GENERATING FUNCTIONS AND ENUMERATION OF SEQUENCES

by

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ABSTRACT

We develop a general theory of enumeration of finite sequences by generating functions which unifies and extends many topics previously studied by various ad hoc techniques. Included in particular are Simon Newcomb's problem and the combinatorial interpretations of the tangent, secant, and Eulerian numbers.

Special features of our approach are the following:

1) A combinatorial interpretation for the reciprocal of a generating function.

2) An explanation of the connection between exponential (and more generally Eulerian) generating functions, and symmetric functions.

3) A unified approach to determinantal formulas for sequence enumeration.

Thesis Supervisor: Richard P. Stanley
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CONTENTS

INTRODUCTION

In this thesis we present a unified approach to the enumeration of permutations and sequences by generating functions. The theory we develop allows us to systematize and generalize a large number of results previously obtained by various *ad hoc* methods.

The first important result of our subject was Desiré André's discovery [7] in 1879 of a combinatorial interpretation of the integers E_n defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x \ ;$$

E_n is the number of permutations $b_1 b_2 \ldots b_n$ of $\{1,2,\ldots,n\}$ satisfying $b_1 < b_2 > b_3 \ldots$.

The first systematic approach to problems of the kind we consider here was taken by Major Percy MacMahon, who worked in the late nineteenth and early twentieth century. In his "Second Memoir on the Compositions of Numbers" [62] in 1908, MacMahon solved "Simon Newcomb's problem," which may be stated as follows: How many permutations $b_1 b_2 \ldots b_k$ are there of the multiset

$\{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ (where $k = \sum_i k_i$) such that

$b_i > b_{i+1}$ for exactly s values of i? MacMahon also solved the more general problem of counting those permutations for which $b_i > b_{i+1}$ iff i belongs to some specified subset of $\{1,2,\ldots,k-1\}$.

MacMahon's work on this problem received little attention in the following forty years, and when Simon Newcomb's problem again became a subject of interest, much of MacMahon's work was independently rediscovered (e.g. [3], [17], [6]). This unfamiliarity with MacMahon's work among combinatorialists may be due to the fact that MacMahon's work is not easy reading; moreover, the generality of many of MacMahon's formulas, especially those involving symmetric functions, may leave the casual reader unconvinced of their usefulness in solving specific problems. In the writer's opinion, one of MacMahon's greatest shortcomings was his ignorance of the enumerative use of exponential generating functions. Many of MacMahon's symmetric-function generating functions for sequences with repetition reduce to simple exponential generating functions for the corresponding sequences without repetition. For example, from a formula on p. 15 of Combinatory Analysis [64], Vol. 1, one can derive the exponential generating function for the Stirling numbers of the second kind; from a formula on p. 212 one can derive the exponential generating function for the Eulerian polynomials. (The connection between symmetric functions and exponential generating functions can be explained by our Theorem 3.5.) MacMahon was apparently unaware of or uninterested in André's result; it was not until sixty years later that Carlitz [17] showed how Andre's formula can be derived from one of MacMahon's [64, Vol. 1, p. 190].

Little work was done on Simon Newcomb's problem until 1946, when Irving Kaplansky and John Riordan investigated the problem from the point of view of rook polynomials [55]. Riordan developed this approach further in his 1958 book _An Introduction to Combinatorial Analysis_ [71], through which the problem reached a wide audience.

In the late sixties Simon Newcomb's became a topic of considerable interest [24], [37], [38], [60], [67], [74], and this interest has increased through the seventies. We note in particular the work of L. Carlitz, who has been by far the most prolific writer on the subject, as a glance at our bibliography (which is not complete) will show.

The authors of the above-mentioned works have used, for the most part, the "classical" method for obtaining generating functions: one derives a recurrence for the numbers in question which translates into an algebraic or differential equation for their generating function. In recent years several authors have tried to develop a "theory" of generating functions, which would explain more directly the connection between a combinatorial problem and the form of its associated generating function.

A fundamental work in this theory is Gian-Carlo Rota's 1964 paper on Möbiussfunctions [76]. The relevance of Rota's paper to our discussion is not the Möbius function per se but rather the important idea that in studying combinatorial objects it is often more productive to work in an algebraic system (in this case the incidence algebra) associated in a natural way with these objects, than to reduce everything immediately to statements about integers.

One of the first important results of this new theory of generating functions was the "exponential formula," which may be roughly paraphrased as saying that if

$A(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{n!}$ is the exponential generating function

for labeled objects of a certain type, then

$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = e^{A(x)}$ is the exponential generating function

for sets of these objects. Although special cases of the exponential formula were known as early as 1939 for permutations [80] and 1953 for graphs [69], a general treatment did not appear until the early 1970's when three different approaches to the exponential formula were published, those of Doubilet-Rota-Stanley [39], Bender-Goldman [11], and Foata-Schützenberger [43].

The last of these is of special significance to our discussion. Foata and Schützenberger's object was to create a "geometric" theory of Eulerian polynomials which would yield enumerative results about permutations as simple consequences of the study of their structural properties. Although their methods are different from ours, their guiding principle is fundamental to our approach.

Our methods are most closely related to those used in two papers by Richard Stanley. In his 1972 study of P-partitions [77], Stanley showed how certain partition problems can be reduced to problems of permutation enumeration. Our interest here is primarily in permutation enumeration; we show, using a similar method, that certain permutation enumeration problems can be reduced to more easily solved problems of counting partitions. In a later paper [79] Stanley showed that permutations can be counted by Möbius inversion on certain posets. Moreover, these posets are closely related to the binomial posets of [39], and thus their Möbius functions can be computed by inverting generating functions. This approach is closely related to our Theorem 4.1, in which Möbius inversion is replaced by inversion of a slightly different kind.

Finally, we mention the very recent work of David Jackson and R. Aleliunas [54] and of James Reilly [68], which has some similarities with the ideas developed here.

We now give an outline of our approach. In Chapter 1 we discuss the algebraic systems in which we work. In the traditional approach to generating functions, one has an analytic function $F(z)$ whose power series expansion $\sum_{n=0}^{\infty} f_n z^n$ "generates" the coefficients f_n which are of interest. It has long been recognized (see for example [10]) that the convergence of the series $\sum_{n=0}^{\infty} f_n z^n$ is for many purposes irrelevant, and that for many purposes the appropriate algebra to work in is the algebra of "formal power series," that is, "formal sums" $\sum_{n=0}^{\infty} f_n z^n$ where the f_n are arbitrary.

It turns out that for our purposes, formal power series are still not general enough. We introduce a more general algebraic system which we call a <u>counting algebra</u>. A counting algebra is an algebra in which each element is a "formal sum" $\sum_{n=0}^{\infty} f_n z_n$ where the z_n

are "basis elements" whose multiplication is arbitrary except for some mild constraints. Examples of useful counting algebras which are not formal power series algebras are power series algebras in noncommuting variables and algebras of matrices.

Since the elements of a general counting algebra can in no way be considered analytic functions which generate their coefficients, we have used in the text the more appropriate term "counting series" instead of generating function.

In Chapter 2 we introduce our basic combinatorial structure. Given any set P, we may consider the set P^* of finite sequences of elements of P. We multiply two elements of P^* by juxtaposing them; in this way P^* becomes a monoid, the <u>free monoid</u> on the set P. We find it useful to consider certain subsets of P^* which we call <u>linear systems</u>. If S is a linear system, then the product of two elements of S is their juxtaposition if this is in S, otherwise the product is "zero."

From the linear system S we construct in Chapter 3 a counting algebra $R[[S]]$, the set of formal sums of elements of S with coefficients in the commutative ring R. Then to any subset W of S we can associate its <u>total counting series</u> which is simply the sum of its elements. Since W is completely determined by its total counting series, we have passed from a combinatorial object to an algebraic object with no loss of information. Less discriminating counting series are obtained as homomorphic images of the total counting series. We explain in this way, for example, the connection between Eulerian counting series and inversions of permutations.

In Chapter 4 we introduce a fundamental object of our study, the linked set. For simplicity we describe here the situation in which the linear system is a free monoid P^*. A <u>link</u> is a sequence in P^* of length two; the links of the sequence $b_1b_2\ldots b_n$ are b_ib_{i+1} for $i = 1,\ldots,n-1$. Now if L is any set of links, we define the <u>linked set</u> C_L to be the set of sequences in P^* all of whose links are in L. Let $\bar{L}$ be the set of links not in L. Our fundamental result is that the total counting series for C_L is the inverse of the total counting series for $C_{\bar{L}}$ with "alternating signs". We give several simple applications of this theorem, then in Chapter 5 we apply it to the enumeration of sequences by runs of each length.

In Chapter 6 we introduce certain linear systems whose elements are associated with paths in digraphs, and whose counting algebras can be represented as algebras of matrices. Through matrix inversion we are led to enumeration formulas involving determinants. We consider first the following problem: given integers $m_1 \geq m_2 \geq \dots \geq m_n$ and $M_1 \geq M_2 \geq \dots \geq M_n$, and a subset F of $\{1,2,\dots,n-1\}$, to count sequences $b_1 b_2 \dots b_n$ of integers with $m_i \leq b_i \leq M_i$ for all i and $b_j > b_{j+1}$ iff $j \in F$. Special cases of our result give several previously unrelated results.

Next we consider sequences with specified beginning and end segments, and periodic middles. These are enumerated by inversion of a certain 3 x 3 matrix. For example, given i, j, and k, we count sequences $b_1 b_2 \dots b_{i+nj+k}$ such that $b_s < b_{s+1}$ if $s \not\equiv i \pmod{j}$ and $b_s > b_{s+1}$ if $s \equiv i \pmod{j}$. Finally we consider a generalization of André's problem: given an integer n and a subset F of $\{1,2,\dots,n\}$, to count sequences $b_1 b_2 \dots b_m$ such that $b_i > b_{i+1}$ iff i is congruent (mod n) to an element of F.

Given a polynomial $p(n)$ of degree k, we may consider the polynomial $B(t)$ defined by

$$\sum_{n=0}^{\infty} p(n) t^n = \frac{B(t)}{(1-t)^{k+1}}$$

In Chapter 7 we study polynomials $p(n)$ for which $B(t)$ has a combinatorial interpretation. First we consider polynomials $p(n)$ which are chromatic polynomials of certain graphs. A special case yields the solution to Simon Newcomb's problem. We then consider the polynomials $p_{2k}(n)$ defined by

$$[1 - t \cosh^2 x]^{-1} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \sum_{n=0}^{\infty} p_{2k}(n) t^n ,$$

ans we show that the corresponding polynomials $B_{2k}(t)$ count permutations of length $2k$ by number of peaks (maxima).

If $b_1 b_2 \ldots b_n$ is a sequence of integers, its <u>greater index</u> is the sum of those i for which $b_i > b_{i+1}$. In Chapter 8 we discuss the greater index from the point of view of the theory of linked sets. It follows from our approach that if $A_n(q,p)$ is the enumerator for permutations of $\{1,2,\ldots,n\}$ by greater index and number of inversions, then $A_n(q,p) = A_n(p,q)$. Finally, we describe how Stanley's theory of P-partitions is related to our theory.

CHAPTER 0

NOTATION

$\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$, and $\mathbb{Q}$ are the sets of integers, positive integers, nonnegative integers, and rational numbers. For $n \in \mathbb{P}$, $[n] = \{1,2,\ldots,n\}$, and $[0]$ is the empty set, $\emptyset$.

A multiset on a set $S = \{s_1,s_2,\ldots,s_m\}$ is a function $M : S \to \mathbb{N}$ which assigns to each element of S a "multiplicity." The sum of two multisets M_1 and M_2 on S is defined by $(M_1+M_2)(s_i) = M_1(s_i) + M_2(s_i)$.

We shall think of M as a set with repeated elements allowed, and shall write $\{s_1^{k_1},\ldots,s_n^{k_n}\}$ for M, where $M(s_i) = k_i$. We shall also use notation such as $\{s_1,s_1,s_2\}$ to denote the multiset $\{s_1^2,s_2\}$.

Wherever there is a possibility of ambiguity as to whether sets or multisets are under discussion, the reader may take it that we refer to sets; we shall always refer to multisets explicitly as such.

A sequence of length n is a function whose domain is $[n]$. (All our sequences have finite length except in our discussion of convergence in Chapter 1.) The unique sequence of length zero is called the empty sequence; we denote it by $\underline{1}$. By $b_1b_2\ldots b_n$ we mean the sequence π

of length n such that $\pi(i) = b_i$. We say that π is a <u>permutation of the multiset</u> $\{b_1, b_2, \ldots, b_n\}$ of <u>length</u> $\ell(\pi) = n$. The word "permutation" without reference to a multiset will always mean a permutation of a set. $\mathfrak{S}_n$ is the set of permutations of $[n]$.

Our sequences will always be sequences of positive integers unless the context clearly indicates otherwise.

CHAPTER 1

COUNTING ALGEBRAS

Let B be a commutative ring with identity. Then the <u>algebra of formal power series</u> $F = B[[u_1,u_2,\dots]]$ in the indeterminates $u_1,u_2,\dots$ is the set of all formal sums

$$\sum_{k_1,k_2\dots} c_{k_1k_2\dots} u_1^{k_1} u_2^{k_2} \dots , \tag{1}$$

where the u_i are commuting indeterminates, the c's are arbitrary elements of B, and the sum is over all $k_1,k_2,\dots$ such that $k_i \neq 0$ for only finitely many i. Note that we allow sums such as $u_1+u_2+\dots$. Multiplication of two elements of F, and scalar multiplication by elements of B are defined in the obvious way: we multiply term-by-term, using an "infinite distributive law."

We can give F a topological structure as follows: a sequence $\{f_i\}_{i=1}^{\infty}$ in F converges to $f \in F$ iff for every monomial $\alpha = u_1^{k_1} u_2^{k_2} \dots$ the coefficient of α in f_i equals the coefficient of α in f for all sufficiently large i. Then with this topology the expression (1) considered as an infinite series converges to the correct value, and the "infinite distributive law" is equivalent to the assertion that multiplication in F and scalar multiplication by elements of B are continuous.

The traditional theory of generating functions takes place in the context of an algebra of formal power series as just described. However we shall find it useful to consider a more general algebraic system which we call a <u>counting algebra</u>.

It will be convenient to fix permanently as our ring of scalars the ring $R = \mathbb{Q}[[p,q,t,t_1,t_2,\ldots]]$ with the topology described above.

Now let A be an (associative) R-algebra with identity $\underset{\sim}{1}$ which is free as an R-module, and let S be a basis for A. To avoid certain irrelevant topological technicalities, we assume that S is countable. Then the structure of A is completely determined by the structure constants $e^{\gamma}_{\alpha,\beta} \in R$ for which $\alpha\beta = \sum_{\gamma \in S} e^{\gamma}_{\alpha,\beta}\, \gamma$ for all $\alpha,\beta \in S$. Note that for fixed α and β, $e^{\gamma}_{\alpha,\beta} = 0$ for all but finitely many γ.

Let us now assume that the following three conditions hold:

(i) $\underset{\sim}{1} \in S$

(ii) For each $\gamma \in S$, $e^{\gamma}_{\alpha,\beta} = 0$ for all but finitely many pairs (α,β).

(iii) There exists a rank function $r : S \to \mathbb{N}$ such that if $e^{\gamma}_{\alpha,\beta} \neq 0$, then $r(\gamma) = r(\alpha) + r(\beta)$; and $r(\alpha) > 0$ for $\alpha \in S - \{\underset{\sim}{1}\}$.

We define $\hat{A}$ to be the set of all formal sums $\sum_{\alpha \in S} c_\alpha \alpha$ for $c_\alpha \in R$. Then $\hat{A}$ can be identified with the set R^S of functions from S to R, and thus can be given the product topology: if $f^{(i)} = \sum_{\alpha \in S} c^{(i)}_\alpha \alpha$ and $f = \sum_{\alpha \in S} c_\alpha \alpha$ then $\{f^{(i)}\}^{\infty}_{i=1}$ converges to f in $\hat{A}$ iff $\{c^{(i)}_\alpha\}^{\infty}_{i=1}$ converges to c_α in R for each $\alpha \in S$.

