequence, the sequence consisting of k bars is weighted by $q^{-m \cdot 0 + k((m-1)\cdot 0+1)}$, which equals q^k. So $\tilde{b}_0^{(m)}(k,q) = q^k$. Since every barred sequence in $B_{n,k}^{(m)}$ must have a bar in its final space, $\tilde{b}_n^{(m)}(0,q) = 0$ for all $n > 0$. We know from Theorem 2.3.4 that we can construct any sequence in $B_{n,k}^{(m)}$ from a unique sequence $\bar{\gamma}$ in $B_{i,k-1}^{(m)}$, for some i such that $0 \leq i \leq n$, by choosing integers c_{i+1}, c_{i+2}, $\ldots$, c_n such that $1 \leq c_j \leq (m-1)j - (m-2)$ for each j that satisfies $i+1 \leq j \leq n$ and such that $c_j > c_{j+1}$ for each j that satisfies $i+1 \leq j \leq n-1$, then inserting the sequence $c_{i+1}c_{i+2}\cdots c_n$ after the final bar in $\bar{\gamma}$, and finally inserting a bar after c_n. We also know that there are $\binom{(m-1)i+1}{n-i}$ ways to choose the integers c_{i+1}, c_{i+2}, $\ldots$, c_n. Note that the sequence $c_{i+1}c_{i+2}\cdots c_n$ will increase the weight of $\bar{\gamma}$ by a factor of $q^{-m(n-i)} \cdot q^{(m-1)n+1} = q^{mi-n+1}$. Hence

$$\tilde{b}_n^{(m)}(k,q) = \sum_{i=\lceil \frac{n-1}{m} \rceil}^{n} \binom{(m-1)i+1}{n-i} \tilde{b}_i^{(m)}(k-1,q)\, q^{mi-n+1}.$$

Since $\tilde{b}_n^{(m)}(k,q)$ and $p_n^{(m)}(k,q)$ satisfy the same initial conditions and the same recurrence, they are equal.

Lemma 3.3 and Theorem 3.4 give us a combinatorial interpretation for the polynomial $P_n^{(m)}(t,q)$:

$$\begin{aligned} P_n^{(m)}(t,q) &= \sum_{\gamma \in \Omega_n^{(m)}} q^{-mn+i(\gamma)}\, t^{d(\gamma)} \\ &= q^{-mn} \cdot \sum_{\gamma \in \Omega_n^{(m)}} q^{i(\gamma)}\, t^{d(\gamma)}. \end{aligned}$$

So if $P_n^{(m)}(t,q) = \sum\limits_{j=0}^{n} P_{n,j}^{(m)}(q) t^j$, then $P_{n,j}^{(m)}(q)$ is a polynomial in q given by

$$P_{n,j}^{(m)}(q) = q^{-mn} \sum_{\substack{\gamma \in \Omega_n^{(m)} \\ d(\gamma)=j}} q^{i(\gamma)}.$$

Table 3 gives $P_{n,j}^{(2)}(q)$ for $0 \le n, j \le 4$.

$n \backslash r$	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	0	$2q$	0	0
3	0	0	1	$5q^3$	0
4	0	0	0	$6q^2+4q^3$	$14q^6$

The polynomials $P_{n,j}^{(2)}(q)$

Table 3

3.5 The case $\left(\dfrac{qx}{1-x^{m-1}}\right)^{<k>}$

We can analyze the formal power series $f(x) = \dfrac{qx}{1-x^{m-1}}$ in a similar fashion. We can write

$$f^{<k>}(x) = \left(\frac{qx}{1-x^{m-1}}\right)^{<k>} = \sum_{n=0}^{\infty} r_n^{(m)}(k,q) x^{(m-1)n+1},$$

where $r_n^{(m)}(k,q)$ is a q-analogue of a polynomial of degree at most n, and we can define a polynomial $R_n^{(m)}(t,q)$ in t of degree at most n by

$$\sum_{k=0}^{\infty} r_n^{(m)}(k,q) t^k = \frac{R_n^{(m)}(t,q)}{\prod_{i=0}^{n}(1-q^{(m-1)i+1}t)}.$$

