A Note On Magic Squares And Magic Constants*

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Abstract

In this note, we investigate integer solutions of a Diophantine equation related to magic squares and a class of quadratic polynomials. As a consequence, we obtain that there are finitely many magic constants in a sequence of s-gonal and centered s-gonal numbers for a fixed integer $s > 2$. Additionally, we also establish several results related to polygonal numbers and centered polygonal numbers using properties of magic squares.

1 Introduction

A magic square is a square array of numbers consisting of distinct positive integers $1, 2, \ldots, n^2$, arranged so that the sum of the numbers in any horizontal, vertical, or main diagonal line is always the same number, known as the magic constant $M_2(n)$ (see sequence A006003 of the OEIS [10]). According to a legend, the oldest known magic square is the Chinese magic square, *lo-shu* (see Figure 1), discovered around 2200 B.C. According to the definition of magic squares [4], we get

4	9	2
3	5	7
8	1	6

Figure 1: lo-shu magic square.

$$M_2(n) = \frac{1 + 2 + \cdots + n^2}{n} = \frac{n(n^2 + 1)}{2}.$$

In Figure 1, the *lo-shu* magic square is a magic square of order 3, with magic constant $M_2(3) = 15$. For $n = 0, 1, 2, 3, 4, 5 \ldots$, the values of the magic constant are $0, 1, 5, 15, 34, 65, 111, \ldots$.

From ancient times, magic squares and puzzles related to them were a source of entertainment not only in royal courts but also among ordinary people. Today, they are still popular among mathematicians, amateurs, and professionals. Several authors have contributed to the study of magic squares and their construction (see [1, 3, 7]). The most recent work includes the construction and enumeration of magic squares of order $4k$, where k is a positive integer by Oboudi [8] in 2022. In the same year, Jitjankarn [5] studied the construction of generalization of magic squares called *sq-corner (or square corner)* magic squares. Magic squares also related to well-known number sequences like the Fibonacci sequence, polygonal, and centered polygonal numbers. In 1964, J.L. Brown Jr. proved there are no magic squares with only Fibonacci entries (see [6]). Magic constants can also be studied in terms of triangular and centered triangular numbers (see Section 2). For example, n copies of the magic constant $M_2(n)$ give a triangular number with a square index. Any magic

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constant is the sum of integers between two consecutive triangular numbers. Any magic constant $M_2(n)$ is the sum of the first n centered triangular numbers. One can see [2] for more details on magic squares.

The study of the intersection of some well-known number sequences with the sequence of polygonal numbers is an interesting problem (see [4] and references therein). The above problem motivates us to investigate magic constants in the sequence of polygonal and centered polygonal numbers. Moreover, we try to answer a more general question:

Does there exist a magic constant in the values of quadratic polynomials? If yes, then they are finitely many or infinitely many?

In this note, we investigate integer solutions of a Diophantine equation related to magic squares and a class of quadratic polynomials. As a consequence, we obtain that there are finitely many magic constants in a sequence of s-gonal and centered s-gonal numbers for a fixed integer $s > 2$. Further, we also establish several results related to polygonal numbers and centered polygonal numbers using properties of magic squares.

2   Preliminaries

We start this section by defining the sequence of polygonal and centered polygonal numbers.

Definition 1 (Polygonal numbers [4]) *For each integer x and $s > 2$, x^{th} s-gonal number is given by*

$$P_s(x) = \frac{(s-2)x(x-1)}{2} + x.$$

For example, for $s = 3$, $P_3(x) = \dfrac{n(n+1)}{2}$ is the sequence of triangular numbers, and for $s = 4$, $P_4(x) = n^2$ are square numbers.

Definition 2 (Centered Polygonal numbers [4]) *For each integer x and $s > 2$, x^{th} centered s-gonal number is given by*

$$CP_s(x) = \frac{sx(x-1)}{2} + 1.$$

For example, for $s = 3$, $CP_3(x) = \dfrac{3n^2 - 3n + 2}{2}$ is the sequence of centered triangular numbers, and for $s = 4$, $P_4(x) = 2n^2 - 2n + 1$ are centered square numbers.

The following result states the relation between magic constants, triangular, and centered triangular numbers.

