Questions about Polytopes

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Why study polytopes? Because they are so beautiful, intriguing, and important, and because there are so many interesting questions about polytopes waiting to be studied and solved.

This answer may be true, but of course it leaves many questions open. In particular, what makes a subject such as polytopes "important," and what makes questions about polytopes "interesting"? As we will see, many basic questions about polytopes have been presented to the theory "from outside," and many of these questions are still waiting for a definitive answer. So, the following panorama of the theory of polytopes, written in the year 2000, will ask "*Who's asking?*" for a selection of seven themes that seem central, important, beautiful, and intriguing. These without doubt will keep us occupied for a while, fascinated, and trying to give answers.

Polytope theory has never been a "theory about the empty set": indeed, there are interesting examples all over the place, and the study of the "key examples" has so often led one to intriguing phenomena, surprising effects, and eventually to some answers. Thus a second question we will pose for each of the seven themes is "*Can you give me a good example?*".

The third leading question I'd love to pose for each theme is "*Where are we going?*" What's the theory going to look like, say, in 2017, at the 50th anniversary of the publication of Grünbaum's classic volume [23] that defined the field? I don't know, of course. But I will try to point to a few directions that seem worthwhile to explore.

1. Symmetry

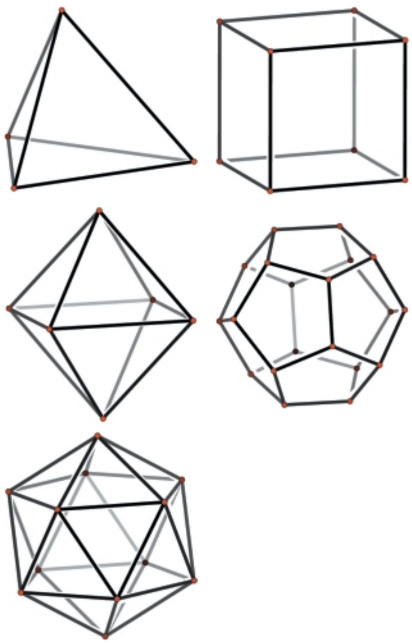
The theory of polytopes has firm roots in classical Greek mathematics: the discovery/construction of the "Platonic solids" can be traced back to Euclid's book XIII of the "Elements" [3], and the discussion of their beauty and universality to Plato, although "The early history of these polyhedra is lost in the shadows of antiquity" [13, p. 13]. They are what we now call the "regular convex 3-dimensional polytopes." The complete list consists of the tetrahedron, the cube (the fundamental building block of real three-space), the octahedron, the dodecahedron and the icosahedron. We owe to the Greeks also the concept of duality, under which the tetrahedron is self-dual, while the cube is dual to the octahedron, and the dodecahedron is dual to the icosahedron. We also owe to the Greeks a lot of mysticism associated with these beautiful objects, which were put into bijection with "the elements," the planets, etc.

"What are the symmetry groups of regular polytopes?" This is a very natural question, and it was asked from many directions, and thus has found many extensions. With hindsight, it may seem strange that the concept of a group was first developed for symmetries of fields (Galois groups), in the context of the solution of polynomial equations by radicals, and not in the study of polytopes. Nevertheless, undergraduate textbooks nowadays start the discussion of groups with the example of the symmetry groups of a hexagon, and of the cube, but not with the roots of a quintic!

The classification of all the regular polytopes was achieved by the ingenious Swiss geometer Ludwig Schläfli around 1852: Besides the "obvious ones," namely the d-dimensional cube and the d-dimensional cross polytope (that is, the unit balls of $\mathbb{R}^d$ in ℓ_∞- respectively ℓ_1-norms) and the Platonic solids, there are three exceptional examples in dimension 4: the 24-cell, the 120-cell, and the 600-cell, whose facets are 24 octahedra, 120 dodecahedra, respectively 600 tetrahedra. One may take the regular polytopes just as "curious objects" of analytical geometry, but we now know that there is much more to them than "meets the eye." For example, consider the classification of the irreducible finite groups generated by reflections, known as the Coxeter groups. All of these occur either as symmetry groups of regular polytopes (if the Coxeter diagram is linear) or as the Weyl group of a root system (if all the weights of the Coxeter graph are in $\{2, 3, 4, 6\}$), or both. The latter case provides also the classification of the simple Lie algebras, and thus the structure theory of simple complex and of compact real Lie groups. In the overlap of the two cases above one has a the "usual suspects" (the symmetry groups A_{d-1} and BC_d of the d-simplices and of the d-cubes), plus one exceptional example: the 24-cell, with symmetry group F_4, which is a remarkable object, interesting from many points of view. For example, the fact that (besides the simplices) the 24-cell is the only self-dual regular polytope, can be made responsible for "special effects" that occur for F_4-buildings. If you ask about key examples, in the theory of regular polytopes this is definitely one to look at. (Coxeter's book [13] is the classical reference on regular polytopes.)