Now $\hat{A}$ has a natural R-module structure, and we can define multiplication in $\hat{A}$ by

$$\left(\sum_{\alpha \in S} c_\alpha \alpha\right)\left(\sum_{\beta \in S} d_\beta \beta\right) = \sum_{\gamma \in S}\left(\sum_{\alpha,\beta \in S} e^{\gamma}_{\alpha,\beta}\, c_\alpha d_\beta\right)\gamma \ .$$

Condition (ii) assures that the inner sum on the right contains only finitely many nonzero terms.

It is easy to see that $\hat{A}$ becomes an R-algebra in which A can be identified with the subalgebra of elements of the form $\sum_{\alpha \in T} c_\alpha \alpha$ where T is a finite subset of S.

1.1. Definition. A counting algebra is a pair $(\hat{A}, S)$ constructed as above and satisfying conditions (i), (ii), and (iii) above. We call S the basis of $\hat{A}$.

We shall often refer to the counting algebra $(\hat{A}, S)$ simply as $\hat{A}$, when the basis is understood.

Our concern with the topological properties of counting algebras is to provide a rigorous foundation for the limiting operations we will want to perform in them. The actual verification that these operations are legitimate in specific instances is straightforward and will generally be omitted.

As examples, we prove the following two propositions.

1.2. Proposition. Let f be any element of the counting algebra $\hat{A}$. Then $\sum_{n=0}^{\infty} (tf)^n$ converges to $(\underset{\sim}{1}-tf)^{-1}$.

Proof. For fixed $\alpha \in S$, let $c_\alpha^{(n)}$ be the coefficient of α in f^n. Then the coefficient of α in $\sum_{n=0}^{\infty} (tf)^n$ is $\sum_{n=0}^{\infty} t^n c_\alpha^{(n)}$ which converges in R whatever the values of $c_\alpha^{(n)}$. It is then easy to see that $(\underset{\sim}{1}-tf) \sum_{n=0}^{\infty} (tf)^n = \underset{\sim}{1}$.

1.3. Proposition. Let f be any element of $\hat{A}$ of the form $\sum_{\alpha \varepsilon S-\{\underset{\sim}{1}\}} c_\alpha \alpha$. Then $\sum_{n=0}^{\infty} f^n$ converges to $(\underset{\sim}{1}-f)^{-1}$.

<u>Proof</u>. Let r be a rank function for $\hat{A}$ whose existence is guaranteed by condition (iii). Then for $\alpha \varepsilon S$, the coefficient of α in $\sum_{n=0}^{\infty} f^n$ is the coefficient of α in $\sum_{n=0}^{r(a)} f^n$. Thus $\sum_{n=0}^{\infty} f^n$ converges, and as before, $(\underset{\sim}{1}-f) \sum_{n=0}^{\infty} f^n = \underset{\sim}{1}$.

CHAPTER 2

LINEAR SYSTEMS

2.1. <u>Definition</u>. For any set P, P^* is the set of sequences of elements of P (including the empty sequence, $\underline{1}$). For $\pi = b_1 b_2 \ldots b_m$, $\sigma = c_1 c_2 \ldots c_n \in P^*$, we define $\pi * \sigma$ to be the sequence $b_1 \ldots b_m c_1 \ldots c_n$.

2.2. <u>Definition</u>. Let S be a subset of P^*. An element π of $S-\{\underline{1}\}$ is a <u>prime</u> of S if $\pi = \alpha * \beta$, with $\alpha, \beta \in S$, implies $\pi = \alpha$ or $\pi = \beta$. We say that S is a <u>linear system</u> if the following three properties hold:

(i) $\underline{1} \in S$

(ii) Every element of $S-\{\underline{1}\}$ has a unique expression in the form $\pi_1 * \pi_2 * \ldots * \pi_n$ where the π_i are primes.

(iii) If $\pi_1 * \pi_2 * \ldots * \pi_n \in S$ where the π_i are primes, then $\pi_j * \ldots * \pi_k \in S$ for $1 \leq j \leq k \leq n$.

The <u>rank</u> $r(\alpha)$ of $\alpha \in S$ is the number of primes in the prime factorization of α, with $r(\underline{1}) = 0$.

Examples of linear systems:

(a) Sequences of positive integers.

(b) Sequences of positive integers of even length.

(c) Permutations of positive integers.

(d) Paths in a digraph, considered as sequences of edges.

Note that in examples (a), (c), and (d) we have $r(\alpha) = \ell(\alpha)$, where $\ell(\alpha)$ is the length of α, and in (b) we have $2r(\alpha) = \ell(\alpha)$.

Now let us assume that "O" is a symbol not in S. We define a multiplication on $S \cup \{O\}$ as follows: For $\alpha, \beta \in S$ we define $\alpha\beta$ to be $\alpha * \beta$ if $\alpha * \beta \in S$, and $\alpha\beta = 0$ otherwise; $\alpha O = O\alpha = O$ for $\alpha \in S \cup \{O\}$.

2.3. Proposition. Let S be a linear system. Then for all $\alpha, \beta, \gamma \in S \cup \{O\}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

Proof. If $\alpha(\beta\gamma) \in S$ then $\alpha * \beta * \gamma \in S$. Thus by (ii) and (iii), $\alpha * \beta \in S$, so $(\alpha\beta)\gamma \in S$. Similarly $(\alpha\beta)\gamma \in S$ implies $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. If neither $(\alpha\beta)\sigma$ nor $\alpha(\beta\sigma)$ is in S then both are O.

We now give two constructions that will be useful later. First let V be a subset of $S - \{\underline{1}\}$. By V^k we mean the set of all nonzero products $v_1 v_2 \ldots v_k$ with $v_i \in V$; $V^0 = \{\underline{1}\}$. By a slight abuse of notation we write V^* for $\bigcup_{i=0}^{\infty} V^i$. Now if every element of $V^* - \{\underline{1}\}$ has a unique factorization into elements of V, then V^* is a linear system. Note that in general the rank function of V^* differs from that of S.

It is often convenient to consider sequences divided into sections by bars, e.g., |23|716|4. In our applications we will want to treat the bar as an actual element of the sequence, rather than as a punctuation mark. Thus the "barred sequence" given above will be an element of P^*, where $P = \mathbb{P} \cup \{|\}$. To avoid the typographical difficulties of expressions such as "1+|", we shall use the symbol "#" instead of "|"; however, we still call it a bar.

Now let S be a linear system, with prime set P, in which the symbol "#" does not occur. We define the linear system $S \oplus \#$ to be the set of those sequences in $(P \cup \{\#\})^*$ of the form $\pi_1 * \pi_2 * \ldots * \pi_k$, where each π_i is either a prime of S or a bar, and such that when all bars are removed, an element of S remains. It is clear that $S \oplus \#$ so defined is a linear system, and its primes are those of S together with #. We call the elements of $S \oplus \#$ <u>barred sequences</u>.

The following combination of the above two constructions will be very useful. Let S be a linear system and let V be any subset of S. Let W be the subset of $S \oplus \#$ consisting of all elements of the form $\#\alpha$ for some $\alpha \in V$.

Then W^* is a linear system whose primes are in one-to-one correspondence with the elements of V.

CHAPTER 3

COUNTING SERIES

1. The Total Algebra

Let S be a linear system and let R be the ring $\mathbb{Q}[[p,q,t,t_1,\ldots]]$. Let A be the set of all formal sums $\sum_{\alpha\in S} c_\alpha \alpha$, with $c_\alpha \in R$, such that $c_\alpha \neq 0$ for only finitely many values of α. Then the multiplication defined on $S \cup \{0\}$ in Chapter 2 extends by linearity to an associative multiplication on A. Then if we define scalar multiplication by elements of R in the obvious way, A becomes an R-algebra which is free as an R-module, with basis S. It is easily verified that A satisfies conditions (i), (ii), and (iii) of Chapter 1, and can therefore be extended to the counting algebra $\hat{A}$ of all formal sums $\sum_{\alpha\in S} c_\alpha \alpha$, with basis S. We call $\hat{A}$ the <u>total algebra</u> of S and denote it by $R[[S]]$.

<u>Remark</u>. It follows from condition (iii) of the definition of linear system that if S is a linear system with prime set P, then $R[[S]]$ is a quotient algebra of $R[[P^*]]$. And in fact our fundamental result about total algebras of linear systems, Theorem 4.1, need only be proved for the special case of free monoids. The

generalization to arbitrary linear systems, though algebraically insubstantial, is nevertheless useful in actually solving problems.

3.1. Definition. If V is any subset of S, its total counting series is $\Gamma(V) = \sum_{\alpha \in V} \alpha$.

Note that V is completely determined by its total counting series.

3.1. Proposition. Let S be a linear system and let P be the set of primes of S. Then

$$\Gamma(S) = [\underline{1} - \Gamma(P)]^{-1}.$$

Proof. $[\underline{1} - \Gamma(P)]^{-1} = \sum_{n=0}^{\infty} [\Gamma(P)]^n$ by Proposition 1.3. This equals $\Gamma(S)$ by unique factorization.

2. Counting Algebras

Useful information about a subset V of S can often be obtained by considering the image of $\Gamma(V)$ under some homomorphism from a subalgebra of $R[[S]]$ containing $\Gamma(V)$ to some other counting algebra B. We call such a B a counting algebra for S. If f is the image of $\Gamma(V)$ in B, we write $\Gamma(V) \twoheadrightarrow f$ and we call f the counting series for V in B. In many cases the homomorphism $R[[S]] \to B$ will be clear from the context and will not be described explicitly.

In this and the next two sections we discuss some of the most important counting algebras for the linear system $\mathbb{P}^*$ of sequences of positive integers. The total algebra $R[[\mathbb{P}^*]]$ is the R-algebra of formal power series in the noncommuting indeterminates "1," "2," etc. To avoid confusion we shall often write $X_1, X_2, \ldots$ for $1, 2, \ldots$ in this context. Thus "$X_1 + X_2$" represents the element of $R[[\mathbb{P}^*]]$ which might otherwise be denoted by the misleading expression "1+2."

One of the simplest and **most** useful counting algebras for $\mathbb{P}^*$ is the algebra $R[[x_1, x_2, \ldots]]$ of formal power series in the (commuting) indeterminates $x_1, x_2, \ldots,$ which we sometimes write as $R[[\underline{x}]]$. Then the homomorphism $R[[S]] \to R[[\underline{x}]]$ is determined by $X_i \mapsto x_i$ and continuity. (Henceforth we shall assume that all homomorphisms between counting algebras are continuous.)

3. Reduced Sequences

Let $\alpha = b_1 b_2 \ldots b_n \in \mathbb{P}^*$ and let $\{b_1, b_2, \ldots, b_n\} = \{c_1, c_2, \ldots, c_k\}$ (as sets) where $c_1 < c_2 < \ldots < c_k$. Then the <u>reduction</u> of α, which we denote by red (α) is obtained by replacing each occurrence of c_i in α with i. For example, red $(33173) = 22132$. If red $(\alpha) = \alpha$, we say that α is <u>reduced</u>.

Let α and β be reduced sequences of lengths $\ell(\alpha) = m$ and $\ell(\beta) = n$. We define their <u>reduced product</u> $\alpha \circ \beta$ to be the set of all reduced sequences $b_1 b_2 \ldots b_{m+n}$ such that red $(b_1 \ldots b_m) = \alpha$ and red $(b_{m+1} \ldots b_{m+n}) = \beta$.

Now for any reduced sequence α, let $\langle\alpha\rangle \in R[[\mathbb{P}^*]]$ be the sum of all sequences in P^* whose reduction is α. Let Sym $(\mathbb{P}^*)$ be the subalgebra of $R[[\mathbb{P}^*]]$ generated by all such $\langle\alpha\rangle$. The following is straightforward.

3.2. <u>Proposition</u>. Sym $(\mathbb{P}^*)$ is a power series algebra with basis $\{\langle\alpha\rangle | \alpha \text{ is reduced}\}$, and

$$\langle\alpha\rangle\langle\beta\rangle = \sum_{\gamma \in \alpha \circ \beta} \langle\gamma\rangle .$$

We introduce here two important sets of elements of Sym $(\mathbb{P}^*)$.

3.3. <u>Definition</u>.

$$H_n = \sum_{i_1 \le i_2 \le \ldots \le i_n} X_{i_1} X_{i_2} \ldots X_{i_n}$$

and

$$A_n = \sum_{i_1 > i_2 > \ldots > i_n} X_{i_1} X_{i_2} \ldots X_{i_n}, \text{ with } H_0 = A_0 = \underline{1}.$$

The images of H_n and A_n in $R[[\underline{x}]]$, which we denote by h_n and a_n, are the complete homogeneous symmetric functions and the elementary symmetric functions of the x_i.

Two homomorphic images of Sym ($\mathbb{P}$*) are of special interest to us. The first is the algebra of symmetric functions in the commuting variables $x_1, x_2, \ldots$, which is the image of Sym (P*) in $R[[\underline{x}]]$. Less obvious is the second, the algebra of Eulerian counting series, which we now discuss.

4. Eulerian Counting Series

3.2. Definition. For $n \in P$ we define $[n]_p$ to be $1 + p + p^2 + \ldots + p^{n-1}$, and for $n \in P$ we define $n!_p$ to be $[1]_p[2]_p \ldots [n]_p$, with $0!_p = 1$. We define the p-binomial (or Gaussian) coefficient $\binom{m+n}{n}_p$ to be $\frac{(m+n)!_p}{m!_p n!_p}$.

It is traditional to use q where we have used p; however, we wish to reserve q for another situation in which its use is also traditional.

3.3. Definition. The algebra of Eulerian counting series is the counting algebra $R[[z]]$ with basis

$$\left\{\frac{z^n}{n!_p}\right\}_{n=0}^{\infty}.$$

Thus an Eulerian counting series is naturally expressed in the form $\sum_{n=0}^{\infty} f_n \frac{z^n}{n!_p}$. Note that on setting $p = 1$, $[n]_p$ becomes n, so $n!_p$ becomes $n!$ and Eulerian counting series become exponential counting series.

3.4. <u>Definition</u>. Let $\alpha = b_1 b_2 \ldots b_n$ be a sequence. An <u>inversion</u> of α is a pair (i,j) with $i < j$ and $b_i > b_j$. The <u>inversion number</u> $I(\alpha)$ of α is the number of inversions of α.

The important connection between Eulerian counting series and inversions of permutations is given by the following theorem.

3.5. <u>Theorem</u>. The map

$$\langle\alpha\rangle \mapsto p^{I(a)} \frac{z^{\ell(\alpha)}}{\ell(\alpha)!_p}, \quad \text{if } \alpha \text{ is a permutation,}$$

$$\langle\alpha\rangle \mapsto 0, \text{ otherwise,}$$

extends by linearity and continuity to a homomorphism $\mathrm{Sym}\,(\mathbb{P}^*) \to \mathbb{R}[[z]]$. (Recall that $\ell(\alpha)$ is the length of α.)

Theorem 3.5 allows us to extract from the total counting series for a set of sequences which is "symmetric," the counting series for permutations by number of inversions. Note that the image of H_n is $\frac{z^n}{n!_p}$ and that of A_n is $p^{\binom{n}{2}} \frac{z^n}{n!_p}$.

To prove Theorem 3.5 we require several lemmas. Recall that $\mathfrak{S}_n$ is the set of reduced permutations of length n.

