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REVISIT OF THE METHOD OF CONSTRUCTION

OF THE FIRST BIMAGIC SQUARES

ABSTRACT

In this paper, I compute all the 8x8 bimagic squares given by what I call "the Coccoz's method". This method of construction is described in a forgiven paper published by the Commandant Victor Coccoz in 1892 and it was used by the few Frenchmen who managed to build the first bimagic squares in the world in the 1890's years (Pfeffermann who built the first one, Coccoz or his pseudonym Luet, Savard who built the first semi-bimagic one, Huber, Portier, etc).

I found that this method gives numerous squares, exactly a set of 10,317 essentially different bimagic squares. This set contains several subsets of bimagic squares which are known today, like the associative bimagic squares, the pandiagonal complete bimagic squares, the Greco-Latin bimagic squares, but it contains also many other squares like many pandiagonal bimagic squares which seem to be new.

THE COCCOZ'S METHOD

In October 2012, when searching on the web the reference [1] for the first 8x8 bimagic square found by G. Pfeffermann in 1890, I saw that most of the issues of the French magazine "Les Tablettes du Chercheur" were available on line in the Bibliothèque Nationale de France site (http://gallica.bnf.fr).

Everyone can now have access directly to the numerous examples of bimagic squares which are presented as puzzles inside the different issues of this magazine. But there is not any information about the way these squares were built. I found however the issue ref. [2] were the editor points out a communication of the Commandant Coccoz to the AFAS (Association Française pour l'Avancement des Sciences) about this subject. And I found too this forgiven communication ref. [3] which is also on line. *These references [2] and [3] are fundamental: the history of the discovery of the first bimagic squares is described and the method of construction of these first squares is also explained!*

This method can be basically summarized as follows.

There are three steps in the general method of building a bimagic square:
- we build a first "generator" with the 64 first numbers distributed into 8 bimagic rows,
- we build a second "generator" with the same properties and which can be "conjugated" with the first generator for giving a SM (semi-bimagic square)
- we permute after the rows and the columns for having two bimagic diagonals (if possible).

Coccoz gives two special features to his SM:

- in the rows and in the columns, he uses only special bimagic series (among the total of 38,039) coming from 5 “groupements” which can be easily calculated by hand (it was important at that time when the computer was unknown)
- the square has 16 subsquares of the same type with a total sum of 130 and with two numbers of sum p and two numbers of sum 130-p, for example for p=57:

1 58 15 56	36 27 46 21	53 14 59 4	24 47 26 33
22 45 28 35	55 16 57 2	34 25 48 23	3 60 13 54
40 31 42 17	5 62 11 52	20 43 30 37	49 10 63 8
51 12 61 6	18 41 32 39	7 64 9 50	38 29 44 19

Each line (row or column) has then a complementary line which can be derived from it.

The parameter p can take the 7 following values: p = 33, 49, 57, 61, 63, 64 and 65 (Coccoz doesn’t demonstrate that these values are the only ones).

ENUMERATION PROGRAMS

With a computer, I enumerated the 8x8 bimagic squares having the second feature of Coccoz, i.e. 16 subsquares with sum p and 130-p (I call them the <u>squares of the Coccoz’s type)</u>. It is not compulsory to limit the enumeration to the first feature (the 5 “groupements”): we can consider <u>all the possible bimagic series</u>.

I did this task which is easier if p is different from 65 because in this case, I demonstrate that we have in the SM:

$$A1 + A3 + A5 + A7 = 2p$$

and the similar relations in the other lines (cf my site ref.[4] for the notations).

The demonstration comes from the equation for the row #2:

$$(p-A1)^2+(p-A3)^2+(p-A5)^2+(p-A7)^2 + (130-p-A2)^2+(130-p-A4)^2+(130-p-A6)^2+(130-p-A8)^2$$
$$= (A1^2+A3^2+A5^2+A7^2) + (A2^2+A4^2+A6^2+A8^2)$$

For p=65, this equation is true whatever the value of the sum A1+A3+A5+A7 and then the computation time is much more long.