Who is asking? Among many others there are the problems posed by crystallography and by crystallographers: to classify the possible symmetries occuring in "nature," to classify the symmetry groups of crystals, to classify the polytopes that tile space (particularly interesting for crystallographers are the tilings of 3-space), etc. Many of these questions have been answered satisfactorily for a long time, such as the classification of the crystallographic space groups in dimensions 3 and 4, although only very recently we got them "into our hands" in a unified way that is also accessible as a database and with the algorithms needed to manipulate them. On the other hand, the question of the possible tilings of 3-space is by far not answered completely, and there are still fascinating questions to be studied. These include:

- What is the maximal number of facets of a convex polyhedron that can be used for a complete face-to-face tiling of 3-space with congruent copies? (The current record, a polyhedron with 34 faces, is due to Engel (1981).)
- Does every tiling of d-space by translates of one single polytope admit a scalar product for which the tiles arise as the Dirichlet-Voronoï cells of the



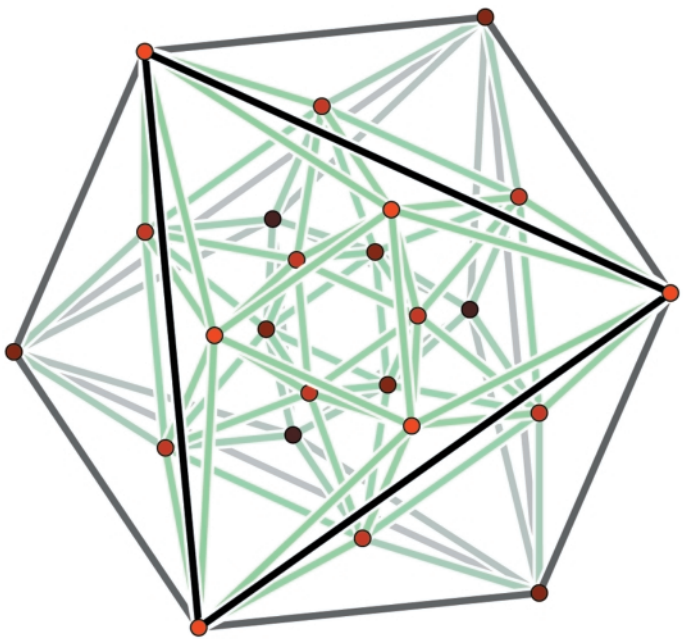


FIGURE 1 (*left*)
The regular 3-polytopes

FIGURE 2 (*right*)
A "Schlegel diagram" of the 24-cell

lattice? (This "geometrization conjecture" of Voronoï (1908) is known to be true in dimensions up to 4, where we understand lattice tilings quite well, and in a number of special cases, but the general question is still open.)

Another question, on a borderline between discrete geometry and crystallography, and thus not easily formulated as a precise "mathematical question," asks for the possible tilings of 3-space that can be "chemically realized." And indeed, interesting new such structures can be found both in the computer and in the laboratory. See [15] for recent "front-page news" in this direction.

2. Counting Faces

The Euler polyhedron formula

$$v + f = e + 2,$$

connecting the numbers of vertices, facets, and edges of a convex 3-polytope, was at various occasions listed as one of the "top ten theorems of mathematics." This formula has an interesting history [18], and it was the starting point for a wealth of further developments. Let us first rewrite the equation as

$$f_0 - f_1 + f_2 = 2,$$

where f_i denotes the number of i-dimensional faces of the polytope. In this context, the "obvious extension" to d dimensions is the relation

$$f_0 - f_1 + f_2 - \ldots + -1^{d-1} f_{d-1} = 1 + (-1)^{d-1}$$

between the entries of the so-called *f-vector* $(f_0, f_1, \dots, f_{d-1})$ of a d-dimensional polytope. It was also found by Schläfli in 1852. However, Schläfli's proof was not complete: It depended on a "good ordering" of the facets, in modern terms "shelling," whose existence was established only in 1970, by Bruggesser and Mani. But a valid proof was given by Poincaré, and this was a starting point of modern algebraic topology.

"What are the possible f-vectors of convex d-polytopes?" This question is easily answered in dimensions 2 and 3: For $d = 2$ the f-vectors are (f_0, f_1) with $f_0 = f_1 \geq 3$. For $d = 3$ the answer given by Steinitz (1906) was that the f-vector (f_0, f_1, f_2) has to satisfy only the conditions $f_2 \leq 2f_0 - 4$ and $f_0 \leq 2f_2 - 4$, in addition to the Euler formula. It is a nice exercise to verify that all integral vectors (f_0, f_1, f_2) that satisfy these conditions are indeed f-vectors of 3-polytopes.

The next step would be to answer the analogous question in dimension $d = 4$, but this is an unsolved problem, even though a lot of work has gone into it – see [7] and [8]. We do know a lot of conditions by now that have to be satisfied for f-vectors (f_0, f_1, f_2, f_3) of 4-polytopes, but there are also substantial "conditions missing." At the moment it seems that we are far from a complete answer from both sides: On the one hand, there inequalities that we don't know or at least cannot prove, such as

$$f_1 + f_2 \leq 6(f_0 + f_3).$$

On the other hand, we do not have enough good construction methods for classes of interesting 4-polytopes. The problem is that most polytopes that you can easily "write down" are either *simple* or *simplicial*: either their facets, or their vertices, are in sufficiently "general position." Perhaps we don't even understand the true complexity and degeneracy that may occur in polytopes of dimensions 4 and higher. We'll have to work harder on this!

If we restrict our attention to the case of simplicial polytopes (where each facet is a simplex), then the situation is much better, and indeed the characterization of the f-vectors of simplicial convex polytopes is a spectacular achievement of modern polytope theory.