3.6. Lemma. (Rodrigues [73])

$$\sum_{\alpha\in\mathfrak{S}_n} p^{I(\alpha)} = n!_p.$$

Proof. We proceed by induction on n. The lemma is true for $n = 0$ and $n = 1$. For $n > 1$ every permutation in $\mathfrak{S}_n$ can be obtained uniquely from some permutation in $\mathfrak{S}_{n-1}$ by inserting "n" in any of n possible positions. The number of new inversions created is equal to the number of integers to the right of the inserted "n." Thus

$$\sum_{\alpha\in\mathfrak{S}_n} p^{I(\alpha)} = [\sum_{\alpha\in\mathfrak{S}_{n-1}} p^{I(\alpha)}]\cdot(1 + p + \ldots + p^{n-1}).$$

3.7. Lemma. Let α and β be reduced permutations of lengths m and n, respectively. Let $[\alpha,\beta] = (\alpha\circ\beta)\cap\mathfrak{S}_{m+n}$. Then

$$\sum_{\gamma\in[\alpha,\beta]} p^{I(\gamma)} = p^{I(\alpha)+I(\beta)} \binom{m+n}{m}_p.$$

Proof. Let $\gamma = b_1b_2\ldots b_mc_1c_2\ldots c_n$ be an element of $[\alpha,\beta]$. Then the inversions of γ can be grouped into three classes: those involving two $\underline{b}$'s, those involving two $\underline{c}$'s, and those involving a $\underline{b}$ and a $\underline{c}$. The numbers of inversions in the first and second classes are clearly $I(\alpha)$ and $I(\beta)$. The number of inversions in the third class depends only on the sets $\{b_1,\ldots,b_m\}$ and $\{c_1,\ldots,c_n\}$ and not on the particular permutations α and β. Thus for each m and n in $\mathbb{N}$ there is a polynomial $B_{m,n}(p)$ such that for all $\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$,

$$\sum_{\gamma\varepsilon[\alpha,\beta]} p^{I(\gamma)} = p^{I(\alpha)+I(\beta)}B_{m,n}(p).$$

Now every $\gamma \in \mathfrak{S}_{m+n}$ is in $[\alpha,\beta]$ for some unique $\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$, thus

$$\sum_{\gamma\varepsilon\mathfrak{S}_{m+n}} p^{I(\gamma)} = \sum_{\substack{\alpha\varepsilon\mathfrak{S}_m \\ \beta\varepsilon\mathfrak{S}_n}} \sum_{\gamma\varepsilon[\alpha,\beta]} p^{I(\gamma)}$$

$$= \sum_{\substack{\alpha\varepsilon\mathfrak{S}_m \\ \beta\varepsilon\mathfrak{S}_n}} p^{I(\alpha)+I(\beta)}B_{m,n}(p)$$

$$= B_{m,n}(p)\Big[\sum_{\alpha\varepsilon\mathfrak{S}_m} p^{I(\alpha)}\Big]\Big[\sum_{\beta\varepsilon\mathfrak{S}_n} p^{I(\beta)}\Big].$$

Thus by Lemma 3.6, $(m+n)!_p = B_{m,n}(p)m!_p n!_p$, whence the lemma follows.

<u>Proof</u> <u>of</u> <u>Theorem</u> 3.5. We need only show that for $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$, we have

$$\sum_{\gamma \in [\alpha,\beta]} p^{I(\gamma)} \frac{z^{m+n}}{(m+n)!_p} = p^{I(\alpha)} \frac{z^m}{m!_p} p^{I(\beta)} \frac{z^n}{n!_p},$$

which follows immediately from Lemma 3.7.

<u>Remark</u>. That inversions of permutations and sequences are related to p-binomial coefficients has long been known ([1], [15], [42], [65]); however, the first systematic use of Eulerian counting series in enumeration of permutations by inversions was made recently by Stanley [79]. We note that Eulerian counting series arise in the study of vector spaces over finite fields [50]. A connection between vector spaces over finite fields and inversions of permutations can be derived from a correspondence of Knuth [57]; we hope to elaborate on this connection elsewhere.

CHAPTER 4

THE INVERSION THEOREM

1. Linked Sets

Let S be a linear system and let P be its set of primes. For any subsets V and W of S let $VW = \{\alpha\beta | \alpha \in V \text{ and } \beta \in W\}$. We call the elements of P^2 links. If $\pi_1\pi_2\ldots\pi_n$ is the prime factorization of $\alpha \in S$ then the links of α are the links $\pi_i\pi_{i+1}$ for $i = 1,2,\ldots,n-1$. Note that if $r(\alpha) < 2$, α has no links. (Recall that $r(\alpha)$ is the number of primes in the prime factorization of α.)

Let L be any subset of P^2. We define C_L to be the set of elements of S all of whose links are in L. Note that C_L contains all elements of S of rank 0 or 1. If $C \subseteq S$ is of the form C_L for some $L \subseteq P^2$ then we call C a linked set. Note that in this case L is uniquely determined by C.

Now let C be a linked set in S, so $C = C_L$ for some $L \subseteq P^2$. Let $\overline{L} = P^2 - L$, and let $\overline{C} = C_{\overline{L}}$.

If V is any subset of S, we define its alternating counting series to be

$$\overline{\Gamma}(V) = \sum_{\alpha \varepsilon V} (-1)^{r(\alpha)} \alpha .$$

4.1. The Inversion Theorem.

Let C be a linked set in S. Then

$$\Gamma(C)\overline{\Gamma}(\overline{C}) = \underline{1}.$$

We prove Theorem 4.1 with the help of the following lemma:

4.2. Lemma. If $\gamma \varepsilon C\overline{C}$ and $\gamma \neq \underline{1}$ then there are exactly two factorizations $\gamma = \alpha_1\beta_1 = \alpha_2\beta_2$ with $\alpha_i \varepsilon C$ and $\beta_i \varepsilon \overline{C}$. Moreover, $r(\beta_1) - r(\beta_2) = \pm 1$.

Proof. The assertion is clearly true for $r(\alpha) = 1$. Otherwise α has a prime factorization $\pi_1\pi_2\ldots\pi_n$ with $n \geq 2$. Then for some k, with $1 \leq\leq k \leq n$, $\pi_{i-1}\pi_i \varepsilon C$ iff $2 \leq i \leq k$. Then the two factorizations of γ are $(\pi_1\ldots\pi_{k-1})(\pi_k\ldots\pi_n)$ and $(\pi_1\ldots\pi_k)(\pi_{k+1}\ldots\pi_n)$.

Proof of Theorem 4.1. We have

$$\Gamma(C)\overline{\Gamma}(\overline{C}) = \sum_{\substack{\alpha\varepsilon C \\ \beta\varepsilon\overline{C}}} (-1)^{r(\beta)}\alpha\beta = \sum_{\gamma\varepsilon C\overline{C}} \gamma \sum_{\substack{\alpha\beta=\gamma \\ \alpha\varepsilon C \\ \beta\varepsilon\overline{C}}} (-1)^{r(\beta)}.$$ But by

the lemma, the last sum is zero for $\gamma \neq \underline{1}$; it is clearly

1 for $\gamma = \underline{1}$.

A result equivalent to Theorem 4.1 has been obtained independently by Carlitz, Scoville, and Vaughan [32]; similar (but somewhat more complicated) reasoning has been used by Robinson [72] and Cartier and Foata [34] on other enumeration problems involving inversion of counting series.

2. A Simple Example

Let C be the set of nondecreasing sequences in $\mathbb{P}^*$. Then C is a linked set: its links are sequences b_1b_2 with $b_1 \le b_2$. Thus $\overline{C}$ is the set of (strictly) decreasing sequences. The total counting series of C and $\overline{C}$ are given by $\Gamma(C) = \sum_{n=0}^{\infty} H_n$ and $\Gamma(\overline{C}) = \sum_{n=0}^{\infty} A_n$, thus from the Inversion Theorem we get

$$\sum_{n=0}^{\infty} H_n = [\sum_{n=0}^{\infty} (-1)^n A_n]^{-1}. \qquad (1)$$

The image of (1) in $R[[\underline{x}]]$ is the well-known symmetric function identity

$$\sum_{n=0}^{\infty} h_n = [\sum_{n=0}^{\infty} (-1)^n a_n]^{-1}.$$

The image of (1) in the algebra of Eulerian series is more interesting. There is one reduced permutation of length n in C for each n; it has no inversions.

There is one reduced permutation of length n in $\overline{C}$; it has $\binom{n}{2}$ inversions. Thus the Eulerian image of (1) is the well-known identity

$$\sum_{n=0}^{\infty} \frac{z^n}{n!_p} = \left[\sum_{n=0}^{\infty} (-1)^n p^{\binom{n}{2}} \frac{z^n}{n!_p}\right]^{-1}.$$

3. Waves

A wave [16] is a sequence of positive integers with adjacent entries unequal. Let C be the set of waves. Then C is a linked set and $\overline{C}$ is the set of sequences all of whose elements are equal. Thus

$$\Gamma(\overline{C}) = \underline{1} + \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (-1)^n X_i^n = \underline{1} + \sum_{i=1}^{\infty} X_i(\underline{1} + X_i)^{-1}, \text{ so}$$

$$\Gamma(C) = [\underline{1} + \sum_{i=1}^{\infty} X_i(1 + X_i)^{-1}]^{-1}.$$

Then the image of $\Gamma(C)$ in $R[[\underline{x}]]$ is

$[1 + \sum_{i=1}^{\infty} \frac{x_i}{1+x_i}]^{-1}$, a result of Carlitz [16]. See also [40].

4. Pairs of Sequences

In this section we consider the problem of counting pairs of sequences $(b_1 b_2 \ldots b_n, c_1 c_2 \ldots c_n)$ with the property that for $1 \le i < n$ either $b_i > b_{i+1}$ or

$c_i > c_{i+1}$ (or both).

In order to rephrase the problem in a form to which our theory applies, let us define $\mathbb{P}_2$ to be the set of ordered pairs $\begin{matrix} b \\ c \end{matrix}$ of positive integers, written vertically. Then an element of $\mathbb{P}_2^*$ can be represented in the form

$$\begin{matrix} b_1 & b_2 & \cdots & b_n \\ c_1 & c_2 & \cdots & c_n \end{matrix} , \tag{1}$$

Now let C be the linked set in $\mathbb{P}_2^*$ of elements of the form (1) such that for each i, $b_i > b_{i+1}$ or $c_i > c_{i+1}$. Then $\overline{C}$ consists of those elements of the form (1) for which $b_1 \leq b_2 \leq \cdots \leq b_n$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Then the Inversion Theorem leads to the following two theorems, detailed proofs of which we omit.

4.2. <u>Theorem</u>. Let $g(x_1,x_2,\ldots;\ y_1,y_2,\ldots)$ $= \sum_{\alpha\in C} x_{b_1}x_{b_2}\cdots x_{b_n}y_{c_1}y_{c_2}\cdots y_{c_n}$ where α has the form (1). Let h_n' be the complete homogeneous symmetric function of degree n in $y_1,y_2,\ldots$. Then

$$g(x_1,x_2,\ldots;\ y_1,y_2,\ldots) = \left[\sum_{n=0}^{\infty} (-1)^n h_n h_n'\right]^{-1}.$$

4.3. Theorem. For $\gamma \varepsilon C$ of the form (1), let $\alpha = b_1 b_2 \ldots b_n$ and let $\beta = c_1 c_2 \ldots c_n$. Let $D = \{\gamma \varepsilon C | \alpha$ and β are reduced permutations$\}$. Then

$$\sum_{\gamma \varepsilon D} p^{I(\alpha)} q^{I(\beta)} \frac{z^{\ell(\gamma)}}{[\ell(\gamma)]!_p [\ell(\gamma)]!_q} = [\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!_p n!_q}]^{-1}.$$

Related results can be found in [31], [32], and [79].

5. Up-Down Sequences

An up-down sequence is a sequence $b_1 b_2 \ldots b_n$ with $b_1 \leq b_2 > b_3 \leq b_4 \ldots$ In this section we consider up-down sequences of even length. Let P be the set of sequences in $\mathbb{P}^*$ of the form $b_1 b_2$ where $b_1 \leq b_2$. Then P^* is a linear system with prime set P. Now let L be the set of links of P^* of the form $b_1 b_2 b_3 b_4$ where $b_1 \leq b_2 > b_3 \leq b_4$. Then $C = C_L$ is the set of up-down sequences of even length, and $\bar{C}$ is the set of nondecreasing sequences of even length. Thus

$$\Gamma(\bar{C}) = \sum_{n=0}^{\infty} H_{2n}, \quad \text{so}$$

$$\Gamma(C) = [\sum_{n=0}^{\infty} (-1)^n H_{2n}]^{-1}.$$

Then the image of $\Gamma(C)$ in $R[[\underline{x}]]$ is $[\sum_{n=0}^{\infty}(-1)^n h_{2n}]^{-1}$ as first obtained (in a somewhat different form) by Carlitz [19]. See also Stanley [79]. The Eulerian counting series for C is $[\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!_p}]^{-1}$ as first shown by Stanley [79]. The special case $p = 1$, in which the series reduces to sec z, is due to André [7].

If $\alpha = b_1 b_2 \ldots b_{2n}$ is an up-down permutation, we define the <u>top line</u> of α to be the sequence $b_2 b_4 \ldots b_{2n}$. Problems involving enumeration of permutations with respect to the top line have been considered by Carlitz in [18] and [23].
We consider here the problem of counting up-down permutations of length divisible by four whose top lines are up-down. (There does not seem to be a simple solution to the analogous problem for sequences.)

Let C be the set of up-down permutations $b_1 b_2 \ldots b_{4n}$ whose top lines are up-down, that is,

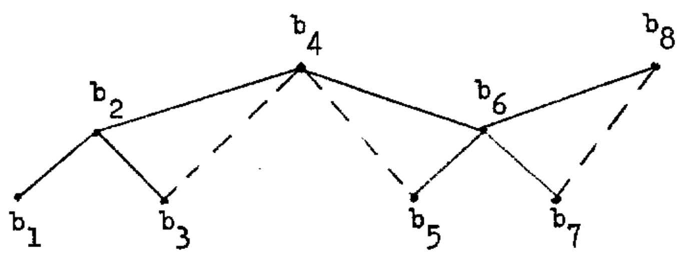
(i) $b_{2i+1} < b_{2i+2}$ $\qquad 0 \leq i \leq 2n-1$

(ii) $b_{2i} > b_{2i+1}$ $\qquad 1 \leq i \leq 2n-1$

(iii) $b_{4i+2} < b_{4i+4}$ $\qquad 0 \leq i \leq n-1$

(iv) $b_{4i} > b_{4i+2}$ $\qquad 1 \leq i \leq n-1$.

These conditions are illustrated by the following diagram:

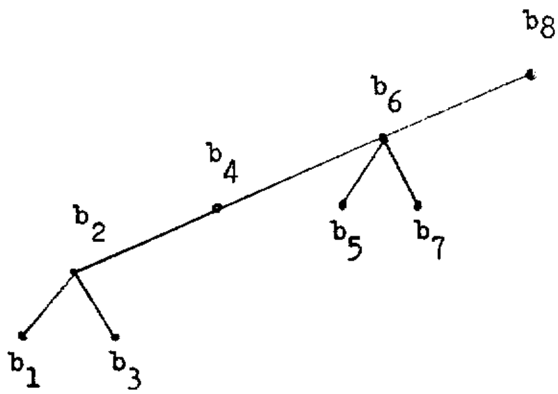


Note that the dotted lines are superfluous, i.e., conditions (i) and (ii) may be replaced by

(i') $\quad b_{4i+1} < b_{4i+2} \qquad 0 \leq i \leq n-1$

(ii') $\quad b_{4i+2} > b_{4i+3} \qquad 0 \leq i \leq n-1.$

Now let S be the set of all permutations of length divisible by four satisfying conditions (i'), (ii') and (iii). Then S is a linear system and C is a linked set in S. The elements of $\bar{C}$ are permutations $b_1 b_2 \ldots b_{4n}$ satisfying (i'), (ii'), (iii), and the condition $b_{4i} < b_{4i+2}$, $1 \leq i \leq n-1$. The conditions on $\overline{C}$ are represented by the following diagram:



Now let $F_n(p) = \sum_{\alpha \in C \cap \mathfrak{S}_{4n}} p^{I(\alpha)}$ and let

$G_n(p) = \sum_{\alpha \in \overline{C} \cap \mathfrak{S}_{4n}} p^{I(\alpha)}$. We leave it to the reader to verify that

$$\sum_{n=0}^{\infty} F_n(p) \frac{z^{4n}}{(4n)!_p} = \left[\sum_{n=0}^{\infty} (-1)^n G_n(p) \frac{z^{4n}}{(4n!)_p} \right]^{-1} ,$$

whence a simple counting argument yields

4.4. <u>Theorem.</u> $\sum_{n=0}^{\infty} F_n(p) \frac{z^{4n}}{(4n)!_p}$

$$= \left[\sum_{n=0}^{\infty} (-1)^n p^n \left\{ \prod_{i=0}^{n-1} [4i+1]_p [4i+2]_p \right\} \frac{z^{4n}}{(4n)!_p} \right]^{-1} .$$

Analogous results for permutations of lengths not divisible by four may be obtained by the methods of Chapter 6.