Now for all $k \geq 0$, $r_0^{(m)}(k,q) = q^k$ and for each $n > 0$, $r_n^{(m)}(o,q) = 0$. An easy adaption of the proof of Lemma 2.4.2 gives us the following recurrence for $n \geq 0$, $k > 0$:

$$r_n^{(m)}(k,q) = \sum_{i=0}^{n} \binom{n+(m-2)i}{(m-1)i} r_i^{(m)}(k-1,q)\, q^{(m-1)i+1}.$$

Let $\Omega^{(m)}$ and $\Omega_n^{(m)}$ be as in Section 3.4. If γ is in $\Omega_n^{(m)}$, define the s-descent set $s(\gamma)$ to be $s(\gamma = \{\, i \in [n] : c_i \geq c_{i+1} - (m-2) \text{ or } i = n \,\}$. Let $B_n^{(m)}$, $B_{n,k}^{(m)}$ and $b_n^{(m)}(k,q)$ be as in Section 3.4. Let $\hat{b}_n^{(m)}(k,q) = q^{-(m-1)n} \cdot b_n^{(m)}(k,q)$, that is, let

$$\hat{b}_n^{(m)}(k,q) = \sum_{\bar{\gamma} \in B_{n,k}^{(m)}} q^{-(m-1)n+\sigma(\bar{\gamma})}.$$

It is straightforward to show that $r_n^{(m)}(k,q)$ and $\hat{b}_n^{(m)}(k,q)$ satisfy the same initial conditions and the same recurrence, so

$$r_n^{(m)}(k,q) = \sum_{\bar{\gamma} \in B_{n,k}^{(m)}} q^{-(m-1)n+\sigma(\bar{\gamma})}.$$

Hence

$$R_n^{(m)}(t,q) = q^{-(m-1)n} \sum_{\gamma \in \Omega_n^{(m)}} q^{i(\gamma)}\, t^{d(\gamma)}.$$

Table 4 gives $R_{n,j}^{(3)}(q)$ for $0 \leq n, j \leq 3$.

$n\backslash r$	0	1	2	3
0	1	0	0	0
1	0	q	0	0
2	0	q	$3q^5$	0
3	0	q	$9q^5 + 6q^8$	$12q^{12}$

The polynomials $R_{n,j}^{(3)}(q)$

Table 4

Recall that in Section 2.4.5 we studied the polynomial $R_n^{(m)}(t) = \sum_{j=0}^{n} R_{n,j}^{(m)} t^j$, and found that the coefficient $R_{n,j}^{(r+1)}$ of t^j in $R_n^{r+1)}(t)$ was the number of multipermutations of the multiset $\{ 1^r, 2^r, \ldots, n^r \}$ with exactly $j-1$ inverse descents. Let $Q_n^{(r)}$ be the set of all multipermutations of the set $\{ 1^r, 2^r, \ldots, n^r \}$. If π belongs to $Q_n^{(r)}$, let $\tilde{d}(\pi)$ be the number of inverse descents of π and define the *inverse index* $\tilde{i}(\pi)$ *of* π to be

$$\tilde{i}(\pi) = \sum_{\substack{l \text{ is an inverse} \\ \text{descent of } \pi}} rl + 1.$$

Then

$$R_n^{(r+1)}(t,q) = q^{-rn} \sum_{\pi \in Q_n^{(r)}} q^{\tilde{i}(\pi)}\, t^{\tilde{d}(\pi)}.$$

So $R_n^{(m)}(t,q)$ counts multipermutations according to the number of inverse descents and the inverse index.

Recall that if $r = 1$, a multipermutation of $\{ 1^r, 2^r, \ldots, n^r \}$ is just a permutation of $[n]$, and that l is an inverse descent of π in $Q_n^{(1)}$ if and only if l is a descent of π^{-1}. Hence $\tilde{i}(\pi) = \mathrm{maj}(\pi^{-1})$, and so $R_n^{(2)}(t,q)$ counts permutations of $[n]$ according to descents and the major index. So $R_n^{(2)}(t,q)$ is q^n times the most well-known q-analogue of the Eulerian polynomial $A_n(t)$ (see [**CA**]).

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