The extremal examples for the simplicial case can be described explicitly: Given a simplicial d polytope with n vertices, Barnette's "Lower Bound Theorem" (1971) says that the number of facets is minimized by any "stacked polytope," which is obtained from a simplex via $n - d - 1$ stellar subdivisions on facets, resulting in a total of $(d + 1) + (d - 1)(n - d - 1)$ facets. At the other extreme, the "Upper Bound Theorem" of McMullen (1970) establishes that the maximal number of facets is obtained by any "neighborly" polytope, for which every set of $\lfloor \frac{d}{2} \rfloor$ vertices form a face. The neighborly polytopes have *many* facets:

$$f_{d-1} = \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} + \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor},$$

to be precise. There are "super-exponentially many" different combinatorial types of neighborly polytopes (Shemer 1982), but the only infinite class of examples that one can construct “in closed form” seems to be that of the cyclic polytopes: take the convex hull of any n distinct points

$$C_d(n) := \mathrm{conv}\{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)\}$$

on a curve γ of order d. For this one can take the “moment curve”

$$\gamma(t) := (t, t^2, \ldots, t^d),$$

for which the combinatorics (“Gale’s evenness criterion”) can be derived from Vandermonde determinants, or – in even dimensions, and that’s the interesting case – one can take a trigonometric curve of the form

$$\gamma(t) := (\cos(t), \sin(t), \cos(2t), \sin(2t), \ldots, \cos(\tfrac{d}{2}t), \sin(\tfrac{d}{2}t)),$$

which results in a symmetric polytope, with a vertex transitive group of rotational symmetries!

The complete characterization of all the f-vectors of simplicial polytopes, between these two extremes, is known as the “g-Theorem.” It consists of four different parts, each of them substantial, surprising, and with its own difficulties. The first one was a conjecture, the “g-Conjecture” of McMullen (1971), proposing an intricate set of linear and nonlinear inequalities that were supposed to give a complete characterization. The second step was a construction by Billera and Lee (1980), which established that every vector that satisfies McMullen’s condition does indeed arise as the f-vector of a simplicial d-polytope. For this, Billera and Lee had to come up with “many examples” of simplicial polytopes with different f-vectors: and they did! They constructed their polytopes as shadow-boundaries of cyclic polytopes – or, equivalently, as d-dimensional central projections of $(d+1)$-dimensional cyclic polytopes. The third step was achieved by Stanley (1980): he established a link to the theory of toric varieties, where to each simplicial polytope one can associate a toric variety that is “reasonably smooth,” and whose rational Betti numbers are given by linear functions of the f-vector. So, Stanley established that the validity of McMullen’s conditions would follow from a “hard Lefschetz theorem” for such toric varieties. Part four was to prove this hard Lefschetz theorem – which turned out to be harder than expected. The first proof was wrong, the second one was extremely technical and difficult, and the third one – by McMullen (1993) – turned things around and used intricate convex geometry methods to establish the algebraic geometry result ... and a definitive “simple” proof, which might perhaps also be true for “star convex” polytopes, still does not seem in sight.

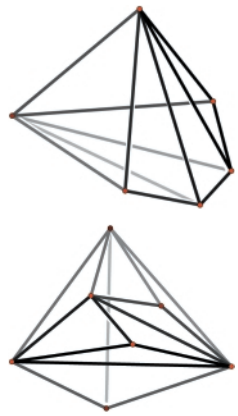


FIGURE 3
A cyclic polytope $C_3(6)$ and a stacked polytope.

Also beyond the simplicial case, the area of “counting faces” features many good open questions, among them:

- Characterize the f-vectors of 4-dimensional polytopes.
- Characterize the f-vectors of centrally-symmetric simplicial polytopes.
- Characterize the f-vectors of cubical polytopes (that is, of polytopes for which all the faces are combinatorial cubes).

The f-vector also includes a lot of information about the combinatorial “size” and complexity of a polytope. It is quite strange to see how little we really know;

so, for example, at the moment we cannot even answer whether

$$f_i \geq \frac{1}{1000} \min\{f_0, f_{d-1}\}$$

holds for all convex polytopes (for $0 < i < d-1$). Or, in other words, could a convex polytope really have a "thin waist"?

3. Paths

Linear programming is an extremely important part of mathematical programming, not only commercially, but also since many other routines (for example for combinatorial optimization) depend on the efficient solution of linear programs. Geometrically, the linear programming problem is to find a "highest point" (with respect to some linear height function) in some polytope or polyhedron that is given as an intersection of half spaces (that is, as the set of solutions to a finite system of linear inequalitities). The simplex algorithm for linear programming, as developed chiefly by George Dantzig starting in 1947, solves this problem by first finding one "feasible" vertex of the polyhedron, and then moving along edges of the polyhedron in such a way that the linear objective function is increased during each step. Linear programming theory (see e.g. [32]) provides a number of methods and transformations that reduce the general case to the study of "nice" problems, which satisfy a number of special properties, namely that

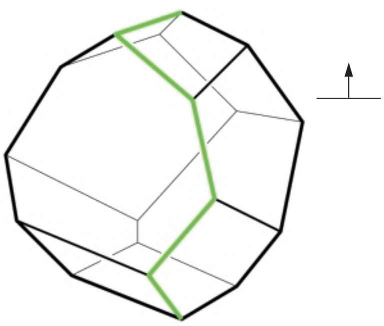


FIGURE 4
A possible path "to the top" taken by a simplex algorithm.

- the polyhedron is bounded (a polytope),
- it is full-dimensional (a d-polytope in d-space),
- it is simple (every vertex is incident to exactly d edges), and
- the objective function is generic (no two vertices of the polytope get the same value, and thus no edge is "horizontal" with respect to the objective function).

For the following, we may even assume that we are provided with one vertex of the polyhedron as our "starting point," and we may assume that this point is the lowest vertex of the polytope. Now the simplex algorithm chooses edges to "walk along them" in order to "reach the top." But which edges to take?