6. A Lattice Path Problem

In this section we prove the following theorem:

4.5. Theorem. For fixed $k \in \mathbb{P}$ and $r \in \mathbb{N}$, let g_n be the number of lattice paths in the plane from the origin to the point $(n, rn + k - 1)$, using unit horizontal and vertical steps, which never pass below the line $y = rx$ nor above the line $y = rx + k - 1$. Then $\sum_{n=0}^{\infty} g_n z^n = [\sum_{n=0}^{\infty} (-1)^n \binom{k - r(n-1)}{n} z^n]^{-1}$, where we adopt the convention that $\binom{a}{b} = 0$ for $a < 0$.

To prove Theorem 4.5 we consider the following problem: how many sequences $b_1 b_2 \ldots b_n$ are there with each $b_i \in [k]$ and satisfying

(i) $b_i \le b_{i+1} + r$?

To solve this problem, let $S = [k]^*$ (the set of sequences of elements of $[k]$) and let C be the subset of S satisfying condition (i). Then C is a linked set in S and $\overline{C}$ consists of those sequences satisfying $b_i > b_{i+1} + r$.

Now let e_n be the number of sequences in $\overline{C}$ of length n. Since all sequences in $\overline{C}$ are decreasing,

e_n is the number of n-element subsets T of $[k]$ such that $i,j \in T$ implies $|i-j| > r$. It is then easily seen that $e_n = \binom{k - r(n-1)}{n}$. It follows from the Inversion Theorem that if f_n is the number of elements of C of length n, then

$$\sum_{n=0}^{\infty} f_n z^n = [\sum_{n=0}^{\infty} (-1)^n \binom{k - r(n-1)}{n} z^n]^{-1} .$$

To prove Theorem 4.5 we need only show that $f_n = g_n$, which we do by exhibiting a bijection between the elements of C of length n and the paths which are counted by g_n.

Let $b_1 b_2 \ldots b_n$ be an element of C. Now let $c_i = b_{i+1} + ri - 1$ for $i = 0,1,\ldots,n-1$. Then condition (i) is equivalent to the condition $c_i \leq c_{i+1}$ and the condition $1 \leq b_i \leq k$ is equivalent to the condition $ri \leq c_i \leq ri + k - 1$. To the sequence $b_1 b_2 \ldots b_n$ we now correspond the path through the lattice points $(0,0)$, $(0,c_0)$, $(1,c_0)$, $(1,c_1)$, $(2,c_1)$, $\ldots$, $(n-1,c_{n-1})$, (n,c_{n-1}), $(n,nr + k - 1)$. It is easy to see that this correspondence is a bijection with the required properties.

Problems involving lattice paths which lie between two lines of slope 1 are discussed in [48] and [52], but even in this case our formula seems to be new.

CHAPTER 5

FURTHER APPLICATIONS OF THE INVERSION THEOREM

1. Runs

5.1. Definition. Let $\alpha = b_1 b_2 \ldots b_n$ be an element of $\mathbb{P}^*$. A run (or nondecreasing run) of α is a maximal nondecreasing subsequence of α of the form $b_i b_{i+1} \ldots b_j$. The empty sequence has no runs.

It is clear that every sequence has a unique factorization into runs.

In this section we give a complete answer to the problem of counting sequences by runs of each possible length. Some related results are given in sections 7 and 8 of [54].

5.2. Theorem. Let θ be the linear operator which takes $\sum_{n=0}^{\infty} c_n z^n$ to $\sum_{n=0}^{\infty} c_n H_n^{\prime}$ for $c_n \in R$. Let $g(z) = \sum_{n=1}^{\infty} t_n z^n$. For $\alpha \in \mathbb{P}^*$, let m_i be the number of runs of α of length i. Then $\sum_{\alpha \in \mathbb{P}^*} \alpha t_1^{m_1} t_2^{m_2} \ldots =$ $\{\theta[(1+g(z))^{-1}]\}^{-1}$.

Before proving Theorem 5.2 we need two definitions.

5.3. Definition. If $\alpha = b_1 b_2 \ldots b_n$ is a sequence, we define the spaces of α to be the integers $0,1,2,\ldots,n$. We think of space i as lying between b_i and b_{i+1} for $1 \leq i \leq n-1$, with space 0 lying to the left of b_1 and space n to the right of b_n. In particular, the phrase "insert a bar in space i" means "insert a bar between b_i and b_{i+1}," etc. In this section we will be concerned only with spaces $0,1,\ldots,n-1$, which we call the proper spaces of α. (The empty sequence has one space, which is not proper.)

5.4. Definition. A fall of α is a proper space i such that b_{i+1} begins a run, that is, $i = 0$ or $b_i > b_{i+1}$. Thus the number of runs of α is equal to the number of falls.

Proof of Theorem 5.2. Let V be the subset of $\mathbb{P}^* \oplus \#$ consisting of all elements of the form $\#\alpha$ where α is a nonempty nondecreasing sequence in $\mathbb{P}^*$. Then V^* is a linear system with prime set V. Let C be the linked set in V^* whose links are of the form $\#\alpha\#\beta$ such that the last entry of α is greater than the first entry of β. Then C is the set of those elements of $\mathbb{P}^* \oplus \#$ which are obtained by inserting a bar in each fall of an element of $\mathbb{P}^*$.

Now $\overline{C}$ is the linked set in V^* whose links are of the form $\#\alpha\#\beta$, where the last entry of α is less than or equal to the first entry of β; in other words, $\alpha\beta$ is nondecreasing. Thus $\overline{C}$ consists of the empty sequence together with those elements of $\mathbb{P}^* \oplus \#$ which are obtained from a nonempty nondecreasing sequence in $\mathbb{P}^*$ by first inserting a bar in space 0 and then inserting a bar in any number of the other proper spaces.

We now define a homomorphism $R[[V^*]] \to R[[\mathbb{P}^*]]$ by $\#\alpha \to \alpha t_{\ell(\alpha)}$. Then under this homomorphism, $\#\alpha_1\#\alpha_2\ldots\#\alpha_k$ goes to $\alpha_1\alpha_2\ldots\alpha_k t_{\ell(\alpha_1)} t_{\ell(\alpha_2)}\ldots t_{\ell(\alpha_k)}$. Now let $G(C)$, $G(\overline{C})$, and $\overline{G}(\overline{C})$ be the images of $\Gamma(C)$, $\Gamma(\overline{C})$, and $\overline{\Gamma}(\overline{C})$ under this homomorphism. To prove Theorem 5.2 we need only evaluate $G(C)$, which by the Inversion Theorem is equal to $[\overline{G}(\overline{C})]^{-1}$.

It is clear that $G(\overline{C}) = \sum_{n=0}^{\infty} f_n H_n$ where

$$f_n = \sum_{m_1,\ldots,m_n} w_n(m_1,m_2,\ldots,m_n) t_1^{m_1}\ldots t_n^{m_n} \quad \text{and}$$

$w_n(m_1,\ldots,m_n)$ is the number of compositions (ordered partitions) of n into m_1 1's, m_2 2's, etc. Then the f_n satisfy

$$\sum_{n=0}^{\infty} f_n z^n = [1 - g(z)]^{-1}$$

where $g(z) = \sum_{n=1}^{\infty} t_n z^n$. Thus if θ is the linear operator that takes $\sum_{n=0}^{\infty} c_n z^n$ to $\sum_{n=0}^{\infty} c_n H_n$, then $G(\overline{C}) = \theta[(1 - g(z))^{-1}]$. Now since each t_i represents one prime in V_*^* to go from $G(C)$ to $\overline{G}(\overline{C})$ we need only change each t_i to $-t_i$, which has the effect of replacing $g(z)$ by $-g(z)$. Thus
$\overline{G}(\overline{C}) = \theta[(1 + g(z))^{-1}]$, whence $G(C) = \{\theta[(1 + g(z))^{-1}]\}^{-1}$, completing the proof.

2. Examples

We give here some applications of Theorem 5.2 which (except for the last) are obtained by specializing the parameters $t_1, t_2, \ldots$. Note that in these examples we can obtain symmetric-function, Eulerian, and exponential counting series by changing H_n to h_n, $\frac{z^n}{n!_q}$, and $\frac{z^n}{n!}$ respectively in the expression for $G(C)$.

Example 1. To count sequences by number of runs, without regard to their lengths we set $t_1 = t_2 = \ldots = t$. Then $g(z) = t\frac{z}{1-z}$, so $[1 + g(z)]^{-1} = (1 + t\frac{z}{1-z})^{-1}$

$$= \frac{1-z}{1-z+tz} = 1 - \frac{tz}{1-z(1-t)} = 1 - t\sum_{n=1}^{\infty} (1-t)^{n-1} z^n.$$ Thus

$$G(C) \mapsto [\underline{1} - t \sum_{n=1}^{\infty} (1-t)^{n-1} H_n]^{-1}. \qquad (1)$$

A formula closely related to the symmetric-function analogue of (1) was given by MacMahon [64, Vol. 1, p. 212]. Setting $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_p}$, the Eulerian analogue of (1) is $[1 - t \sum_{n=1}^{\infty} (1-t)^{n-1} \frac{z^n}{n!_p}]^{-1} = \frac{1 - t}{1 - te[(1-t)z]}$, which is equivalent to a formula of Stanley [79]. Setting $p = 1$ yields the exponential counting series $\frac{1 - t}{1 - te^{(1-t)z}}$ for the Eulerian polynomials, apparently first given by Riordan [70].

<u>Example 2</u>. To count sequences in which every run has length d, we set $t_d = 1$, $t_i = 0$ for $i \neq d$. Thus $g(z) = z^d$, so $[1 + g(z)]^{-1} = \sum_{n=0}^{\infty} (-1)^n z^{nd}$, so $G(C) \mapsto [\sum_{n=0}^{\infty} (-1)^n H_{nd}]^{-1}$. The corresponding exponential series is due to Carlitz [17], [21]. See also Stanley [79].

<u>Example 3</u>. To count sequences in which every run has length less than k, we set $t_1 = t_2 = \ldots = t_{k-1} = 1$, $t_i = 0$ for $i \geq k$. Then $1 + g(z) = 1 + z + \ldots + z^{k-1} = \frac{1-z^k}{1-z}$. Thus $[1+g(z)]^{-1} = \frac{1-z}{1-z^k} = 1 - z + z^k - z^{k+1} + z^{2k} \ldots$,

so $G(C) \mapsto [\underline{1} - H_1 + H_k - H_{k+1} + H_{2k} - H_{2k+1} + \ldots]^{-1}$. The corresponding exponential counting series is due to David and Barton [36, pp. 156-157].

<u>Example 4</u>. To count sequences with every run of length at least two, set $t_1 = 0$, $t_i = 1$ for $i \geq 2$. Then $1 + g(z) = 1 + \frac{z^2}{1-z} = \frac{1-z+z^2}{1-z}$, so

$$[1 + g(z)]^{-1} = \frac{1-z}{1-z+z^2} = \frac{1-z^2}{1+z^3} = 1 - z^2 - z^3 + z^5 + \ldots,$$

so $G(C) \mapsto [\underline{1} - H_2 - H_3 + H_5 + H_6 - H_8 \ldots]^{-1}$.

<u>Example 5</u>. To count sequences in which every run has odd length, set $t_1 = t_3 = t_5 = \ldots = 1$, $t_2 = t_4 = t_6 = \ldots = 0$. Then $g(z) = z + z^3 + z^5 + \ldots$ $= \frac{z}{1-z^2}$. Thus $[1 + g(z)]^{-1} = (\frac{1+z-z^2}{1-z^2})^{-1} = \frac{1-z^2}{1+z-z^2}$

$$= 1 - \frac{z}{1+z-z^2} = 1 + \sum_{n=1}^{\infty} (-1)^n F_n z^n,$$ where F_n is the n^{th} Fibonacci number. ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$.) Thus $G(C) \mapsto [1 + \sum_{n=1}^{\infty} (-1)^n F_n H_n]^{-1}$.

<u>Example 6</u>. To count sequences of 1's and 2's only, we map $X_i \mapsto 0$ for $i > 2$. If we further map $X_1 \mapsto x$ and $X_2 \mapsto y$, where x and y are commuting indeterminates,

then we have $H_n \mapsto x^n + x^{n-1}y + \ldots + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}$.

Thus $\theta(z^n) \mapsto \frac{x^{n+1} - y^{n+1}}{x - y}$, so by linearity we have for any power series $f(z)$,

$\theta[f(z)] \mapsto \frac{1}{x - y}[xf(x) - yf(y)]$. Thus

$\theta[(1 + g(z))^{-1}] \mapsto \frac{1}{x - y}[\frac{x}{1 + g(x)} - \frac{y}{1 + g(y)}]$, so

$G(C) \mapsto (x-y)[\frac{x}{1 + g(x)} - \frac{y}{1 + g(y)}]^{-1}$.

3. Counting Waves by Falls

In this section we count waves (sequences with adjacent entries unequal) according to the number of falls.

Let W be the subset of $\mathbb{P}^* \oplus \#$ consisting of all elements of the form $\#\alpha$ where α is a nonempty (strictly) increasing sequence in $\mathbb{P}^*$. Let C be the linked set in W^* whose links are of the form $\#\alpha\#\beta$ where the last entry of α is greater than the first entry of β. Then the elements of C are waves with bars in their falls; evaluating $\Gamma(C)$ will solve our problem.

5.3. Theorem.

$$\Gamma(C) = \{\underline{1} - \# \prod_{i=1}^{\infty} (\underline{1} + X_i)(\underline{1} + \#X_i)^{-1}\}^{-1}(\underline{1} - \#).$$

<u>Proof</u>. We define a <u>level</u> of the sequence $b_1 b_2 \ldots b_n$ to be a space i such that $b_i = b_{i+1}$. (Then $1 \leq i \leq n-1$.) Now the links of $\overline{C}$ are of the form $\#\alpha\#\beta$ where the last entry of α is less than or equal to the first entry of β. Thus the nonempty elements of $\overline{C}$ are those elements of W^* obtained from a nonempty nondecreasing sequence in P^* by inserting a bar in space 0, in every level, and in any number of the other proper spaces.

We now claim that

$$\Gamma(\overline{C}) = (\underline{1} + \#)^{-1}(\underline{1} + \#B), \tag{1}$$

where $B = \prod_{i=1}^{\infty} (\underline{1} + X_i)(\underline{1} - \#X_i)^{-1}$. To see this, observe that B is the total counting series for the set D of barred sequences obtained from a nondecreasing sequence by inserting a bar in every level, and then inserting a bar in any number of the other proper spaces. Now $\overline{C}$ consists of the empty sequence together with those nonempty elements of D which begin with a bar. Thus $\Gamma(\overline{C}) - \underline{1} = \#(\underline{1} + \#)^{-1}(B - \underline{1})$, whence (1) follows.

Now since the rank of an element of W^* is the number of bars in it, to get $\Gamma(\overline{C})$ from $\Gamma(C)$ we need only change $\#$ to $-\#$ in (1). Thus

$$\bar{\Gamma}(\bar{C}) = (\underline{1} - \#)^{-1}\{\underline{1} - \# \prod_{i=1}^{\infty} (\underline{1} + X_i)(1 + \#X_i)^{-1}\},$$

whence the theorem follows by the Inversion Theorem.

5.4. Corollary. Under the homomorphism $R[[\mathbb{P}^* \oplus \#]] \to R[[\underline{x}]]$ induced by $\# \mapsto t$, $X_i \mapsto x_i$, we have

$$\Gamma(C) \to \frac{1 - t}{1 - t \prod_{i=1}^{\infty} \frac{1 + x_i}{1 + tx_i}} \tag{2}$$

An equivalent result was given by Carlitz [16]. If we call the expression on the right side of (2) $g(t;x_1,x_2,\ldots)$ then Carlitz's counting series is

$$x[g(\tfrac{y}{x};\ xz_1,xz_2,\ldots) - 1]$$

$$= xy \frac{\prod_{i=1}^{\infty} (1 + yz_i) - \prod_{i=1}^{\infty} (1 + yz_i)}{x \prod_{i=1}^{\infty} (1 + yz_i) - y \prod_{i=1}^{\infty} (1 + xz_i)} .$$

By similar reasoning we can count sequences by levels and falls. (See [16] and [53].)

