Identities involving binomial-coefficients,
Bernoulli- and Stirlingnumbers

Mathematical Miniatures

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List of matrices (uncommented)

Abstract: The most prominent vectors and matrices are introduced.

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Project-Index	http://go.helms-net.de/math/binomial/index
Intro/notation	http://go.helms-net.de/math/binomial/00_0_intro.pdf

1. List (not yet complete) of important identities

This list is (weakly) ordered for the type of matrices, row-sums and column-sums, Right & leftmultiplication by powerseries-vectors and factorial-vectors, sometimes in connection with powerseries.

1.1. Unsigned and Pascal-matrix and variants

1.1.1. Unsigned and signed Pascalmatrix

(1.1.1.1.) $P := P_{r,c} = binomial(r,c)$ (if $c>r$ $P_{r,c} = 0$)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

(1.1.1.2.) $P_J := P_{J\ r,c} = (-1)^c * binomial(r,c)$ // if r>=c
$= P * J$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & -1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ 1 & -3 & 3 & -1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ 1 & -5 & 10 & -10 & 5 & -1 \end{bmatrix}$$

(1.1.1.3.) $_JP := P_{J\ r,c} = (-1)^r * binomial(r,c)$ // if r>=c
$= J * P$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ -1 & -1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ -1 & -3 & -3 & -1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ -1 & -5 & -10 & -10 & -5 & -1 \end{bmatrix}$$

(1.1.1.4.) $P^{-1} = J*P*J$
$\Sigma_{k=1..r} [(-1)^{c-k} * bi(r,k) * bi(k,c)] = \delta_{r,c}$ where δ is the Kronecker-delta.

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$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ -1 & 1 & . & . & . & . \\ 1 & -2 & 1 & . & . & . \\ -1 & 3 & -3 & 1 & . & . \\ 1 & -4 & 6 & -4 & 1 & . \\ -1 & 5 & -10 & 10 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{bmatrix}$$

(1.1.1.5.) $P^{\,n} = P * P^{\,n-1}$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 4 & 4 & 1 & . & . & . \\ 8 & 12 & 6 & 1 & . & . \\ 16 & 32 & 24 & 8 & 1 & . \\ 32 & 80 & 80 & 40 & 10 & 1 \end{bmatrix}$$

and also

(1.1.1.6.) $P^{\,n} = {}^dV(n) * P * {}^dV(1/n) = P ☼ (V(n) * V(1/n)\sim) = P ☼ Toeplitz(n)$

(1.1.1.7.) $P = exp(L)$

$$\exp\left(\begin{bmatrix} . & . & . & . & . & . \\ 1 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 3 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 5 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ 1 & 1 & . & . & . & . \\ 1 & 2 & 1 & . & . & . \\ 1 & 3 & 3 & 1 & . & . \\ 1 & 4 & 6 & 4 & 1 & . \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix}$$

(1.1.1.8.) $P^a = exp(L*a)$
$= P ☼ ShT(V(a)) = P ☼ Toeplitz(a)$

$$\exp\left(\begin{bmatrix} . & . & . & . & . & . \\ a & . & . & . & . & . \\ . & 2*a & . & . & . & . \\ . & . & 3*a & . & . & . \\ . & . & . & 4*a & . & . \\ . & . & . & . & 5*a & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . & . \\ a & 1 & . & . & . & . \\ a\hat{}2 & 2*a & 1 & . & . & . \\ a\hat{}3 & 3*a\hat{}2 & 3*a & 1 & . & . \\ a\hat{}4 & 4*a\hat{}3 & 6*a\hat{}2 & 4*a & 1 & . \\ a\hat{}5 & 5*a\hat{}4 & 10*a\hat{}3 & 10*a\hat{}2 & 5*a & 1 \end{bmatrix}$$

1.2. Pascalmatrix-variants

A shifted version:

(1.2.1.1.) $L = Sd(1,Z(-1)+1)$

$$\begin{bmatrix} . & . & . & . & . & . \\ 2 & . & . & . & . & . \\ . & 3 & . & . & . & . \\ . & . & 4 & . & . & . \\ . & . & . & 5 & . & . \\ . & . & . & . & 6 & . \end{bmatrix}$$

(1.2.1.2.) $Pc = exp(L)$

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix}$$

(1.2.1.3.)