This is a question asked by linear programming to polytope theory. In fact, there are three different main questions, none of them answered satisfactorily by now, and each of them intriguing:

- "Is there always a 'short' path?"
- "How long can a monotone path be?"
- "Would a random monotone path be short?"

For each of them, the 'short' and 'long' is meant in terms of the size of the input polytope, which is given by the dimension d, and the number n of facets/inequalities.

The first question was apparently asked by Warren Hirsch of George Dantzig at a conference in 1957, and it appears in Dantzig's fundamental book from 1963 [14]: "Is there always a short path (specifically, a path using at most $n-d$ edges) from each vertex to every other vertex?" So the Hirsch conjecture – intensively studied but still unsolved [28] – claims that the combinatorial diameter of a simple d-polytope with n facets is bounded by $n - d$. This conjecture is best possible: There are example polytopes of diameter $n - d$ for all $n > d \geq 8$ (Holt & Klee 1998; Fritzsche & Holt 1999). However, we have no upper bound available that would even be polynomial in n and d; the best result to date, due to Kalai and Kleitman (1992), amounts to an upper bound of $n^{1+\log d}$.

The Hirsch conjecture is of great importance because of its close connection to the complexity of the simplex method. In particular, if there is no polynomial upper bound on this diameter, then the simplex algorithm can not provide a strongly polynomial method for linear programming. But a positive solution of the Hirsch conjecture could also provide us with a method to *find* short paths. And thus, we are dealing with the complexity of the linear programming problem itself [10] [32].

The second question, "How many edges can a monotone path on a simple polytope have in the worst case?", is a beautiful example from mathematics history where the question that was asked, and the answer that was finally provided for it, don't really fit together. On the way to an answer, Gale (1963) found/constructed the duals of the cyclic polytopes, which have *many* vertices. And then the answer that was given to our question is composed of two parts:

- The number of vertices on a monotone path is bounded by the number of all vertices of the polytope.
- The number of vertices of a d-polytope with n facets is bounded by the number of facets of the cyclic polytope $C_d(n)$, according to the Upper Bound Theorem.

Both bounds are sharp – the second one for the duals of cyclic polytopes, the first one for example for a sequence of deformed cubes constructed by Klee and Minty (1972): The Klee-Minty cubes are "deformed" so that there is a monotone path through all the vertices. Many other examples were given of polytopes with long monotone paths, but they all rely on the "deformed product" idea of Klee and Minty [1].

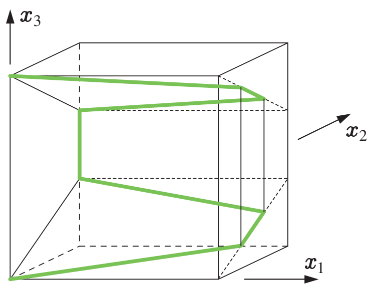


FIGURE 5
A 3-dimensional Klee-Minty cube.

However, to answer the original question, we must find out whether both bounds above can be *simultaneously* tight: It is not clear that for a dual of a cyclic polytope, there can be a monotone path through all, or most, of the vertices. Thus there is still a huge gap between the upper bound provided by the Upper Bound Theorem and the length of a monotone path that one can actually construct via deformed products ...

If – for the third question – we look at a random path from the bottom to the top vertex of a polyhedron, how long will that be? Exponentially long paths exist, as we see from the Klee-Minty cubes. However, it is still possible that the expected length of a (suitably selected) random path will be polynomial, or even quadratic in n and d. This would provide a strongly polynomial (randomized) method for linear programming and thus settle the question about the complexity of linear programming in all kinds of complexity models that admit a probabilistic method. However, the method of taking "randomly the next increasing edge" is easy to describe, but seems nasty to analyze. There are other probabilistic rules that are harder to describe, but easier to analyze

[25]. A crucial point seems to be that purely combinatorial games/arguments will not suffice to settle the problem; rather, we have to see whether we can get genuinely geometric arguments and properties to help [20].

All three basic questions about paths on polytopes are important, but at the moment we cannot answer them. The solution will be that we need to use new tools, from other parts of Mathematics. On the one hand, probability theory enters as soon as we consider the simplex algorithm (with random pivots) as a stochastic process: Analyze it!

A different point of view is that polytopes should be considered as "round" in quite a variety of different ways: A suitable coordinate transformation can be computed that eliminates "thin" and "flat" polytopes, so we can assume that our polytope is "round." For large and complex polytopes, perhaps convex geometry versions of paths on reasonably smooth bodies could be applied [30]. But there are also discrete models of curvature (see [19]) according to which boundaries of polytopes could be positively or at least non-negatively curved, which could imply small diameters, one might hope.

4. 0/1-Polytopes

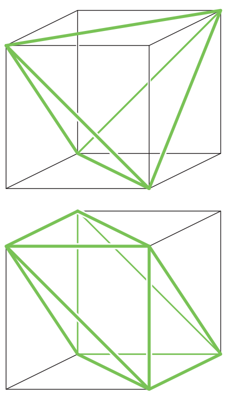


FIGURE 6
A regular 0/1-tetrahedron and a non-regular 0/1-octahedron.

Combinatorial optimization has had remarkable success in the exact solution of large instances of (NP-)hard optimization problems, such as travelling salesman and max-cut problems. This success is based on investigations in "polyhedral combinatorics," the study of the combinatorics (in particular, the facets) of specific classes of 0/1-polytopes, such as the travelling salesman polytopes and the cut polytopes.