CHAPTER 6

MATRICES

1. G-Systems

In this chapter we consider certain linear systems associated with digraphs (directed graphs) whose total algebras can be represented as algebras of matrices. Application of the Inversion Theorem to these linear systems leads to formulas involving matrix inversion, and we are thus led to solutions of enumeration problems expressed as determinants and quotients of determinants.

Let G be a digraph on the vertices $v_1, v_2, \ldots, v_s$. Then the edges of G are ordered pairs of vertices. We write (i,j) for the edge (v_i, v_j). For each edge e of G, let P_e be a set of sequences. If $e = (i,j)$, we may write P_{ij} for P_e. Let P be the set of all pairs (π, e) where e is an edge of G and $\pi \in P_e$, and let S be the subset of P^* of all sequences $(\pi_1, e_1)(\pi_2, e_2)\ldots(\pi_n, e_n)$ such that $e_1 e_2 \ldots e_n$ is a path in G. (We include in S the empty sequence.) It is clear that S is a linear system with prime set P.

Given an element $\alpha = (\pi_1, e_1)(\pi_2, e_2)\ldots(\pi_n, e_n)$ in S, let $\underline{\alpha}$ be the sequence $\pi_1 * \pi_2 * \ldots * \pi_n$ (recall that the stars denote juxtaposition) and for $\alpha \neq \underline{1}$ let $E(\alpha)$

be the ordered pair (i,j) such that $e_1e_2 \ldots e_n$ is a path from v_i to v_j. (We leave $E(\underline{1})$ undefined, although we may think of it as being all pairs (i,i).)

6.1. Definition. The linear system S as defined above is a G-system if for $\alpha, \beta \in S - \{\underline{1}\}$, $\underline{\alpha} = \underline{\beta}$ and $E(\alpha) = E(\beta)$ implies $\alpha = \beta$.

If W is any subset of the G-system S, let W_{ij} be the set of all nonempty sequences $\underline{\alpha}$ such that $E(\alpha) = (i,j)$, together with the empty sequence $\underline{1}$ iff $\underline{1} \in W$ and $i = j$. It follows from the definition of a G-system that W is uniquely determined by the sets W_{ij}. Note that this notation is consistent with our earlier use of the notation P_{ij}. We will sometimes, by abuse of notation, refer to elements of S_{ij} as elements of S.

Now let $\underline{P} = \bigcup_e P_e$ and let B be the R-algebra of $s \times s$ matrices with entries in $R[[\underline{P}^*]]$. We define the map $\lambda : S \to B$ as follows: For $\alpha \in S - \{\underline{1}\}$, $\lambda(\alpha)$ is the matrix with $\underline{\alpha}$ in the (i,j) position, where $(i,j) = E(\alpha)$, and with zeroes elsewhere; and $\lambda(\underline{1})$ is the $s \times s$ identity matrix. We extended λ by linearity (and continuity) to a map $R[[S]] \to B$. Then the following theorem is immediate.

6.2. <u>Theorem</u>. The map λ is an isomorphism from $R[[S]]$ onto a subalgebra of B.

In view of Theorem 6.2, for any $W \subseteq S$ we shall identify $\Gamma(W)$ with its image in B. Then by our conventions, $\Gamma(W_{ij})$ is the (i,j) entry of $\Gamma(W)$.

In applying Theorem 6.2 we shall use implicitly the fact that a homomorphism between two algebras induces a homomorphism between the corresponding $s\times s$ matrix algebras.

We shall use the following notation: $(f_{ij})_s$ denotes the $s\times s$ matrix whose (i,j) entry is f_{ij}; $|f_{ij}|_s$ denotes the corresponding determinant.

2. Restricted Sequences

In this section we prove a theorem that generalizes several previously unrelated results.

6.3. <u>Definition</u>. Let α be a sequence in P^*. The <u>fall set</u> of α is the set of nonzero falls of α.

6.4. <u>Definition</u>. For integers m,M we define

$$H_n(m,M) = \sum_{m \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq M} X_{i_1} X_{i_2} \cdots X_{i_m}$$

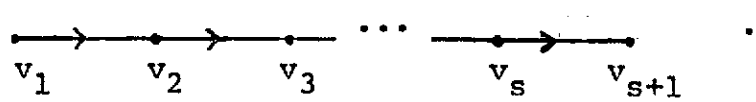
with $H_0(m,M) = \underline{1}$ and $H_n(m,M) = 0$ for $n < 0$.

We write $H_n(M)$ for $H_n(1,M)$, and we let $h_n(m,M)$ and $h_n(M)$ be the images in $R[[\underline{x}]]$ of $H_n(m,M)$ and $H_n(M)$.

6.5. Theorem. Let $m_1 \geq m_2 \geq \cdots \geq m_n$ and $M_1 \geq M_2 \geq \cdots \geq M_n$ be integers and let $F = \{f_2 \leq f_3 \leq \cdots \leq f_s\}$ be a subset of $[n-1]$, with $f_1 = 0$ and $f_{s+1} = n$. Then the number of permutations $b_1 b_2 \ldots b_n$ of the multiset $\{1^{n_1}, 2^{n_2}, \ldots\}$ satisfying $m_i \leq b_i \leq M_i$ for each i, and with fall set F, is the coefficient of $x_1^{n_1} x_2^{n_2} \ldots$ in

$$|h_{f_{j+1}-f_i}(m_{f_i+1}, M_{f_{j+1}})|_s. \tag{1}$$

Proof. Let G be the digraph



We define a G-system S by taking $P_{j,j+1}$ to be the set of nondecreasing sequences $\alpha = b_1 b_2 \ldots b_{f_{j+1}-f_j}$ satisfying

(*) $\quad m_{f_j+i} \leq b_i \leq M_{f_j+i}$

for each i. It will be convenient to set $b_i = d_{f_j+i}$, then α becomes $d_{f_j+1}d_{f_j+2}\cdots d_{f_{j+1}}$ and (*) becomes $m_i \leq d_i \leq M_i$. It is clear that this construction does in fact yield a G-system.

Now let C be the linked set in S such that C_{ij} is the set of sequences

$d_{f_i+1}\ d_{f_i+2}\ \cdots\ d_{f_j}$ for $1 \leq i \leq j \leq s + 1$ such that

$$d_k > d_{k+1} \quad \text{if} \quad k \in F$$

$$d_k \leq d_{k+1} \quad \text{if} \quad k \notin F.$$

Then $\overline{C}_{ij}$ is the set of nondecreasing sequences $d_{f_i+1}\cdots d_{f_j}$ in S.

We now claim that $\Gamma(\overline{C}_{ij}) = H_{f_j-f_i}(m_{f_i+1}, M_{f_j})$.

To see this, we observe that $\overline{C}_{ij}$ is the set of sequences $d_{f_i+1}\cdots d_{f_j}$ satisfying

(i) $d_k \leq d_{k+1}$

(ii) $m_k \leq d_k \leq M_k$

for $f_i + 1 \leq k \leq f_j$. But since $m_1 \geq m_2 \geq \cdots \geq m_n$ and $M_1 \geq M_2 \geq \cdots \geq M_n$, condition (ii) can be replaced by the condition $m_{f_i+1} \leq d_k \leq M_{f_j}$ for $f_i + 1 \leq k \leq f_j$, whence our claim follows.

Thus by the Inversion Theorem,

$\Gamma(C) = ((-1)^{i-j}H_{f_j-f_i}(m_{f_i+1},M_j))^{-1}_{s+1}$. Now the total counting series for sequences $d_1d_2\ldots d_n$ in C is $\Gamma(C_{1,n+1})$. Applying the formula for the inverse of a matrix to the image of $\Gamma(C)$ in the algebra of matrices over $R[[\underline{x}]]$, we get

$$(-1)^s\left|(-1)^{i-j+1}h_{f_{j+1}-f_i}(m_{f_i+1},M_{f_{j+1}})\right|_s$$

as the image in $R[[\underline{x}]]$ of $\Gamma(C_{1,n+1})$. This is easily seen to be equal to the expression given in the statement of the theorem.

6.6. Corollary. If we set each $x_i = 1$ in Theorem 6.5, the determinant becomes

$$\left|\binom{M_{f_{j+1}}-m_{f_i+1}+f_{j+1}-f_i}{f_{j+1}-f_i}\right|_s .$$

Proof. When we set each $x_i = 1$, $h_k(m,M)$ becomes the number of ways of choosing k elements, with repetition, from an M-m+1 element set, which is $\binom{M-m+k}{k}$.

If we set $m_i = 1$ and let M_i be infinite for all i, then (1) becomes

$$|h_{f_{j+1}-f_i}|_s \tag{2}$$

as obtained by MacMahon [62], [64, Vol. 1, p. 200]. The image of (2) in the algebra of exponential counting series is

$$\left| \frac{z^{f_{j+1}-f_i}}{(f_{j+1}-f_i)!} \right|_s , \tag{3}$$

where by convention, $\frac{z^k}{k!} = 0$ for $k < 0$. Now (3) is equal to $N \frac{z^n}{n!}$ where N is the number of permutations in $\mathfrak{S}_n$ with fall set F. Thus

$$N = n! \left| \frac{1}{(f_{j+1}-f_i)!} \right|_s , \tag{4}$$

as also found by MacMahon [62], [64, Vol. 1, p. 190], who apparently did not realize it could be derived from (2).

Formulas equivalent to (4) were independently rediscovered by Niven [67] and Carlitz [17]. See also [5], [12].

Equation (4) is closely related to a result of Forcade [47] on paths in tournaments, which we now describe.

We may define a <u>tournament</u> T on [n] to be a subset of [n]×[n] such that

(i) For all $i,j \in [n]$ with $i \neq j$, $(i,j) \in T$ iff $(j,i) \notin T$.

(ii) For all $i \in [n]$, $(i,i) \notin T$. Given a permutation $b_1 b_2 \ldots b_n$ of [n], we may define its <u>T-fall set</u> to be the set of those i for which $(b_i, b_{i+1}) \notin T$. According to a theorem of Redei, for any tournament T on [n], the number of permutations of [n] with empty T-fall set is odd. It follows that for any set $F \subseteq [n-1]$, the parity of the number of permutations of [n] with T-fall set F is independent of T, and thus is the parity of N given by (4). In particular, if N is odd, then for any tournament T on [n] there exists a permutation of of [n] whose T-fall set is F.

3. Inversion Sequences

6.7. <u>Definition</u>. Let $\alpha = b_1 b_2 \ldots b_n$ be a reduced permutation. Let $c_i = \text{card}\,\{j \mid i \leq j \text{ and } b_i \geq b_j\}$ for $1 \leq i \leq n$. Then we call $\text{Inv}(\alpha) = c_1 c_2 \ldots c_n$ the <u>inversion sequence</u> of α.

For example, the inversion sequence of 614352 is 613221. Note that if $\text{Inv}(\alpha) = c_1 c_2 \ldots c_n$ then α has

$\sum_{i=1}^{n} (c_i - 1)$ inversions.

It is clear that

$$(*) \qquad 1 \le c_i \le n - i + 1,$$

and that any sequence $c_1 c_2 \ldots c_n$ satisfying (*) is the inversion sequence of a unique permutation in $\mathfrak{S}_n$. Note that the falls of Inv(α) are the same as those of α. Now let $\omega(\alpha) = x_{c_1} x_{c_2} \ldots x_{c_n}$, where $\text{Inv}(\alpha) = c_1 c_2 \ldots c_n$. Applying Theorem 6.5 we have

6.8. Theorem. Let K be the set of permutations in $\mathfrak{S}_n$ with fall set $F \subseteq [n-1]$. Then

$$\sum_{\alpha \varepsilon k} \omega(\alpha) = |h_{f_{j+1}-f_i}(n-f_{j+1}+1)|_s, \qquad (6)$$

where f_i and s are as in Theorem 6.5.

Setting $x_i = 1$ in (6) yields $\left|\binom{n - f_i}{f_{j+1} - f_i}\right|_s$,

which is easily seen to be equal to (4). Setting $x_i = p^{i-1}$ (so that $\omega(\alpha) = p^{I(\alpha)}$) yields $\left|\binom{n - f_i}{f_{j+1} - f_i}_p\right|_s$,

a result of Stanley [79]. (This formula can also be derived analogously to formula (4).)

Setting $x_i = i$, so that $\omega(\pi) = \prod_{i=1}^{n} c_i$, yields $|S(n-f_i+1, n-f_{j+1}+1)|_s$, where $S(m,k)$ is the Stirling number of the second kind. These numbers are related to the "Π-Eulerian numbers" of [78].

4. Nondecreasing Sequences.

Theorem 6.5 can easily be dualized: we may define the "non-fall set" of a sequence of length n to be the set of those spaces in [n-1] which are not falls. Then given integers $m_1 \leq m_2 \leq \cdots \leq m_n$ and $M_1 \leq M_2 \leq \cdots \leq M_n$ we can find the counting series for sequences $b_1 b_2 \ldots b_n$ satisfying $m_i \leq b_i \leq M_i$ with given non-fall set. We state here only the most interesting special case, that in which the non-fall set is all of [n-1]. The proof, which we omit, is analogous to that of Theorem 6.5.

6.9. <u>Theorem</u>. Let $m_1 \leq m_2 \leq \cdots \leq m_n$ and $M_1 \leq M_2 \leq \cdots \leq M_n$ be integers. Then the number of nondecreasing sequences $b_1 b_2 \ldots b_n$ which satisfy $m_i \leq b_i \leq M_i$ for $1 \leq i \leq n$ and which are permutations of the multiset $\{1^{k_1}, 2^{k_2}, \ldots\}$ is the coefficient of $x_1^{k_1} x_2^{k_2} \cdots$ in

$$|a_{j-i+1}(m_j, M_i)|_n,$$

where $a_k(m,M) = \sum_{M \geq i_1 > i_2 > \cdots > i_k \geq m} x_{i_1} x_{i_2} \cdots x_{i_k}$

6.10. Corollary. If we set each $x_i = 1$ in Theorem 6.9, the determinant becomes

$$\left| \binom{M_i - m_j + 1}{j - i + 1} \right|_n .$$

A result equivalent to Corollary 6.10 was obtained by Kreweras [59, p. 55]. See also [25], [49], and [66].

5. Inverse Identities

5.11. Theorem. Let M_i be positive integers such that $M_1 \leq M_2 \leq \cdots \leq M_s$ and such that $M_j - M_i \leq j-i$ for $1 \leq i \leq j \leq s$. Then $\sum_{i \leq j \leq k} (-1)^{j-i} a_{j-i}(M_{j-1}) h_{k-j}(M_j) =$

$$\begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i<k. \end{cases}$$ (Here i and k are fixed and we sum on j.)

Proof. (Sketch) With the terminology of the proof of Theorem 6.5, let S_{ij} be the set of sequences $d_i d_{i+1} \cdots d_{j-1}$ with $1 \leq d_i \leq M_i$. Let C be the linked set of strictly increasing sequences in S. Note that if $d_i < d_{i+1} < \cdots < d_j$ then $d_i \leq d_j - (j-i)$. Thus

$d_j < M_j$ implies $c_i \leq M_j - (j-i) \leq M_i$, so the conditions

$$c_i < c_{i+1} < \dots < c_j \quad \text{and} \quad c_k \leq M_k \quad \text{for all } k$$

may be replaced by the conditions

$$c_i < c_{i+1} < \dots < c_j \quad \text{and} \quad c_j \leq M_j.$$

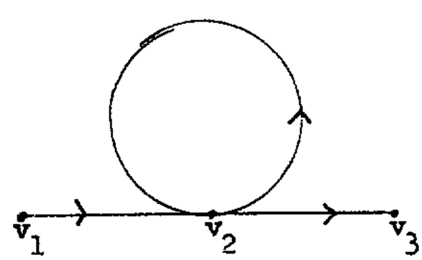
The theorem then follows on taking the image of the equation $\Upsilon(C)\Gamma(\bar{C}) = \underline{1}$ in $R[[\underline{x}]]$.

The most important special case of Theorem 6.11 is that in which $M_i = i+1$. Here the theorem reduces to a result of Comtet [35] which includes as special cases inverse identities for binomial coefficients, p-binomial coefficients, Stirling numbers, and central factorial numbers.