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 1 & . & . & . & . \\ 3 & 3 & 1 & . & . & . \\ 4 & 6 & 4 & 1 & . & . \\ 5 & 10 & 10 & 5 & 1 & . \\ 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 21 \\ 85 \\ 341 \\ 1365 \end{bmatrix} or= \begin{bmatrix} (x\hat{}1-1)/(x-1) \\ (x\hat{}2-1)/(x-1) \\ (x\hat{}3-1)/(x-1) \\ (x\hat{}4-1)/(x-1) \\ (x\hat{}5-1)/(x-1) \\ (x\hat{}6-1)/(x-1) \end{bmatrix}$$

Versions of different orders

* by powers of the values in subdiagonal 1:

(1.2.1.4.) $\mathbf{P}_0$ $= exp(Sd(-1,Z(0))$

(1.2.1.5.) $:= P_{0\ r,c} = 1/(r-c)!$ //=0 if c>r

$= F^{-1} * P * F$

(1.2.2.) $$P_0 := P_{0\ r,c} = \left(\frac{r!}{c!}\right)^0 * \frac{1}{(r-c)!}$$

$$exp\left(\begin{bmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1/2 & 1 & 1 & . & . \\ 1/6 & 1/2 & 1 & 1 & . \\ 1/24 & 1/6 & 1/2 & 1 & 1 \end{bmatrix}$$

(1.2.2.1.) $\mathbf{P}_1$ $= exp(Sd(-1,Z(1))$

(1.2.2.2.) $:= P_{1\ r,c} = (r!/c!)^1/(r-c)!$ //=0 if c>r

$= F * P_0 * F^{-1}$

(1.2.3.) $$P_1 := P_{1\ r,c} = \left(\frac{r!}{c!}\right)^1 * \frac{1}{(r-c)!}$$

$$exp\left(\underset{\text{P1}}{\begin{bmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 2 & . & . & . \\ . & . & 3 & . & . \\ . & . & . & 4 & . \end{bmatrix}}\right) = \underset{\text{P}}{\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}}$$

(1.2.3.1.) P_2 $= exp(Sd(-1,Z(-2))$

(1.2.3.2.) $:= P_{2\ r,c} = (r!/c!)^2*1/(r-c)!$ //=0 if c>r

$= F^2 * P_0 * F^{-2}$

(1.2.4.) $$P_2 := P_{2\ r,c} = \left(\frac{r!}{c!}\right)^2 * \frac{1}{(r-c)!}$$

$$exp\left(\begin{bmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 4 & . & . & . \\ . & . & 9 & . & . \\ . & . & . & 16 & . \end{bmatrix}\right) = \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 2 & 4 & 1 & . & . \\ 6 & 18 & 9 & 1 & . \\ 24 & 96 & 72 & 16 & 1 \end{bmatrix}$$

$P_2 = F * P_1 * F^{-1}$

$$\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 2 & . & . \\ . & . & . & 6 & . \\ . & . & . & . & 24 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 2 & 1 & . & . \\ 1 & 3 & 3 & 1 & . \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} * \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1/2 & . & . \\ . & . & . & 1/6 & . \\ . & . & . & . & 1/24 \end{bmatrix}$$

$P_2 = F^2\ P_0\ F^{-2}$

$$\left(\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 2 & . & . \\ . & . & . & 6 & . \\ . & . & . & . & 24 \end{bmatrix}\right)^2 * \begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1/2 & 1 & 1 & . & . \\ 1/6 & 1/2 & 1 & 1 & . \\ 1/24 & 1/6 & 1/2 & 1 & 1 \end{bmatrix} * \left(\begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & . & . \\ . & . & 1/2 & . & . \\ . & . & . & 1/6 & . \\ . & . & . & . & 1/24 \end{bmatrix}\right)^2$$

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by powers of the logarithm:

"Gauss"-matrix

$Gsl = - Pl^2/2$

$$\left(\underset{\text{(P1)}}{\begin{bmatrix} . & . & . & . & . \\ 1 & . & . & . & . \\ . & 2 & . & . & . \\ . & . & 3 & . & . \\ . & . & . & 4 & . \end{bmatrix}}\right)2 / 2 \qquad \underset{\text{1²/2}}{\begin{bmatrix} . & . & . & . & . \\ . & . & . & . & . \\ 1 & . & . & . & . \\ . & 3 & . & . & . \\ . & . & 6 & . & . \end{bmatrix}}$$

$Gs = exp(Gsl)$

$$exp\left(\underset{\text{GS1}}{\begin{bmatrix} . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -1 & . & . & . & . & . & . \\ . & -3 & . & . & . & . & . \\ . & . & -6 & . & . & . & . \\ . & . & . & -10 & . & . & . \\ . & . & . & . & -15 & . & . \end{bmatrix}}\right) = \underset{\text{GS}}{\begin{bmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ -1 & . & 1 & . & . & . & . \\ . & -3 & . & 1 & . & . & . \\ 3 & . & -6 & . & 1 & . & . \\ . & 15 & . & -10 & . & 1 & . \\ -15 & . & 45 & . & -15 & . & 1 \end{bmatrix}}$$