The 0/1-cube $[0,1]^d = \text{conv}(\{0,1\}^d)$ is a very basic, simple d-polytope. The 0/1-polytopes arise as the convex hulls of subsets of the vertex set of the 0/1-cube, that is, as $\text{conv}(V)$ for subsets $V \subseteq \{0,1\}^d$. Some of these 0/1-polytopes are quite intricate beasts, have extremely complicated structure, and pose lots of difficult and interesting questions.

The past ten years have given us a good overview over *extremal* properties of 0/1-polytopes. The next ten years should provide us with a picture of the *typical* 0/1-polytopes – in terms of the same parameters and properties that were and are interesting for extremal 0/1-polytopes.

To produce a "typical" 0/1-polytope one could look at a random 0/1-polytope, that is, a polytope whose vertices are chosen with equal probability from the vertices of the d-dimensional 0/1-cube. Random 0/1-polytopes are particularly interesting since they provide a natural model of random polytopes that are typically neither simple nor simplicial. The number of vertices would be fixed in advance – say $2d$, or d^2, or $2^{d/2}$ vertices. You might complain that such a choice is quite arbitrary: Well, it is; what we'd really want to understand is the *evolution* of a random polytope, the development of its expected parameters when the number of vertices grows from $d+1$ to 2^d, with the random vertices added one by one. (One might take the evolution of the random graph, which is quite well understood by now, as a role model.)

Thus each of the exciting extremal results that we now know about 0/1-polytopes becomes a much more difficult question for the evolution of a random

0/1-polytope. We here name just a few, and refer to [34] for references and details.

We know that 0/1-polytopes may have lots of edges; for example, there are examples such as the cut polytopes with an exponential number of vertices that are 2-neighborly, that is, any two vertices are connected with an edge. We also know that vertex degrees may be as large as 2^{d-1} and even larger that that, from an example found by O. Aichholzer.

We know that the integral facet-defining inequalities of 0/1-polytopes may have huge coefficients: The upper bound provided by Hadamard's inequality applied to 0/1-determinants is essentially sharp (Alon & Vũ 1997).

And, perhaps most strikingly, let's consider the basic question "How many facets can a d-dimensional 0/1-polytope have?" It appears for example in a paper by Grötschel and Padberg (1988) about the travelling salesman polytopes, where they asked for an a priori estimate for the travelling salesman polytopes. Only very recently we got closer to convincing answers: On the one hand, there is an upper bound of roughly $(d-2)!$, due to Fleiner, Kaibel & Rote (2000). On the other hand, Bárány & Pór (2000) produced a first superexponential lower bound for the number of suitable random 0/1-polytopes with a rather large (exponential) number of vertices. Their investigation includes a lot of information about the geometry of these random 0/1-polytopes, including quite precise descriptions about the points that "with high probability" lie in the random 0/1-polytope.

"Do the examples of interest in combinatorial optimization, such as travelling salesman polytopes and cut polytopes, really "look" and "behave" like random 0/1-polytopes?" In fact, these two examples behave quite differently. They both arise from the following construction: Let K_m denote a complete graph with m vertices and $d := \binom{m}{2}$ edges. If we identify the d edges of K_m with the coordinates $1, 2, \ldots, d$, then the subsets of the edge set of K_m correspond to the 0/1-vectors of length d, that is, to the vertices of $[0,1]^d$. The travelling salesman polytope T_m is now defined as the convex hull of all the 0/1-vectors that correspond to the travelling salesman tours in K_m, while the cut polytope C_m is defined as the convex hull of those 0/1-vectors that correspond to the edge sets of cuts through the graph K_m.

What can we say about these examples? One can say quite a lot. For example, the travelling salesman polytope T_m has dimension $\frac{m(m-3)}{2}$, it is very symmetric (with a vertex transitive symmetry group), it has lots of facets (large classes of which can be explicitly described), and it is *very complicated*. This is strikingly apparent from the result of Billera and Sarangarajan (1996) that *every* 0/1-polytope appears as a face of a travelling salesman polytope. Thus, there is no hope for a complete description or perfect analysis of these polytopes. Nevertheless, many of the facts that were found out about the combinatorics of these polytopes (in particular, about their classes of facets) have been used in practice: Embedded into cutting plane procedures this has led to striking successes, such as the exact solution of travelling salesman problems with more than 10 000 nodes [2].

And the cut-polytopes? They are full-dimensional, they are also very symmetric (even though this is not quite apparent from the definition!), they have extremely many facets (as one can see from explicit computations for $m \leq 9$), and they are rather neighborly: Any two vertices determine an edge, any three of them determine a triangle. Thus, the combinatorial types of the low-dimensional faces are very restricted: They are all simplices! Again, large classes of facets of cut polytopes can be described – via a "functional analysis connection" to the study of finite metric spaces [16] – and this was used successfully in the solution of max-cut problems occuring in practice.

So, this is a success story for polytopes, giving the basis for powerful optimization methods. But, let's not get too excited about this: Great challenges do remain! To mention just one, while travelling salesman problems of large size and max-cut problems of medium size can be solved successfully these days, stable set and coloring problems are much harder: they seem computationally quite intractable. There are 0/1-polytopes associated with these coloring problems, such as the stable set polytopes. And those wait for further investigation from a polytope theory point of view.

5. Integer Points

"How can one count the number of integral points in a polytope?" The counting of lattice points in polytopes is a difficult matter [6] – much more complicated than one might think, as one can see from some of the answers that are more complicated than one might think. Take, for example, the example of an orthogonal tetrahedron $\Delta[a, b, c]$, with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$. This is a tetrahedron of volume $\frac{1}{6}abc$, and thus one would expect that for large a, b, c the number of lattice points is roughly $\frac{1}{6}abc$. Indeed it is, but the precise number is not easy to get into one's hands. Furthermore, we know from complexity theory and combinatorial optimization that the number of integer points in a polytope is extremely hard to compute if the dimension is not fixed.