6. Sequences with Initial and Final Segments

In this section we consider sequences with specified initial and final segments, and periodic middles.

Let G be the following digraph.



6.12. Theorem. Let S be a G-system and let C be a linked set in S. Then

(a) $\Gamma(C_{22}) = \overline{\Gamma}(\overline{C}_{22})^{-1}$

(b) $\Gamma(C_{12}) = -\overline{\Gamma}(\overline{C}_{12})\overline{\Gamma}(C_{22})^{-1}$

(c) $\Gamma(C_{23}) = -\overline{\Gamma}(\overline{C}_{22})^{-1}\overline{\Gamma}(\overline{C}_{23})$

(d) $\overline{\Gamma}(C_{13}) = \overline{\Gamma}(\overline{C}_{13}) - \overline{\Gamma}(\overline{C}_{12})\overline{\Gamma}(\overline{C}_{22})^{-1}\overline{\Gamma}(\overline{C}_{23})$.

Proof. Compute the inverse of the matrix

$$\overline{\Gamma}(\overline{C}) = \begin{pmatrix} \underline{1} & \overline{\Gamma}(\overline{C}_{12}) & \overline{\Gamma}(\overline{C}_{13}) \\ 0 & \overline{\Gamma}(\overline{C}_{22}) & \overline{\Gamma}(\overline{C}_{23}) \\ 0 & 0 & \underline{1} \end{pmatrix}.$$

A result similar to Theorem 6.12 has been used by Jackson and Aleliunas in counting permutations by maxima and minima [54, §§10-12].

6.13. Theorem. Let i,j, and k be nonnegative integers with $j > 0$ and $i,k < j$. For $n \geq 0$ let u_{i+nj+k} be the number of sequences of elements of $[m]$ $b_1b_2\ldots b_{i+nj+k}$ such that

$$b_s < b_{s+1} \quad \text{if} \quad s \not\equiv i \pmod{j}$$

$$b_s > b_{s+1} \quad \text{if} \quad s \equiv i \pmod{j}.$$

Let $U = \sum_{n=0}^{\infty} u_{i+nj+k} x^{i+nj+k}$. Then

(a) If $i = k = 0$,

$$U = [\sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{nj} x^{nj}]^{-1}.$$

(b) If $i > 0$, $k = 0$,

$$U = [\sum_{n=0}^{\infty} (-1)^n \binom{m+n}{i+nj} x^{i+nj}] / [\sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{nj} x^n].$$

(c) If $i = 0$, $k > 0$,

$$U = [\sum_{n=0}^{\infty} (-1)^n \binom{m+n}{nj+k} x^{nj+k}] / [\sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{nj} x^{nj}].$$

(d) If $i > 0$, $k > 0$,

$$U = \sum_{n=0}^{\infty} (-1)^n \binom{m+n+1}{i+nj+k} x^{i+nj+k}$$

$$-[\sum_{n=0}^{\infty} (-1)^n \binom{m+n}{i+nj} x^{i+nj}] [\sum_{n=0}^{\infty} (-1)^n \binom{m+n-1}{nj} x^{nj}]^{-1}$$

$$\cdot [\sum_{n=0}^{\infty} (-1)^n \binom{m+n}{nj+k} x^{nj+k}].$$

<u>Proof</u>. Let S be the G-system with prime set P such that P_{12}, P_{22}, and P_{23} are the sets of increasing sequences of elements of $[m]$ of lengths i, j, and k, respectively. Let C be the linked set in S such that

C_{12}, C_{22}, C_{23}, and C_{13} are respectively sequences of the forms $b_1b_2\ldots b_{i+nj}$, $b_{i+1}b_{i+2}\ldots b_{i+nj}$, $b_{i+1}b_{i+2}\ldots b_{i+nj+k}$, and $b_1b_2\ldots b_{i+nj+k}$ such that

$$b_s < b_{s+1} \quad \text{if} \quad s \not\equiv i \pmod{j}$$

$$b_s > b_{s+1} \quad \text{if} \quad s \equiv i \pmod{j}.$$

Let $\overline{V}_{12}$ be the image in $R[[x]]$ of $\overline{\Gamma}(\overline{C}_{12})$ where we take $X_i \mapsto x$, and similarly for $\overline{V}_{22}$, $\overline{V}_{23}$, and $\overline{V}_{13}$.

Then $\overline{V}_{12} = \sum_{n=0}^{\infty} (-1)^{n+1} v_{i+nj} x^{i+nj}$, where v_{i+nj} is the number of sequences $b_1b_2\ldots b_{i+nj}$ such that

$$b_s < b_{s+1} \quad \text{if} \quad s \not\equiv i \pmod{j}$$

$$b_s \leq b_{s+1} \quad \text{if} \quad s \equiv i \pmod{j}.$$

Now for such a sequence, let r be the number of occurrences of $b_s = b_{s+1}$. Then the sequence contains $i + nj - r$ different elements, which can be chosen from $[m]$ in $\binom{m}{i + nj - r}$ different ways. Given a choice of these $i + nj - r$ elements, the sequence is determined by specifying those r values of s for which $b_s = b_{s+1}$. Since there are n possible such values of s, this can be done in $\binom{n}{r}$ ways. Thus

$$v_{i+nj} = \sum_r \binom{m}{i + nj - r}\binom{n}{r} = \binom{m + n}{i + nj},$$

by Vandermonde's convolution. By similar reasoning we find that

$$\overline{V}_{22} = \sum_{n=0}^{\infty} (-1)^n \binom{m + n - 1}{nj} x^{nj}$$

$$\overline{V}_{23} = \sum_{n=0}^{\infty} (-1)^{n+1} \binom{m + n}{nj + k} x^{nj+k}$$

$$\overline{V}_{13} = \sum_{n=0}^{\infty} (-1)^n \binom{m + n + 1}{i + nj + k} x^{i+nj+k}.$$

The theorem then follows from Theorem 6.12.

The case $j = 2$ of Theorem 6.13 was obtained by Carlitz and Scoville [26].

By similar reasoning we derive the following theorem, which solves a special case of a more general problem we consider in the next section.

6.14. Theorem. Let i, j, and k be nonnegative integers with $j > 0$ and $i, k < j$. Let B be the total counting series for sequences $b_1 b_2 \ldots b_{i+nj+k} \in \mathbb{P}^*$ for $n \geq 0$ such that

$$b_s \leq b_{s+1} \quad \text{if} \quad s \not\equiv i \pmod{j}$$

$$b_s > b_{s+1} \quad \text{if} \quad s \equiv i \pmod{j}.$$

Let $\Psi_r = \sum_{n=0}^{\infty} (-1)^n H_{nj+r}$. Then

(a) If $i = k = 0$, then $B = \Psi_0^{-1}$.

(b) If $i > 0$, $k = 0$, then $B = \Psi_i \Psi_0^{-1}$.

(c) If $i = 0$, $k > 0$, then $B = \Psi_0^{-1}\Psi_k$.

(d) If $i > 0$, $k > 0$, then $B = \Psi_{i+k} - \Psi_i \Psi_0^{-1} \Psi_k$.

Note that this generalizes the result of André [7] mentioned in the introduction. The exponential counting series corresponding to (a), (b), and (c) were found by Carlitz [17], [21].

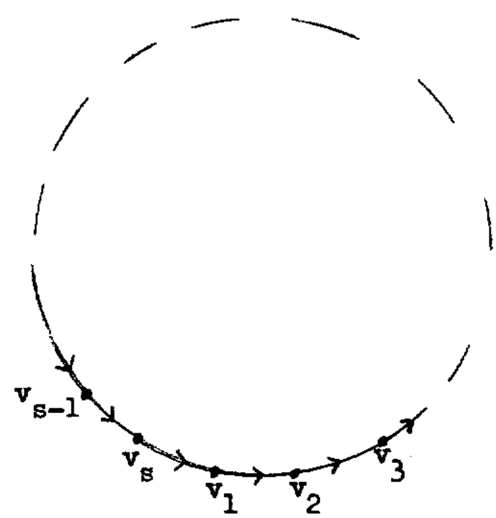
7. Sequences with Periodic Pattern

In this section we count sequences whose run-lengths are periodic. More specifically, let m be a positive integer and let F be any subset of $\{0\} \cup [m-1]$. Then we want to find the counting series in $R[[\underline{x}]]$ for sequences whose fall set consists of those integers (in the appropriate range) which are congruent (mod m) to elements of F.

We find it convenient to classify these sequences according to their length (mod m). Although we do not give an explicit formula for the general case (such a formula is possible, but would be awkward to write down), the method we describe here easily yields the solution in any particular case.

We first consider the special case in which $0 \in F$ and the length of the sequence is congruent (mod m) to an element of F.

Let $F = \{f_1, f_2, \ldots, f_s\}$, where $0 = f_1 < f_2 < \ldots < f_s < m$. Let G be the digraph



Let S be the G-system whose prime set P is given by $\Gamma(P_{i,i+1}) = H_{f_i - f_{i-1}}$ for $1 \leq i < s$, and $\Gamma(P_{s,1}) = H_{m-f_s}$. Now let

$$d_{ij} = \begin{cases} f_j - f_i & \text{for } i \leq j \\ f_j - f_i + m & \text{for } i > j. \end{cases}$$

Let $\underline{C}$ be the linked set in S such that

$$\Gamma(C_{ij}) = \sum_{n=0}^{\infty} H_{d_{ij}+mn}.$$

Then if we set

$$\nu_{ij} = \begin{cases} j-i & \text{for } i \leq j \\ j-i+s & \text{for } i > j, \end{cases}$$

we have

$$\overline{\Gamma}(\overline{C}_{ij}) = (-1)^{\nu_{ij}} \sum_{n=0}^{\infty} (-1)^{ns} H_{d_{ij}+mn}.$$

Now $\Gamma(C_{1j})$ is the total counting series for sequences of length congruent to $f_j \pmod m$ whose falls are those spaces congruent (mod m) to an element of F. Thus $\Gamma(C_{1j})$ is an entry of the inverse of the known matrix $\overline{\Gamma}(\overline{C})$, and its homomorphic image in any commutative counting algebra can be explicitly expressed as a quotient of determinants.

We give an example. Let $m = 9$ and $F = \{0,2,3,7\}$, so $s = 4$. Let $\Phi_k = \sum_{n=0}^{\infty} H_{9n+k}$.

Then

$$\overline{\Gamma}(\overline{C}) = \begin{pmatrix} \Phi_0 & -\Phi_2 & \Phi_3 & -\Phi_7 \\ -\Phi_7 & \Phi_0 & -\Phi_1 & \Phi_5 \\ \Phi_6 & -\Phi_8 & \Phi_0 & -\Phi_4 \\ -\Phi_2 & \Phi_4 & -\Phi_5 & \Phi_0 \end{pmatrix}.$$

Now let $\phi_k = \sum_{n=0}^{\infty} h_{9n+k}$. Then the image in R[[x]] of $\Gamma(C_{12})$, which counts sequences of length congruent to 2 (mod 9) with falls congruent to 0, 2, 3, 7 (mod 9), is

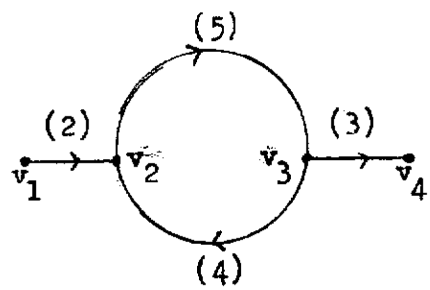
$$- \frac{\begin{vmatrix} -\phi_2 & \phi_3 & -\phi_7 \\ -\phi_8 & \phi_0 & -\phi_4 \\ \phi_4 & -\phi_5 & \phi_0 \end{vmatrix}}{\begin{vmatrix} \phi_0 & -\phi_2 & \phi_3 & -\phi_7 \\ -\phi_7 & \phi_0 & -\phi_1 & \phi_5 \\ \phi_6 & -\phi_8 & \phi_0 & -\phi_4 \\ -\phi_2 & \phi_4 & -\phi_5 & \phi_0 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} \phi_2 & \phi_3 & \phi_7 \\ \phi_8 & \phi_0 & \phi_4 \\ \phi_4 & \phi_5 & \phi_0 \end{vmatrix}}{\begin{vmatrix} \phi_0 & \phi_2 & \phi_3 & \phi_7 \\ \phi_7 & \phi_0 & \phi_1 & \phi_5 \\ \phi_6 & \phi_8 & \phi_0 & \phi_4 \\ \phi_2 & \phi_4 & \phi_5 & \phi_0 \end{vmatrix}}.$$

The analogous formulas for exponential counting series are due to Richard Stanley (unpublished).

In the general case, we may wish to begin and end our sequence with a "nonperiodic" segment. For simplicity, we describe here only a typical example, which illustrates the general case.

We want to count sequences whose run-lengths have the pattern 2, 5, 4, 5, 4, ..., 5, 4, 3. Let G be the digraph.



Here the numbers in parentheses indicate run-lengths. We define a G-system S with prime set P by $\Gamma(P_{12}) = H_2$, $\Gamma(P_{23}) = H_5$, $\Gamma(P_{32}) = H_4$, and $\Gamma(P_{34}) = H_3$.

Let Φ_k and ϕ_k be as in the previous example. Then if we let $\overline{C}$ be the linked set of nondecreasing sequences in S, the answer to our problem is given by the image in $R[[\underline{x}]]$ of the (1,4) entry of $\Gamma(C)$. We have

$$\Gamma(\bar{C}) = \begin{pmatrix} \underline{1} & \phi_2 & \phi_7 & \phi_{10} \\ 0 & \phi_0 & \phi_5 & \phi_8 \\ 0 & \phi_4 & \phi_0 & \phi_3 \\ 0 & 0 & 0 & \underline{1} \end{pmatrix}.$$

Then the image of $\Gamma(\bar{C})$ in $R[[\underline{x}]]$ is

$$\begin{pmatrix} 1 & -\phi_2 & \phi_7 & -\phi_{10} \\ 0 & \phi_0 & -\phi_5 & \phi_8 \\ 0 & -\phi_4 & \phi_0 & -\phi_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the desired counting series is

$$\frac{-\begin{vmatrix} -\phi_2 & \phi_7 & -\phi_{10} \\ \phi_0 & -\phi_5 & \phi_8 \\ -\phi_4 & \phi_0 & -\phi_3 \end{vmatrix}}{\begin{vmatrix} 1 & -\phi_2 & \phi_7 & -\phi_{10} \\ 0 & \phi_0 & -\phi_5 & \phi_8 \\ 0 & -\phi_4 & \phi_0 & -\phi_3 \\ 0 & 0 & 0 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} \phi_2 & \phi_7 & \phi_{10} \\ \phi_0 & \phi_5 & \phi_8 \\ \phi_4 & \phi_0 & \phi_3 \end{vmatrix}}{\begin{vmatrix} \phi_0 & \phi_5 \\ \phi_4 & \phi_0 \end{vmatrix}}.$$

From the same matrix we find that the counting series series in $R[[\underline{x}]]$ for sequences with run-length pattern 5, 4, 5, 4, ..., 5, 4 is

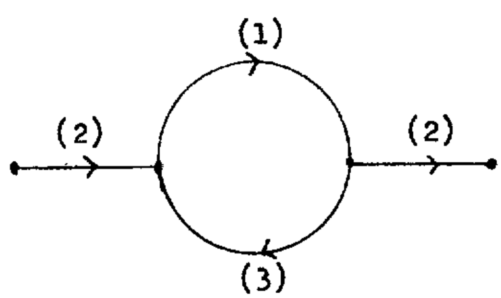
$$\frac{\begin{vmatrix} 1 & \phi_7 & -\phi_{10} \\ 0 & \phi_0 & -\phi_3 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} \phi_0 & \phi_5 \\ \phi_4 & \phi_0 \end{vmatrix}} = \frac{\phi_0}{\begin{vmatrix} \phi_0 & \phi_5 \\ \phi_4 & \phi_0 \end{vmatrix}}$$

and the counting series for sequences with run-length pattern 2, 5, 4, 5, 4, ..., 5, 4 is

$$\frac{-\begin{vmatrix} -\phi_2 & \phi_7 & -\phi_{10} \\ -\phi_4 & \phi_0 & -\phi_3 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} \phi_0 & \phi_5 \\ \phi_4 & \phi_0 \end{vmatrix}} = \frac{\begin{vmatrix} \phi_2 & \phi_7 \\ \phi_4 & \phi_0 \end{vmatrix}}{\begin{vmatrix} \phi_0 & \phi_5 \\ \phi_4 & \phi_0 \end{vmatrix}},$$

and so on. Eulerian counting series are obtained analogously.