1.3. Gp-matrices and variants

(1.3.1.1.) Gp

$$\begin{bmatrix} 1 & . & . & . & . \\ 1/2 & 1/2 & . & . & . \\ 1/6 & 1/2 & 1/3 & . & . \\ 0 & 1/4 & 1/2 & 1/4 & . \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 \end{bmatrix}$$

(1.3.1.2.) Gp^{-1}

$$\begin{bmatrix} 1 & . & . & . & . \\ -1 & 2 & . & . & . \\ 1 & -3 & 3 & . & . \\ -1 & 4 & -6 & 4 & . \\ 1 & -5 & 10 & -10 & 5 \end{bmatrix}$$

(1.3.1.3.) Gm

$$\begin{bmatrix} 1 & . & . & . & . \\ -1/2 & 1/2 & . & . & . \\ 1/6 & -1/2 & 1/3 & . & . \\ 0 & 1/4 & -1/2 & 1/4 & . \\ -1/30 & 0 & 1/3 & -1/2 & 1/5 \end{bmatrix}$$

(1.3.1.4.) Gm^{-1}

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 2 & . & . & . \\ 1 & 3 & 3 & . & . \\ 1 & 4 & 6 & 4 & . \\ 1 & 5 & 10 & 10 & 5 \end{bmatrix}$$

1.4. Stirling-matrices

(1.4.1.1.) St_1

$$\begin{bmatrix} 1 & . & . & . & . \\ -1 & 1 & . & . & . \\ 2 & -3 & 1 & . & . \\ -6 & 11 & -6 & 1 & . \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix}$$

(1.4.1.2.) St_2

$$\begin{bmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & 3 & 1 & . & . \\ 1 & 7 & 6 & 1 & . \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}$$

1.5. Vandermonde-matrix

(1.5.1.1.) ZV

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{bmatrix}$$

(1.5.1.2.) ZV^r

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 \\ 1 & 1/4 & 1/16 & 1/64 & 1/256 \\ 1 & 1/5 & 1/25 & 1/125 & 1/625 \end{bmatrix}$$

2. References

[Project-Index] http://go.helms-net.de/math/binomial/index

[Intro/Notation] http://go.helms-net.de/math/binomial/00_0_intro.pdf

[ListOfMatrices] http://go.helms-net.de/math/binomial/00_1_ListOfMatrices.pdf

[binomialmatrix] http://go.helms-net.de/math/binomial/01_1_binomialmatrix.pdf

[signed binomial] http://go.helms-net.de/math/binomial/01_2_signedbinomialmatrix.pdf

[Stirlingmatrix] http://go.helms-net.de/math/binomial/01_3_stirlingmatrix.pdf

[Gaussmatrix] http://go.helms-net.de/math/binomial/01_5_gaussmatrix.pdf

[GenBernRec] (Generalized Bernoulli-recursion)
http://go.helms-net.de/math/binomial/02_1_GeneralizedBernoulliRecursion.pdf

[SumLikePow] (Sums of like powers)
http://go.helms-net.de/math/binomial/04_3_SummingOfLikePowers.pdf

[Erdoes] http://go.helms-net.de/math/binomial/10_1_erdoes.pdf

[Hasse] http://go.helms-net.de/math/binomial/10_2_recihasse.pdf

[InvVandermonde] http://go.helms-net.de/math/binomial/10_3_InverseVandermonde.pdf

Projekt ***Bernoulli-numbers***, first versions of the above, contain a ***first rough exploratory*** course but are already cover most topics and contain also the basic material about $\boldsymbol{G_p}$ and $\boldsymbol{G_m}$ which is still missing in the above list:

[Bernoulli] http://go.helms-net.de/math/binomial/bernoulli_en.pdf

[Summation] http://go.helms-net.de/math/binomial/pmatrix.pdf

[Matexp] Matrixexponential Alan Edelman & Gilbert Strang, MIT
http://web.mit.edu/18.06/www/pascal-work.pdf

[Laguerre] Laguerrematrix
http://mathworld.wolfram.com/LaguerrePolynomial.html

[Roots] "Polynomials From Pascal's Triangle" Mathforum at Drexel
http://mathpages.com/home/kmath304.htm

[Toeplitzmatrix] Toeplitz-matrices Wikipedia
http://en.wikipedia.org/wiki/Toeplitz_matrix

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