My programs show that p can take only the 7 indicated values (the demonstration comes from the way to distribute the 64 first numbers into 16 couples of sum p and 16 couples of sum 130-p).

I have the fundamental result:

there are exactly 10,317 essentially different 8x8 bimagic squares of the Coccoz's type above defined

The total number of bimagic squares generated by this method is then
10,317*1,536 when counting all the squares
or 10,317*192 unique bimagic squares.

Here are the detailed results of my enumeration programs:

p	Nb of elem. SM sq.	Nb of elem. SM giving bimagic sq.	Nb of ess. diff. bimagic sq.			
			Total	A	B	C
33	598	266	2,726	2,188	136	402
49	554	182	2,444	2,188	60	196
57	574	201	2,531	2,188	74	269
61	534	201	2,465	2,188	69	208
63	570	227	2,523	2,188	60	275
64	598	245	2,687	2,188	137	362
65	74,222	3,421	8,605	2,188	536	5,881
Total of sq.			23,981	15,316	1,072	7,593
Total of different sq.			10,317	2,188	536	7,593

A given bimagic solution (in standard position) can appear several times according to the value of p. There are:
2,188 sq. of type A which appear 7 times in 7 different files
536 sq. of type B which appear 2 times in 2 different files
7,593 sq. of type C which appear 1 time in 1 file only.

For example, the sq # 7 (out of 10,317) which is a sq. of type A:

1	8	55	46	50	43	28	29
23	18	33	60	40	61	14	11
38	35	20	9	21	16	63	58
30	27	44	49	45	56	7	2
12	13	62	39	59	34	17	24
52	53	6	31	3	26	41	48
57	64	15	22	10	19	36	37
47	42	25	4	32	5	54	51

comes from the SM (p=33) # 315 (out of 598):

1	50	8	55	28	43	29	46
47	32	42	25	54	5	51	4
12	59	13	62	17	34	24	39
38	21	35	20	63	16	58	9
23	40	18	33	14	61	11	60
57	10	64	15	36	19	37	22
30	45	27	44	7	56	2	49
52	3	53	6	41	26	48	31

but also from the SM (p=49) # 201 (out of 554):

1	29	8	28	43	55	46	50
52	48	53	41	26	6	31	3
12	24	13	17	34	62	39	59
57	37	64	36	19	15	22	10
38	58	35	63	16	20	9	21
23	11	18	14	61	33	60	40
47	51	42	54	5	25	4	32
30	2	27	7	56	44	49	45

etc until p=65.

I have in my files all the different identifications of each bimagic square and the number of bimagic solutions for a given SM sq. I have also the list of all the elementary SM for each p.

I found that a given SM sq. can generate 0, or 1, or 2, …or a maximum of 42 bimagic solutions by the (8!)*(8!) possible permutations of rows and columns.

I put in attachment

1 the list of the 10,317
2 the list of the 2,188 appearing 7 times
3 the list of the 536 appearing 2 times.

The set of 2,188 comes from 120 SM and from the “5 groupements” (in his paper, Coccoz gives - more or less clearly - this number of 120).

Note: I have said that the computation time is very long for p=65 (several months in fact). But if p is different from 65 and if we consider the squares p=65* with the different relations A1+A3+A5+A7=130, the computation time is shorter. We find a subset of 5,485 squares which can be more rapidly computed (several days). Here are the results for this subset of 5,485 squares:

p	Nb of elem. SM sq.	Nb of elem. SM giving bimagic sq.	Nb of ess. diff. Bimagic sq.			
			Total	A	B	C
33	598	266	2,726	2,188	0	538
49	554	182	2,444	2,188	0	256
57	574	201	2,531	2,188	0	343
61	534	201	2,465	2,188	0	277
63	570	227	2,523	2,188	0	335
64	598	245	2,687	2,188	0	499
65*	866	356	3,237	2,188	0	1,049
Total of sq.			18,613	15,316	0	3,297
Total of different sq.			5,485	2,188	0	3,297

Cf attachment 4 for the list of the 5,485 sq.

VERIFICATIONS

I made a lot of verifications for my result of 10,317 squares.