Geometric arguments about integer points in polytopes have been studied as a (successful) tool for number-theoretic investigations since the time of Gauß. It was Minkowski who (around 1900) created a whole subject called the "geometry of numbers" that treats the geometry of convex bodies – and thus in particular polytopes – in relation to the lattice of integral points in $\mathbb{R}^d$. A classical theorem by Minkowski says that a centrally-symmetric convex body which does not contain any lattice points other than its center cannot have a volume that is larger than 2^d. If we look at convex bodies that do not contain *any* interior lattice points, then we know that they have to be "flat" in some lattice direction. This is a classical theorem of Khintchine (1948). But how flat exactly such a body must be is not clear: The latest results (Banaszczyk et al. 1998) give an upper bound of the order of $d^{3/2}$ for the number of adjacent lattice hyperplanes that contain the body, but we seem to have examples only of lattice width d, as given by

$$\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i,\ x_1 + \ldots + x_d \leq d\}.$$

Many investigations in this context concern quite general convex bodies. But the geometry of numbers has also produced specific results about lattice polytopes:

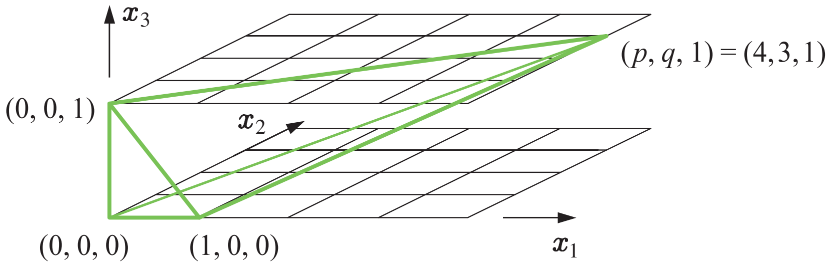


FIGURE 7
White's tetrahedron $\Delta(4, 3)$.

Polytopes all of whose vertices have integral coordinates. With this extra restriction, even tetrahedra become interesting. For example, a result found by White (1964), rediscovered and reproved many times afterwards, states that a lattice tetrahedron which contains no integral points other than its vertices, must be "of width one": It can be constructed by taking vertices from two adjacent lattice planes and thus it is equivalent to one of the tetrahedra $\Delta(p, q)$ obtained by taking the points $(0, 0, 0)$ and $(1, 0, 0)$ at "height zero" and the points $(0, 0, 1)$ and $(p, q, 1)$ at "height one," for $0 < q < p$ and $\gcd(q, p) = 1$.

In higher dimension, we have no complete classification of the empty lattice simplices, and we do not know their maximal width [24].

The theory of toric varieties, created by Demazure (1970), by Mumford et al. (1973) [27], and others, has revived and intensified the study of lattice polytopes. The reason is that toric geometry associates interesting complex algebraic varieties with lattice polytopes, and all the simple or subtle questions about those varieties can be translated into problems on lattice polytopes via a well-established dictionary [17]! Using this connection, Pommersheim (1993) discovered a quite remarkable formula for the lattice points in the the tetrahedron $\Delta[a, b, c]$ in terms of a, b, c and certain number theoretical functions, namely, Dedekind sums $s(ab, c)$, $s(ac, b)$ and $s(bc, a)$. Pommersheim's formula was derived from the Todd characteristic class of the associated toric variety:

$$\begin{aligned}\#(\Delta[a, b, c] \cap \mathbb{Z}^3) = {} & \frac{abc}{6} + 2 + \frac{ab + ac + bc + a + b + c}{4} \\ & + \frac{(ab)^2 + (ac)^2 + (bc)^2 + 1}{12abc} \\ & - s(ab, c) - s(ac, b) - s(bc, a),\end{aligned}$$

where $s(p, q) = \sum_{i=1}^{q} ((\frac{i}{q}))((\frac{pi}{q}))$ and $((x)) = x - \lfloor x \rfloor - x - \frac{1}{2}$ for $x \notin \mathbb{Z}$, $((x)) = 0$ for $x \in \mathbb{Z}$.

The dictionary between lattice polytopes and toric varieties has also led to new questions about lattice polytopes, for example to the study of unimodular triangulations: "Which lattice polytopes have triangulations into simplices of minimal volume (determinant 1) that use only integral vertices?" Every 2-dimensional polytope has such a triangulation, but many 3-dimensional ones do not: For example, the empty lattice simplices $\Delta(p, q)$ don't have such triangulations, for $p > 1$. Knudsen, Mumford and Waterman [27] have an intriguing theorem which states that every lattice polytope has such a unimodular triangulation after you enlarge it by some suitable large integral factor. This theorem is a gem that has never been polished and displayed in full light, and which would deserve that! Moreover, we should answer a few questions: For example,

– Would *every* sufficiently large integral factor be good enough?
– If the dimension is 3, does a factor of 4 always suffice? (We know that $k\Delta(p,q)$ has a unimodular triangulation for every $k \geq 4$.)
– In any dimension d, is there some factor that is good enough for all lattice d-polytopes?

On the other hand, Bruns, Gubeladze & Trung (1997) have shown (quite easily) that for every lattice polytope P, the expanded version kP will have a covering by unimodular simplices, if k is large enough.