Carlitz and Scoville [27], [30] have found the exponential counting series corresponding to the digraph



CHAPTER 7

GENERALIZED EULERIAN POLYNOMIALS

1. The classical Eulerian polynomials

The classical Eulerian polynomials $A_k(t)$ may be defined by

$$\sum_{n=0}^{\infty} n^k t^n = \frac{A_k(t)}{(1-t)^{k+1}} . \tag{1}$$

More generally, if $p(n)$ is any polynomial in n of degree k, we may define a polynomial $A(t)$ of degree at most k by

$$\sum_{n=0}^{\infty} p(n) t^n = \frac{A(t)}{(1-t)^{k+1}} . \tag{2}$$

The Eulerian polynomials (1) have a simple combinatorial interpretation: the coefficient of t^d in $A_k(t)$ is the number of permutations of $[k]$ with d runs. In this chapter we discuss conditions under which the polynomial $A(t)$ defined by (2) has an analogous interpretation.

We note that the coefficients of $A(t)$ are easily computed from the values of $p(n)$:

7.1. Lemma. Let $A(t) = A_0 + A_1 t + \dots + A_k t^k$ and let

$$\sum_{n=0}^{\infty} p(n)t^n = \frac{A(t)}{(1-t)^{k+1}} . \qquad (2)$$

Then $A_m = \sum_{j=0}^{m} (-1)^j \binom{k+1}{j} p(m-j)$.

Proof. Multiply both sides of (2) by $(1-t)^{k+1}$ and equate coefficients of t^m.

2. C-descents

Let S be a linear system. If α is an element of S, with prime factorization $\pi_1\pi_2\dots\pi_k$, we define the S-spaces of α to be those spaces lying between adjacent primes, together with the zeroth and last spaces. For example, if S is the set of sequences of even length, the S-spaces of an element of S are the even spaces. In general a sequence of rank m has $m+1$ S-spaces. The empty sequence has one S-space.

Let C be a linked set in S. With α as above, we define the C-descents of α to be those spaces lying between π_i and π_{i+1} for which $\pi_i\pi_{i+1} \notin C$, together with space 0 if $\alpha \neq \underline{1}$. (The empty sequence has no C-descents.) We define the C-runs of α to be the maximal subsequences of α in C of the form $\pi_i\pi_{i+1}\cdots\pi_j$. Thus the number of C-descents of α is equal to the number of C-runs.

7.2. Theorem. Let C be a linked set in S. For each $\alpha \in S$ let $d(\alpha)$ be the number of C-descents of α. Then

$$[\underline{1} - t\Gamma(C)]^{-1} = \sum_{\alpha \in S} \frac{t^{d(\alpha)}}{(1-t)^{r(\alpha)+1}}\alpha.$$

(Here $r(\alpha)$ is the rank of α.)

Proof. Working in $S \oplus \#$, we consider the set D of elements of the form $\#\beta_1\#\beta_2\cdots\#\beta_n$ with $\beta_i \in C$. (Here β_i may be empty.) It is clear that

$$\Gamma(D) = [\underline{1} - \#\Gamma(C)]^{-1}.$$

Applying the homomorphism determined by $\# \mapsto t$ yields

$$\Gamma(D) \mapsto [\underline{1} - t\Gamma(C)]^{-1}.$$

Now an element of D can be viewed as an element of S with bars inserted in some of its S-spaces. In fact, every element of D is obtained exactly once from some $\alpha \in S$ by the following construction: given $\alpha \in S$, first insert a bar in each C-descent, then insert an arbitrary number of bars in each S-space. Mapping $\# \mapsto t$, we obtain $t^{d(\alpha)}(1 + t + t^2 + \ldots)^{r(\alpha)+1}$ $= t^{d(\alpha)}/(1 - t)^{r(\alpha)+1}$ as the coefficient of α in $[\underline{1} - t\Gamma(C)]^{-1}$.

The basic idea of this proof goes back to MacMahon, in his investigations of the "Lattice Function" and "Permutation Function" [64]. The idea was systematically applied by Stanley [77] in his study of P-partitions. (MacMahon and Stanley actually considered a more general situation that we discuss in the next chapter.)

We remark that Theorem 7.2 enables us to count elements of S in which every S-space is a C-descent; thus Theorem 4.1 can be derived from Theorem 7.2. Conversely, Theorem 7.2 can easily be derived from Theorem 4.1. (Consider the linked set whose primes are of the form $\#\alpha$ with $\alpha \in C$ and whose links are of the form $\#\alpha\#\beta$ with $\alpha\beta \notin C$.)

As a simple example, we use Theorem 7.2 to solve Simon Newcomb's problem. Here $S = \mathbb{P}^*$ and C is the set of nondecreasing sequences. Then

$$\Gamma(C) = \sum_{n=0}^{\infty} H_n = \prod_{i=1}^{\infty} (\underline{1} - X_i)^{-1},$$

so

$$[\underline{1} - t\Gamma(C)]^{-1} = [\underline{1} - t\prod_{j=1}^{\infty}(\underline{1} - x_i)^{-1}]^{-1}$$

$$= \sum_{\alpha\varepsilon\mathbb{P}^*} \frac{t^{d(\alpha)}}{(1-t)^{\ell(\alpha)+1}}\alpha, \qquad (3)$$

where $d(\alpha)$ is the number of falls of α. Thus if $A_{1^{k_1}2^{k_2}\cdots}(t)$ is the enumerator for permutations of the multiset $\{1^{k_1},2^{k_2},\ldots\}$ by falls, with $k = \sum_i k_i$, then

$$[1 - t\prod_{i=1}^{\infty}(1-x_i)^{-1}]^{-1}$$

$$= \sum_{k_1,k_2,\ldots} \frac{A_{1^{k_1}2^{k_2}\cdots}(t)}{(1-t)^{k+1}} x_1^{k_1} x_2^{k_2}\cdots \qquad (4)$$

From (4) an explicit formula for the coefficients of $A_{1^{k_1}2^{k_2}\cdots}(t)$ can easily be obtained. We derive a generalization of this explicit formula in the next section.

Passing to the algebra of Eulerian counting series, we obtain from (3)

$$[1 - te(z)]^{-1} = \sum_{n=0}^{\infty} \frac{A_n^{(p)}(t)}{(1+t)^{n+1}} \frac{z^n}{n!_p} ,$$

where $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_p}$ and $A_n^{(p)}(t)$ is the enumerator for permutations of $[n]$ by falls and inversions.

Changing z to $(1-t)z$ yields

$$\sum_{n=0}^{\infty} A_n^{(p)}(t) \frac{z^n}{n!_p} = \frac{1 - t}{1 - te[(1-t)z]},$$

as obtained in Chapter 5.

3. Chromatic polynomials

We now discuss a generalization of Simon Newcomb's problem that can be described most easily in the language of graph theory.

Let G be a graph with vertex set $[m]$, where we allow loops, but not multiple edges. We say that a multiset M on $[m]$ is <u>independent</u> (with respect to G) if M contains no two adjacent vertices of G, and contains no vertex with a loop more than once.

A <u>coloring</u> of the multiset M in <u>n colors</u> (with respect to G) is a sequence $N_1, N_2, \ldots, N_n$ of independent multisets whose multiset sum is M. Intuitively, we think of N_i as the multiset of vertices assigned color i. It is easily seen that our definition is equivalent to the usual one when M is a set and G has no loops.

Now let $p_M(n)$ be the number of colorings of M with n colors. We call $p_M(n)$ the chromatic polynomial of M (with respect to G).

For the rest of this section we assume that G satisfies the following condition:

(*) If $\{i,\ell\}$ is an edge of G, then so is $\{j,k\}$ whenever $i \leq j \leq k \leq \ell$.

(Here a loop at i is the edge $\{i,i\}$.)

7.3. Lemma. Let

$$M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\},$$

let E be the set of edges of G, and let

$$f_j = \begin{cases} \sum\limits_{\substack{i<j \\ \{i,j\}\in E}} k_i & \text{if } \{j,j\} \in E \\ -k_j + 1 & \text{if } \{j,j\} \notin E. \end{cases}$$

Then $p_M(n) = \prod_{j=1}^{m} \binom{n - f_j}{k_j}$.

Proof. A coloring of M in n colors is constructed by successively assigning k_1 1's, k_2 2's, etc., to the multisets $N_1, N_2, \ldots, N_n$ so that at every stage each N_i is independent. Let us assume that this has been done for $i = 1, 2, \ldots, j-1$.

First assume $\{j,j\} \in E$ and let s be the least integer such that $\{s,j\} \in E$. Then by (*), the f_j elements of the multiset $\{s^{k_s}, \ldots, (i-1)^{k_{i-1}}\}$ have been assigned to f_j different multisets N_i, thus there remain $n-f_j$ multisets N_i from which we can pick k_j in

$\binom{n - f_j}{k_j}$ ways. If $\{j,j\} \notin E$, then by (*) there are no edges of G incident to j, so we may pick k_j N_i's arbitrarily, with repetition allowed, which can be done in $\binom{n + k_j - 1}{k_j}$ ways.

Now let S be the set of sequences of elements of $[m]$. We define the linked set C in S to be the set of sequences $b_1b_2\ldots b_n$ such that $b_i \leq b_{i+1}$ and b_i and b_{i+1} are not adjacent as vertices of G. Equivalently, by (*), $b_1b_2\ldots b_n$ is in C iff $b_1 \leq b_2 \leq \ldots \leq b_n$ and $\{b_1, b_2, \ldots, b_n\}$ is an independent multiset.

7.4. Theorem. Let M be a multiset on $[m]$ and let $A_M(t)$ be the enumerator for permutations of M by C-descents. Then

$$\sum_{n=0}^{\infty} p_M(n)t^n = \frac{A_M(t)}{(1-t)^{|M|+1}},$$

where $|M|$ = card M.

Proof. For any multiset

$M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ on $[m]$,

let $\omega(M) = x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$. Then by Theorem 7.2,

$$[1 - t\sum_N \omega(N)]^{-1} = \sum_M \frac{A_M(t)}{(1-t)^{|M|+1}}\,\omega(M), \tag{5}$$

where the sum on the left is over all independent multisets N on $[m]$ and the sum on the right is over all multisets M on $[m]$. But the left side of (5) is

$$\sum_{n=0}^{\infty} t^n \sum_{N_1,N_2,\ldots,N_n} \omega(N_1)\omega(N_2)\ldots\omega(N_n),$$

where the sum is over all n-tuples of independent multisets, hence is equal to

$$\sum_{n=0}^{\infty} t^n \sum_M p_M(n)\omega(M) = \sum_M \omega(M) \sum_{n=0}^{\infty} t^n p_M(n),$$

where the sum is over all multisets M on $[m]$.

We note that Theorem 7.4, together with Lemma 7.1 and Lemma 7.3, yields an explicit formula for the coefficients of $A_M(t)$. The special case of Theorem 7.4 in which M is a set is due to Stanley (unpublished). An analog of the theorem for arbitrary graphs can be obtained using the methods of [34].

The simplest case of Theorem 7.4 is that in which G has no edges (or loops). Then C-descents are just falls and the coefficients of $A_M(t)$ give the solution to Simon Newcomb's problem. If $M = \{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$, then by Lemma 7.3 we have $p_M(n) = \prod_{j=1}^{n} \binom{n+k_j-1}{k_j}$.

Then applying Lemma 7.1, we have explicitly:

7.5. Proposition. The number of permutations of the multiset $\{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$ with d falls is

$$\sum_{i=0}^{d} (-1)^i \binom{k+1}{i} \prod_{j=1}^{m} \binom{d-i+k_j-1}{k_j}, \text{ where } k = \sum_i k_i.$$

This formula was first given by MacMahon [62] in 1908. He had earlier [61] given a less explicit expression for these numbers.

As another application of Theorem 7.4, fix an integer $r \geq 1$ and let the edges of G be those pairs $\{i,j\}$ with $|i-j| < r$. Then from Theorem 7.4 and Lemma 7.1 and 7.3 we can write down an explicit formula for the number of permutations $b_1b_2\ldots b_k$ of the multiset M with a given number of occurrences of $b_{i+1} - b_i < r$. This problem has been considered in [2], [4], [43], and [71, p. 235]; however, our explicit formula for the general case seems to be new.

Other work on Simon Newcomb's problem and its generalizations can be found in [3], [8], [13], [16], [24], [37], [38], [53], [55], [58, pp. 34-48], [60], [70], and [74].

4. Peaks

We define a peak (or maximum) of the sequence $b_1b_2\ldots b_n$ to be an index i such that

(i) $i = 1$ or $b_i \geq b_{i-1}$

and (ii) $i = n$ or $b_i > b_{i+1}$.

By definition, the empty sequence has no peaks.

In this section we count sequences of even length by number of peaks. Let S be the linear system of sequences in $\mathbb{P}^*$ of even length. Then the primes of S have length two and the links of S have length four.

Now let C be the subset of S of all sequences of the form

(*) $b_1b_2\ldots b_{2m}\ c_1c_2\ldots c_{2m}$, where $b_1b_2\ldots b_{2m}$ is nondecreasing and $c_1c_2\ldots c_{2n}$ is decreasing.

Here m, n, or both may be zero. It is easily verified that C is a linked set.

7.6. Lemma. For $\alpha \in S$, the number of peaks of α is equal to the number of C-runs of α.

Proof. If ρ is a C-run of α written in the form (*), then there is a peak corresponding to either b_{2m} or c_1, but not both, and there are no other peaks of α in ρ. Thus there is exactly one peak in each C-run of α.

To count sequences by peaks using Theorem 7.2, we need only evaluate $\Gamma(C)$. It is easy to see that the representation (*) of an element of C is unique; thus

$$\Gamma(C) = \left(\sum_{n=0}^{\infty} H_{2n}\right)\left(\sum_{n=0}^{\infty} A_{2n}\right). \qquad (6)$$

Then Theorem 7.2 yields:

7.7. Theorem. Let $g(\alpha)$ be the number of peaks of the sequence α. Then

$$\sum_{\alpha \in S} \frac{t^{g(\alpha)}}{(1-t)^{r(\alpha)+1}} \alpha = [\underline{1} - t\Gamma(C)]^{-1},$$

where $\Gamma(C)$ is given by (6). Note that here $r(\alpha) = \frac{1}{2}\ell(\alpha)$.

7.8. Corollary. Let $B_{1^{k_1}2^{k_2}\cdots}(t)$ be the enumerator for permutations of the multiset $\{1^{k_1}, 2^{k_2}, \ldots\}$ by number of peaks. Let $k = \sum_i k_i$. Then

$$\sum_{\substack{k_1, k_2, \ldots \\ k \text{ even}}} \frac{B_{1^{k_1}2^{k_2}\cdots}(t)}{(1-t)^{\frac{k}{2}+1}} x_1^{k_1} x_2^{k_2} \cdots$$

$$= [1 - t(\sum_{n=0}^{\infty} h_{2n})(\sum_{n=0}^{\infty} a_{2n})]^{-1}.$$

7.9. Corollary. Let $B_m(t)$ be the enumerator for permutations of $[m]$ by number of peaks. Then

$$\sum_{m=0}^{\infty} \frac{B_{2m}(t)}{(1-t)^{m+1}} \frac{z^{2m}}{(2m)!} = [1 - t \cosh^2 z]^{-1}.$$

7.10. Corollary. Let $B_m^{(p)}(t)$ be the enumerator for permutations of [m] by peaks and inversions. Then

$$\sum_{m=0}^{\infty} \frac{B_{2m}^{(p)}(t)}{(1-t)^{m+1}} \frac{z^{2m}}{(2m)!_p}$$

$$= [1 - t(1 + \sum_{n=1}^{\infty} (1+p)(1+p^2)\ldots(1+p^{2n-1})\frac{z^{2n}}{(2n)!_p}))]^{-1}.$$

Proof. All we need is the identity

$$(\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!_p})(\sum_{n=0}^{\infty} p^{\binom{2n}{2}} \frac{z^{2n}}{(2n)!_p})$$

$$= 1 + \sum_{n=1}^{\infty} (1+p)(1+p^2)\ldots(1+p^{2m-1})\frac{z^{2n}}{(2n)!_p}$$

which follows easily from the "p-binomial theorem"

$$e(az)\bar{e}(bz) = 1 + \sum_{n=1}^{\infty} (a+b)(a+bp)\ldots(a+bp^{n-1})\frac{z^n}{n!_p} ,$$

where $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_p}$ and $\bar{e}(z) = \frac{1}{e(-z)} = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{n!_p}$.