- I verified that the squares printed in the articles of Coccoz are in my list (I had to put before each square in standard position). Idem for several squares printed in “Les Tablettes du Chercheur”.
  For example, the first bimagic square ref. [1] found by Pfeffermann is the # 2361 of my list. The Tarry’s square (ref.[5] and [6]) is the square # 8991 of my list.

- I filtered the 10,317 sq. by programs searching sets which are manifestly of Coccoz’s type:

- the ess. diff. associative bimagic squares. I found 841 ess. diff. squares. Cf attach. 5. This number of 841 is the same as the one found by Walter Trump in March 2011 (and the total number of unique squares is 841*192=161,472)

- the ess. diff. bimagic sq. which are pandiagonal complete or isomorphic to these pandiag. complete by application of G1,536 (group of the geometric transformations working on all the 8x8 magic squares). When taking into account the conditions of reduction on the 1st diagonal, i.e. A1=min (A1,B2,C3,D4,E5,F6,G7,H8), B2<C3<D4<E5, C3<F6, B2<G7, I found:
843 squares of type pandiag. complete (A1+E5=65 in an abbreviated notation)
538 squares of type pandiag. complete*$(12436578)_{all}$ (A1+F6=65 in an abbreviated notation)
455 squares of type pandiag. complete*$(14327658)_{all}$ (A1+G7=65 in an abbreviated notation)
Total: 1,836 ess. diff. squares among 10,317. Cf attach. 6, 7, 8. We can then state that, for the set of the bimagic pandiag. complete sq., there are also 1,836 ess.diff. sq.
The number of 1,836 is the same as the one found by Walter Trump in March 2011 (In fact, Walter found the total number of 29,376 unique squares, and I found the number of ess. diff. squares from the total file he kindly sent to me). When applying the group G1,536 we find 12 isomorphic sets of 29,376 sq.

With two different enumerations made by two different persons, the numbers for the associative and for the pandiagonal complete bimagic squares can be definitively considered as established.

- I verified also that the 1,344 ess. diff. Greco-Latin bimagic squares I enumerated in February 2012 (cf ref.[6]) are inside the 10,317 squares. Cf attach. 9 for the ordered list.
  The 1,344 squares are also inside the above mentioned 2,188 squares! We have:
  1,344 sq. → 2,188 sq. → 5,485 sq. → 10,317 sq.
  The above mentioned Tarry’s square is one of the 1,344 sq.

All these verifications are good hints for considering sure the number of 10,317. A second enumeration by a different person should naturally be welcome for considering this number as totally established.

OTHER INVESTIGATIONS

I drove several other investigations into the set of 10,317 squares.

- I enumerated the pandiagonal sq. among the 10,317: there are 860 solutions (cf attach. 10):

843 complete as indicated above (cf attach. 6)
17 not complete (cf attach. 11)

With these 17 not complete squares, it seems we find new types of 8x8 pandiagonal squares, for example the type A1+B8=65 for the 9th:

3	26	49	24	63	38	13	44
21	16	39	2	41	52	27	62
34	59	29	60	30	7	33	8
57	36	6	35	5	32	58	31
14	23	64	25	50	43	4	37
28	1	42	15	40	61	22	51
47	54	20	53	19	10	48	9
56	45	11	46	12	17	55	18

(I have also enumerated all the 8x8 pandiagonal bimagic squares of the type A1+D6=65 like the 12th)

The enumeration of the pandiagonal squares among the 10,317*192 unique sq. is more tricky. I began this task and I found many other types than the classical complete type A1+E5=65, at least 29 types for the moment (A1+A2=65, A1+A6=65, A1+A8=65, etc).

- I enumerated the 600 bimagic series (out of 38,039) made with the "5 groupements". Most of these series are Latin (low or high), but not all of them. It should be possible to enumerate the squares generated by these 600 bimagic series, but the interest should be chiefly historical.
  Idem for the special bimagic series used by Coccoz for the search of the 2 diagonals, i.e. the series with 2 complementary pairs (cf ref.[7]).