Physicists have also been asking questions in this context: For building "string theories" they need so-called Calabi-Yau manifolds; examples for such manifolds can be constructed as generic hypersurfaces in toric varieties; and the dictionary tells us that these toric Calabi-Yau manifolds correspond to "reflexive lattice polytopes": These are polytopes with integral vertices, with the origin in their interior, such that all facets determine lattice hyperplanes that are adjacent to a lattice hyperplane through the origin. Or, equivalently: We look at integral polytopes such that the polar dual polytope is integral as well. It follows from general results in the geometry of numbers that in every dimension there are only finitely many examples of such reflexive polytopes. In dimension 2 there are sixteen non-equivalent types. Kreuzer and Skarke [29] have classified the reflexive polytopes in dimensions 3 and 4: There are 4319 respectively 473800776 of them. Among these "tons of" examples there are a few really fascinating ones. For example, in their classification they have found remarkable symmetric coordinates for the 24-cell, with integral coordinates both for the polytope and its polar dual!

6. Combinatorial Types

"Can't one just enumerate all the combinatorial types of d-dimensional polytopes with n vertices?" One can, in principle, since it is easy to see that for every n and d there are only finitely many combinatorial types. But how many are they?

Let's first look for explicit answers, for small n and d. The case $d = 2$ is not really interesting: For every $n > 2$ there is exactly one combinatorial type, the convex n-gon. For $d = 3$ the situation already becomes more complicated: For $n = 4$ we have the tetrahedron, for $n = 5$ one gets the square pyramid and the bi-pyramid over a triangle, for $n = 6$ there are seven combinatorial types, and for $n = 7, 8, 9, 10$ one has found that there are 34, 257, 2606, resp. 32300 combinatorial types. More enlightening than these exact values (which have been computed up to $n = 17$), the asymptotics for this have been derived in a very fruitful interaction between polytope theory and graph theory. The translation into graph theoretical problems is done via a classical theorem of Steinitz: The combinatorial type of a 3-polytope is given by an arbitrary simple, planar, 3-connected graph. Tutte, Bender and others have produced beautiful formulas for the numbers of "rooted maps," and from these obtained sharp asymptotics for the numbers of combinatorial types. For example,

$$N(f_0 = n+1,\, f_2 = m+1) \sim \frac{1}{972mn(m+n)}\binom{2m}{n+3}\binom{2n}{m+3}$$

is an excellent approximation for the number of distinct types of 3-polytopes with $n+1$ vertices and $m+1$ facets, due to Bender and Wormald (1988). So much for small dimensions: In “small co-dimensions” one can also get explicit answers. Indeed, for small $n-d$ linear algebra comes to the rescue, and provides the method of Gale diagrams (developed by Perles [23]); from this one gets that for $n = d+1$ there is just the d-simplex, for $n = d+2$ there are exactly $\lfloor \frac{d^2}{4} \rfloor$ combinatorial types of polytopes; even for $n = d+3$, with the help of Gale diagrams one can completely enumerate the combinatorial types of polytopes. But for $n = d+4$ things become really interesting: This is the “threshold for counter-examples.” In dimension $d = 4$ this is the case $n = 8$, where Altshuler and Steinberg (1985) have produced a complete classification of all the combinatorial types of 4-dimensional polytopes with 8 vertices: there are exactly 1294 combinatorial types.

This result is based on massive computer work, but also on interesting theoretical foundations. One of them is the theory of oriented matroids [9] [11], which provides one with a combinatorial description of the type of a polytope together with its “internal structure.” This is the combinatorial basis for an enumeration, and from this basis one has to decide whether a type that has been generated is “realizable” or not. This realizability problem for types of polytopes and for oriented matroids is difficult and still provides many challenges, in particular in the “non-uniform” case, when the polytope is neither simple nor simplicial. It is a consequence of Richter-Gebert’s universality theorem for 4-polytopes [31] that the problem of realizability of spheres as polytopes is in fact computationally difficult – it is equivalent to the existential theory of the reals (ETR), the problem to decide whether a system of real polynomial equations and inequalities has a solution. Nevertheless, the oriented matroid approach to the realizability problem for polytopes – pioneered by Bokowski – has been very successful [12].

“How many combinatorial types of d-dimensional polytopes with n vertices are there, if n and d are large?” We can’t expect exact answers, but we can try to get the asymptotics right. Again, the theory of oriented matroids, and its “magnifying glass” obtained by taking the internal structure of polytopes into account, provides one with rather sharp and slightly surprising answers. The main result is, to quote the title of a paper by Goodman and Pollack (1986), that “There are asymptotically far fewer polytopes than we thought.” The result is interesting, the method to get it is interesting as well: The main point is that the combinatorial type of a polytope is determined if we know the signs of all the determinants spanned by the $(d+1)$-tuples of vertices of the polytope. Thus the combinatorial type of a polytope corresponds to one or several strata in the stratification of $\mathbb{R}^{dn}$ given by the $\binom{n}{d+1}$ determinant functions. There are classical theorems by Oleinik, Petrovky, Warren, Milnor and Thom which give bounds on the numbers of these strata. From this Alon (1986) derived that the