(See [9, pp. 17-19].)

Analogous results can be obtained for sequences of odd length (e.g., using the methods of Chapter 6). The basic result, which we state without proof, is the following.

7.11. Proposition. Let T be the set of sequences in $\mathbb{P}^*$ of odd length. Then

$$\sum_{\alpha \varepsilon T} \frac{t^{g(\alpha)}}{(1-t)^{\frac{\ell(\alpha)+1}{2}}} \alpha$$

$$= [\underline{1} - t(\sum_{n=0}^{\infty} H_{2n})(\sum_{n=0}^{\infty} A_{2n})]^{-1} t(\sum_{n=0}^{\infty} H_{2n})(\sum_{n=0}^{\infty} A_{2n+1}).$$

There has been a significant amount of work on the enumeration of sequences by peaks [20], [28], [29], [33], [41], [43], [44], [46], [54], [56]. However, the fact that the counting series takes an especially simple form when sequences of even and odd length are considered separately seems not to have been noticed before.

CHAPTER 8

THE GREATER INDEX

1. C-indices

In this chapter we develop a "q-analog" of the subject of Chapter 7.

Let C be a linked set in the linear system S. For simplicity we assume that the primes of S have length one. (The general case requires only minor notational changes.)

8.1. Definition. The C-index $i(\alpha)$ of $\alpha \in S$ is the sum of the C-descents of α.

For example, if $S = \mathbb{P}^*$ and C is the set of nondecreasing sequences, the C-index is the greater index, first studied by MacMahon [63], [64].

Now let W be any subset of S. We define $\Gamma_n(W)$ to be $\sum_{\alpha \in W} q^{n\ell(\alpha)} \alpha$, and we define $\Gamma^{[n]}(W)$ to be $\Gamma_{n-1}(W)\Gamma_{n-2}(W)\ldots\Gamma_0(W)$, with $\Gamma^{[0]}(W) = \underline{1}$.

Let $(t;q)_n = (1-t)(1-tq)\ldots(1-tq^{n-1})$, and let $(q)_n = (q;q)_n = (1-q)(1-q^2)\ldots(1-q^n) = (1-q)^n n!_q$.

8.2. Theorem. Let C be a linked set in S. Then

$$\sum_{\alpha\varepsilon S} \frac{t^{d(\alpha)}q^{i(\alpha)}}{(t;q)_{\ell(\alpha)+1}}\alpha = \sum_{n=0}^{\infty} t^n\Gamma^{[n]}(C),$$

where $d(\alpha)$ is the number of C-descents of α and $i(\alpha)$ is the C-index of α.

Proof. As in the proof of Theorem 7.2 we work in $S \oplus \#$ and consider the set D of elements of the form $\#\beta_1\#\beta_2\ldots\#\beta_m$, where $\beta_i \varepsilon C$. As before,

$$\Gamma(D) = \sum_{n=0}^{\infty} [\#\Gamma(C)]^n. \tag{1}$$

Now consider the R-algebra $A = R[[S]][Q]$ where Q satisfies the commutativity relationship $\pi Q = qQ\pi$ for all primes π of S. Let μ be the homomorphism $R[[S \oplus \#]] \to A$ that takes $\#$ to tQ. Then we have

$$\#\beta_1\#\beta_2 \mapsto t^2q^{\ell(\beta_1)}Q^2\ \beta_1\beta_2,$$

$$\#\beta_1\#\beta_2\#\beta_3 \mapsto t^3q^{2\ell(\beta_1)+\ell(\beta_2)}Q^3\ \beta_1\beta_2\beta_3\ , \text{ etc. so}$$

$$\Gamma(D) \mapsto \sum_{n=0}^{\infty} t^nQ^n\Gamma^{[n]}(C).$$

Now every monomial in A can be expressed uniquely in the form $Q^n\alpha$ for some $\alpha \varepsilon S$. Thus we may define a linear transformation (which is not a homomorphism) $\nu : A \to R[[S]]$ by $\nu(Q^n\alpha) = \alpha$. Let θ be the composite map $\nu \cdot \mu : R[[S \oplus \#]] \to R[[S]]$. Then

$$\theta[\Gamma(D)] = \sum_{n=0}^{\infty} t^n \Gamma^{[n]}(C).$$

As in the proof of Theorem 7.2, we observe that every element of D is obtained uniquely from some $\alpha \in S$ by first inserting a bar in every C-descent and then inserting an arbitrary number of bars in each space. A bar inserted in space j corresponds to a factor of tq^j after θ is applied. Thus the coefficient of α in $\theta[\Gamma(D)]$ is

$$\frac{t^{d(\alpha)}q^{i(\alpha)}}{(1-t)(1-tq)\ldots(1-tq^{\ell(\alpha)})}.$$

This completes the proof.

8.3. <u>Corollary</u>.

$$\sum_{\alpha \in S} \frac{q^{i(\alpha)}}{(q)_{\ell(\alpha)}} \alpha = \lim_{n\to\infty} \Gamma^{[n]}(C).$$

<u>Proof</u>. By Theorem 8.2,

$$\sum_{\alpha \in S} \frac{t^{d(\alpha)}q^{i(\alpha)}}{(1-tq)\ldots(1-tq^{\ell(\alpha)})} \alpha = (1-t) \sum_{m=0}^{\infty} t^m \Gamma^{[m]}(C)$$

$$= \underline{1} + \sum_{m=1}^{\infty} t^m [\Gamma^{[m]}(C) - \Gamma^{[m-1]}(C)].$$

Setting t = 1, we have

$$\sum_{\alpha \in S} \frac{q^{i(\alpha)}}{(q)_{\ell(\alpha)}} \alpha = \underline{1} + \lim_{n\to\infty} \sum_{m=1}^{n} [\Gamma^{[m]}(C) - \Gamma^{[m-1]}(C)]$$

$$= \underline{1} + \lim_{n\to\infty} [\Gamma^{[n]}(C) - \underline{1}] = \lim_{n\to\infty} \Gamma^{[n]}(C).$$

2. The Greater Index

We now apply the theory of the last section to the case in which $S = \mathbb{P}^*$ and C is the linked set of nondecreasing sequences. Thus C-descents are falls and the C-index is the greater index.

Here we have

$$\Gamma(C) = \sum_{n=0}^{\infty} H_n = \prod_{j=1}^{\infty} (\underline{1} - x_i)^{-1}.$$

Passing to $R[[\underline{x}]]$, we have

$$\Gamma^{[n]}(C) \mapsto \prod_{j=1}^{\infty} \prod_{s=0}^{n-1} (1 - x_j q^s)^{-1}.$$

Then from the well-known identity (a special case of Theorem 2.1, p. 17 of [9])

$$\prod_{s=0}^{n-1} (1-xq^s)^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q x^k, \text{ we get}$$

$$\Gamma^{[n]}(C) \mapsto \prod_{j=1}^{\infty} \sum_{k_j=0}^{\infty} \binom{n+k_j-1}{k_j}_q x_j^{k_j}$$

$$= \sum_{k_1, k_2, \ldots 0}^{\infty} \binom{n+k_1-1}{k_1}_q \binom{n+k_2-1}{k_2}_q \cdots x_1^{k_1} x_2^{k_2} \cdots,$$

where the last sum is over all $k_1, k_2, \ldots$ of which only finitely many are nonzero. Now let

$A_M(t,q) = \sum_{\alpha} t^{d(\alpha)} q^{i(\alpha)}$, where the sum is over all permutations α of the multiset $\{1^{k_1}, 2^{k_2}, \ldots, m^{k_m}\}$. Then it follows from Theorem 8.2 that (with $k = \sum_i k_i$)

$$\frac{A_M(t,q)}{(t;q)_{k+1}} = \sum_{n=0}^{\infty} t^n \binom{n+k_1-1}{k_1}_q \binom{n+k_2-1}{k_2}_q \cdots \binom{n+k_m-1}{k_m}_q$$

as first shown by MacMahon [64, Vol. 2, p. 211]. See also [14] and [22]. Now $\lim_{n\to\infty} \binom{n+j-1}{j}_q$

$$= \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+j-1})}{(1-q)(1-q^2)\cdots(1-q^j)} = \frac{1}{(q)_j}, \quad \text{so by}$$

Corollary 8.3, $\dfrac{A_M(1,q)}{(q)_k} = \prod_{i=1}^{m} \dfrac{1}{(q)_{k_i}}$, thus

$$A_M(1,q) = \frac{k!_q}{k_1!_q k_2!_q \cdots k_m!_q},$$ as first shown by MacMahon [63], [64, Vol. 2, p. 206]. See also [42].

Now let us see what happens in the algebra of Eulerian counting series. Let $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_p}$ and let

$A_n(t,q,p) = \sum_{\alpha \in \mathfrak{S}_n} t^{d(\alpha)} q^{i(\alpha)} p^{I(\alpha)}$. Then Theorem 8.2

yields the following.

8.4. <u>Theorem</u>. $\sum_{n=0}^{\infty} \frac{A_n(t,p,q)}{(t;q)_{n+1}} \frac{z^n}{n!_p}$

$$= \sum_{n=0}^{\infty} t^n e(z)e(qz)\ldots e(q^{n-1}z).$$

To apply Corollary 8.3, we observe that

$$e(\tfrac{z}{1-p}) = \sum_{n=0}^{\infty} \frac{z^n}{(p)_n} = \prod_{m=0}^{\infty} (1-zp^m)^{-1}$$ [9, Corollary 2.2, p. 19].

Thus changing z to $\frac{z}{1-p}$ and applying Corollary 8.2 we obtain:

8.5. <u>Theorem</u>.

$$\sum_{n=0}^{\infty} \frac{A_n(1,q,p)}{(q)_n (p)_n} = \prod_{m,n=0}^{\infty} \frac{1}{1-zp^m q^n} .$$

It follows from Theorem 8.5 that $A_n(1,q,p) = A_n(1,p,q)$. This fact was conjectured by Alter, Curtz, and Wang [6], and was first proved by Foata and Schützenberger [45], using a combinatorial correspondence. If we let $B_n(q,p) = \sum_{\alpha \in \mathfrak{S}_n} q^{i(\alpha)} p^{i(\alpha^{-1})}$, where α^{-1} is the inverse permutation of α (in the group-theoretic sense), then Foata and Schützenberger's correspondence shows that $A_n(1,q,p) = B_n(q,p)$.

The infinite product $\prod_{m,n=0}^{\infty} \frac{1}{1-zp^m q^n}$ has been studied by several authors, and B. Gordon [51] first showed that the polynomials $C_n(q,p)$ defined by

$$\prod_{m,n=0}^{\infty} \frac{1}{1-zp^m q^n} = \sum_{n=0}^{\infty} \frac{C_n(q,p)}{(q)_n (p)_n} z^n$$

have nonnegative coefficients. In fact, Gordon's proof shows that $C_n(q,p) = B_n(q,p)$ as defined above, but Gordon did not state this explicitly. This combinatorial interpretation for the coefficients of $C_n(q,p)$ was first stated explicitly (in a slightly different form) by Roselle [75].

We remark that Gordon's proof implies the following: Let $B_n^{(j)}(q_1,q_2,\ldots,q_j) = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_j} q_i^{i(\alpha_1)} q_2^{i(\alpha_2)} \cdots q_j^{i(\alpha_j)}$,

where the sum is over all permutations $\alpha_1,\alpha_2,\ldots,\alpha_j$ in $\mathfrak{S}_n$ whose product $\alpha_1\alpha_2\ldots\alpha_j$ (in the group-theoretic sense) is the identity permutation. Then

$$\sum_{n=0}^{\infty} \frac{B_n(q_1,q_2,\ldots,q_j)}{(q_1)_n (q_2)_n \cdots (q_j)_n} z^n$$

$$= \prod_{m_1,m_2,\ldots,m_j=0}^{\infty} \frac{1}{1-zq_1^{m_1} q_2^{m_2} \cdots z_j^{m_j}} .$$

We hope to show elsewhere how Gordon's method can be extended to count these permutations by greater index and number of falls.

3. P-partitions

In this section we indicate briefly how R. Stanley's theory of P-partitions can be developed in the framework we have presented. For an exposition of the theory of P-partitions, with examples and applications, see [77].

Let P be a partial ordering on $[m]$. We use the symbol "$\leq_P$" to denote the partial ordering of P and "$\leq$" to denote the usual total order on $\mathbb{Z}$.

Let S be the set of sequences $b_1 b_2 \ldots b_k$ of elements of $[m]$ such that if $i < j$ then $b_i \not>_P b_j$.

Now let M be a multiset on $[m]$, and let n and s be nonnegative integers.

8.6. Definition. For $n > 0$, an $(M;n)$-partition of s is a sequence $N_{n-1}, N_{n-2}, \ldots, N_0$ of multisets on $[m]$ such that

(i) $N_{n-1} + N_{n-2} + \ldots + N_0 = M$ (multiset sum).

(ii) If $b \in N_i$ and $c \in N_j$ with $i > j$, then $b \not>_P c$.

(iii) $\sum_i i|N_i| = s$.

For $n = 0$, the only $(M;n)$ partition is the empty sequence.

To see the intuitive meaning of our definition, consider the case in which $M = [m]$. Then an $(M;n)$-partition can be identified with a map $\sigma : [m] \to \{0,1,\ldots,n-1\}$ such that $b \leq_p c$ implies $\sigma(b) \geq_p \sigma(c)$. (If $b \in N_i$ we define $\sigma(b) = i$.) Then $\sum_i i|N_i| = \sum_{b\in[m]} \sigma(b)$. Our definition is a convenient way of generalizing the concept of an order-reversing map to multisets; moreover, its form makes the application of Theorem 8.2 easy and points out the similarity to colorings of graphs.

<u>Remark</u>. We note that our definition differs slightly from Stanley's: our $([m];n)$-partition corresponds to his $(P;n-1)$ partition, and for simplicity we have omitted condition (ii) on page 5 of [77]. (Thus we consider only the case in which P is "naturally labeled.")

Now let $\mathcal{L}(M)$ be the set of elements of S which are permutations of the multiset M. Let $V_{M(n)} = \sum_{N_{n-1},N_{n-2},\ldots,N_0} q^s$, where the sum is over all $(M;n)$-partitions and $s = \sum_i i|N_i|$.

8.7. Proposition. (Analogous to Proposition 8.3, p. 24, of [77].)

$$\sum_{n=0}^{\infty} V_{M(n)} t^n = \sum_{\alpha \in \mathcal{L}(M)} \frac{t^{d(\alpha)} q^{i(\alpha)}}{(t;q)_{|M|+1}}, \quad \text{where} \quad |M| = \text{card } M,$$

$d(\alpha)$ is the number of falls of α, and $i(\alpha)$ is the greater index of α.

Proof. Let C be the linked set of nondecreasing sequences in S. By Theorem 8.2,

$$\sum_{\alpha \in S} \frac{t^{d(\alpha)} q^{i(\alpha)}}{(t;q)_{\ell(\alpha)+1}} \alpha = \sum_{n=0}^{\infty} t^n \Gamma^{[n]}(C).$$

Now any nonzero term on the right is of the form

$$t^n q^{(n-1)\ell(\beta_1)+(n-2)\ell(\beta_2)+\ldots+\ell(\beta_{n-1})} \beta_1\beta_2\ldots\beta_n,$$

where $\beta_1\beta_2\ldots\beta_n$ is an element of S. Each such term corresponds to an $(M;n)$-partition of $(n-1)\ell(\beta_1)+\ldots+\ell(\beta_{n-1})$, where $\beta_1\beta_2\ldots\beta_n$ is a permutation of the multiset M. Then the proposition follows by collecting all terms corresponding to permutations of M.

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