- Many other investigations are possible:
  We can study the distribution of the 10,317 according to the type of each square (with the isomorphisms between the different resulting subsets).
  We can also study the sets of squares having the same number of induced squares by permutation, for example the squares having 42 induced squares.
  Etc.
  There are truly many investigations still to do!

CONCLUSION

The enumeration of the Coccoz's squares is very fruitful: it gives a large set of squares and it is not surprising after the event that this set was used by the Frenchmen who built the first bimagic squares. We can cheer these men who built these squares without any computer!

This enumeration allowed the validation of the previous enumerations of the associative and of the pandiagonal complete bimagic squares of order 8. It allowed also the discovery of 8x8 pandiagonal squares which seem to be new.

But the Coccoz's squares are only a very little part of all the 8x8 bimagic squares, other methods of construction do exist. After his first article ref.[3], Coccoz himself published later (ref.[7]) improvements of his method by transformation, he published also the method of Rilly (ref [9], [10], [11]). I will treat this subject in a future note.

REFERENCES

[1] Tablettes du Chercheur, 15 Janvier 1891 et 1er Fevrier 1891

[2] Sur les carrés à deux degrés, Tablettes du Chercheur, 1893, p.182-183

[3] Des carrés de 8 et de 9 magiques aux deux premiers degrés. Des carrés de mêmes bases en nombres triangulaires, par M. Coccoz, Compte-rendu de la 21ème session de l'AFAS, Congrès de Pau 1892, séance du 17 septembre 1892. Compte-rendu, seconde partie, p.136-148

[4] Site http://www.gaspalou.fr/magic-squares/

[5] Carrés panmagiques de base 3n, par M. G. Tarry, Compte-rendu de la 32ème session de l'AFAS, Congrès d'Angers 1903, séance du 6 août 1903. Compte-rendu, seconde partie, p.130-142 (see particularly the p.141)

[6] Francis Gaspalou, how many squares are there, Mr. Tarry? February 10, 2012 http://www.multimagie.com/GaspalouTarry.pdf

[7] Des variations qu'on peut apporter aux carrés de huit magiques aux deux premiers degrés, par M. Coccoz, Compte-rendu de la 22ème session de l'AFAS, Congrès de Besançon 1893, séance du 4 août 1893. Compte-rendu, seconde partie, p.171-183

[8] Construction des carrés magiques avec des nombres non consécutifs, etc…, par M. Coccoz, Compte-rendu de la 23ème session de l'AFAS, Congrès de Caen 1894, séance du 10 août 1894. Compte-rendu, seconde partie, p.163-183

[9] Quelques exemples de carrés de huit magiques aux deux premiers degrés dont les lignes et surtout les diagonales sont de composition qui, n'étant point connues, n'ont pas été mentionnées en 1892 et 1893 aux congrès de Pau et Besançon, par M. V. Coccoz, Compte-rendu de la 31ème session de l'AFAS, Congrès de Montauban 1902, séance du 9 août 1902. Compte-rendu, seconde partie, p.137-157

[10] Carrés magiques, par M. le Commandant Coccoz, Compte-rendu de la 32ème session de l'AFAS, Congrès d'Angers 1903, séance du 6 août 1903. Compte-rendu, seconde partie, p.142-157

[11] Transformations dont sont susceptibles certains carrés bimagiques, par M. Achille Rilly, Compte-rendu de la 36ème session de l'AFAS, Congrès de Reims 1907, séance du 3 août 1907. Compte-rendu, seconde partie, p.42-48

All these documents are on line.

IN ATTACHMENT: zip file *with*:

1 list of the 10,317 squares
2 list of the 2,188 squares appearing 7 times
3 list of the 536 squares appearing 2 times
4 list of the 5,485 squares
5 list of the 841 ess. diff. associative bi. sq.
6 list of the 843 ess. diff. bi. sq. A1+E5=65
7 list of the 538 ess. diff. bi. sq. A1+F6=65
8 list of the 455 ess. diff. bi. sq. A1+G7=65
9 list of the 1,344 ess. diff. Greco-Latin bi. sq.
10 list of the 860 ess. diff. pandiag. bi. sq. (843 complete + 17 not complete)
11 list of the 17 ess. diff. pandiag. bi. not complete sq.