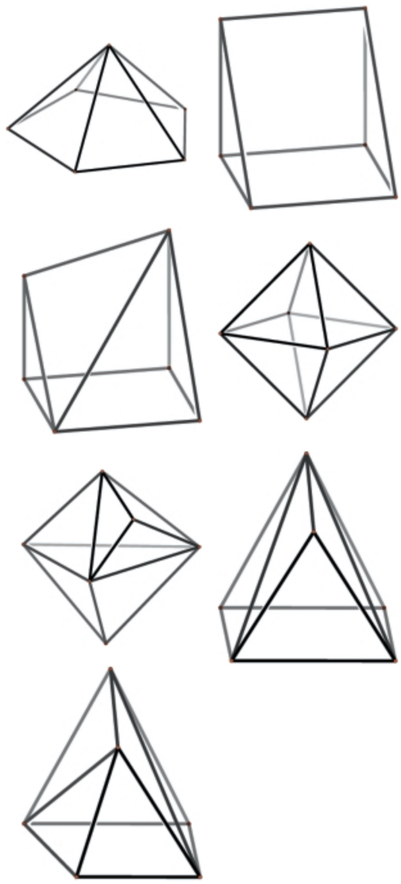


FIGURE 8
The seven 3-polytopes with 6 vertices.

total number of polytopes with n vertices (of any dimension!) is bounded by

$$2^{n^3+O(n^2)},$$

where $O(n^2)$ denotes a function that grows at most quadratically in n. Why is this result surprising? Because it implies that there are so many more "combinatorial types" of simplicial spheres that cannot be realized as polytopes. Indeed, Kalai (1988) has found a very simple construction of such simplicial spheres, as follows: One starts with a cyclic $(d + 1)$-polytope with enough vertices and combinatorially constructs lots of different d-dimensional balls in its boundary. These balls of dimension d are proved to be shellable, and their boundaries are thus piecewise-linear simplicial spheres. Thus one obtains that there are more than

$$2^{2^{n/3}}$$

different types of simplicial spheres with n vertices, *most of which* are thus not realizable. This gap may make one wonder also about generalization of results such as the g-Theorem: If we know that it is true for simplical polytopes, how much basis do we really have to think that the same statement might be true for the so much larger class of simplicial spheres?

7. Algorithms

"Given a polytope ... " may mean many different things. For example, it may mean that you are given the vertices of a polytope as in the case of $0/1$-polytopes: Then one usually wants to compute the facets, that is, a complete list of facet-defining linear inequalities. It may also mean that one is given a list of inequalities, not necessarily facet-defining, as in the case of linear programs. Then one might ask for "the highest vertex," as a linear programming problem would do, or less modestly for a complete list of the vertices of the polytope or polyhedron. It turns out that the two computational problems "given the vertices, compute all the facets" and "given the facets, compute all the vertices" are essentially equivalent, and known as the convex hull problem. Once we have the vertices and the facets of a polytope, we can compute "the rest": We can remove redundancies, compute the vertex-facet incidences, and from these derive all the other combinatorial properties, such as the graph, the diameter, f-vectors, pictures, etc. Thus the convex hull problem is a key computational problem, *to be solved*. It turns up whenever one is trying to "compute examples," and thus has been posed and studied from different directions and for various types of problems.

Theoretically, the problem "is solved" if one considers the dimension d as fixed: Then, given the vertices one can compute the facets (or vice versa) in a time that is proportional to the number $n^{\lfloor d/2 \rfloor}$, which is proportional to the maximal number of facets/vertices that one could get as the output (Chazelle 1993). However, if one does not consider the dimension d as fixed, then we don't have a polynomial time algorithm. We know that the answer of a convex hull problem may be extremely large, indeed exponentially large in the size of the input: So what we want is an algorithm that would produce the facets/vertices of the output one by one, in a computation time "per output element" that is polynomial in n and d. But none of the many interesting algorithms that have been proposed and analyzed for the convex hull problem will really guarantee this, as was demonstrated by Avis, Bremner & Seidel [4]. Even worse (in fact, equivalently) we do not have a "verification algorithm" that would in polynomial time answer the following question: "Given all the vertices and a list of facets, tell me whether the list of facets is complete!" If we had a method to solve this problem efficiently, then we could use this as a subroutine to construct an output-polynomial algorithm for the convex hull problem.

One reason, I think, why we can't really solve the convex hull problem, is that we do not really understand the complexity of polytopes, even in constant dimension. This is not a problem in dimensions 2 or 3: Here we do understand the complexity, and we can efficiently solve the convex hull problem, which is in particular important because of applications in computational geometry, computer graphics, robot motion control, etc. But as soon as the dimension is 4, we don't really know how "complex" things can be. So, if you have a 4-polytope with n vertices and m facets, how many vertex-facet incidences can it have? How many edges and 2-faces can there be then? In other words, are the ratios

$$\text{intricacy}(P) := \frac{f_{03}}{f_0 + f_3} \qquad \text{and} \qquad \text{fatness}(P) := \frac{f_1 + f_2}{f_0 + f_3}$$

really bounded for 4-polytopes P? This is the question about "intricate" and "fat-lattice" 4-polytopes [4].

In practice, we do however have a number of algorithms and a number of efficient implementations at hand "that work." Thus, for "reasonably sized" examples of polytopes, we can compute them and study them. So we have the `polymake` software framework of Gawrilow & Joswig [21, 22], which allows one to work with examples on the computer. `polymake` will help to generate examples, and – provided it can solve the convex hull problem – it can compute parameters, analyze properties, and (last, not least) produce pictures. Thus, I believe that a new imperative of polytope theory is: *Experiment*!

The ability to construct, enumerate, compute and visualize examples on a computer has definitely changed polytope theory. It has led to examples illustrating old questions, to solutions of some of these, and to new questions, some simple, some simple-sounding, some being just proven or destroyed while I was writing these lines. The field of discrete geometry is very much alive.

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