Proposition 1 ([4, page-293]) *Let $M_2(n)$ be the magic constant associated with $n \times n$ magic square, $P_s(x)$ be the xth s-gonal number, and $CP_s(x)$ be the xth centered s-gonal number. Then*

(i) *for every natural number n, $nM_2(n) = P_3(P_4(n))$. In other words, n copies of the magic constant $M_2(n)$ give a triangular number with a square index.*

(ii) *for every natural number n, $M_2(n) = (P_3(n-1)+1) + (P_3(n-1)+2) + \cdots + P_3(n)$. In other words, any magic constant is the sum of integers between two consecutive triangular numbers.*

(iii) *for every natural number n, $M_2(n) = CP_3(1) + CP_3(2) + \cdots + CP_3(n)$. In other words, any magic constant $M_2(n)$ is the sum of the first n centered triangular numbers.*

We use the following well-known results for the number of integral points from the theory of cubic curves and *elliptic curves* [9, 11]:

Proposition 2 (Siegel's Theorem [9, page-146]) *Let C be a non-singular cubic curve given by an equation $f(x,y) = 0$ with integer coefficients. Then C has only finitely many points with integer coordinates.*

Proposition 3 (Baker's Theorem [9, page-176]) *Let $a, b, c \in \mathbb{Z}$ and let $H = \max\{|a|, |b|, |c|\}$, then every point (x, y) on the elliptic curve $y^2 = x^3 + ax^2 + bx + c$ with integer coordinates $x, y \in \mathbb{Z}$ satisfies $\max\{|x|, |y|\} \leq \exp\left(10^6 H\right)^{10^6}$.*

3   Main Results

In this section, we give an affirmative answer to the problem proposed in Section 1 for a class of quadratic polynomials. The following result is in this direction.

Theorem 1 *There are at most finitely many positive integers x and y satisfying the Diophantine equation*

$$Ay^2 + By + C = \frac{x(x^2+1)}{2}, \tag{1}$$

where $A \neq 0$, B, and $C \in \mathbb{R}$ such that $2A$, $2B$, and C are integers. Further, let

$$H = \max\left\{|(2A)^2|, |((2AB)^2 - 2(2A)^3C)|\right\},$$

then integer solutions of (1) satisfies $\max\{|x|, |y|\} \leq \exp(10^6 H)^{10^6}$.

Proof. We multiply both sides of (1) by $16A^3$ and after simplification, we obtain

$$(4A^2y + 2AB)^2 = (2Ax)^3 + (2A)^2(2Ax) + (2AB)^2 - 2(2A)^3C.$$

On substituting $Y = 4A^2y + 2AB$ and $X = 2Ax$ in above equation, we obtain

$$Y^2 = X^3 + (2A)^2X + (2AB)^2 - 2(2A)^3C. \tag{2}$$

Equation (2) represents an elliptic curve, which is a cubic curve. The positive integer solutions of equation (1) correspond to some integer points on the elliptic curve given by (2). Since $A \neq 0$, therefore the discriminant of a cubic polynomial on the right-hand side of (2) is non-zero, i.e., $\Delta = -4^2 A^2((4A)^2 + 27(B^2 - 4AC)^2) \neq 0$. Hence, the elliptic curve given by (2) is a non-singular cubic curve. Therefore, using Proposition 2, there are finitely many points (X, Y) with integer coordinates. As a consequence, (1) has a finite number of positive integer solutions whenever they exist. By Proposition 3, we deduce that integer points (X, Y) on an elliptic curve (2) satisfy

$$\max\{|X|, |Y|\} \leq \exp\left(10^6 H\right)^{10^6}.$$

Clearly, $|x| \leq |X|$ and $|y| \leq |Y|$, therefore the positive integer solutions to (2) satisfy

$$\max\{|x|, |y|\} \leq \exp\left(10^6 H\right)^{10^6}.$$

∎

Remark 1 *Theorem 1 does not guarantee the existence of positive integer solutions of (1), but it ensures that there are only finitely many positive integer solutions whenever they exist.*

As an application to Theorem 1, we deduce the following results related to magic squares, polygonal numbers, and centered polygonal numbers.

Corollary 1 *Let $s > 2$ be a fixed positive integer, and $M_2(n)$ be the magic constant associated with $n \times n$ magic square. Then there are finitely many magic constants in the sequence of s-gonal numbers.*

Proof. Clearly, by Definition 1, the yth s-gonal number is given by

$$P_s(y) = \frac{(s-2)y(y-1)}{2} + y = \frac{(s-2)y^2}{2} - \frac{(s-4)y}{2}.$$

In order to find magic constants in sequence of s-gonal numbers, we find positive integers x and y such that $P_s(y) = M_2(x)$ which can be written as

$$\frac{(s-2)y^2}{2} - \frac{(s-4)y}{2} = \frac{x(x^2+1)}{2}. \tag{3}$$

The above equation is of the form of the Diophantine equation given by (1), where

$$A = \frac{(s-2)}{2} \neq 0, \quad B = -\frac{(s-4)}{2} \quad \text{and} \quad C = 0.$$

Note that $(x, y) = (1, 1)$ satisfies (3). Hence, by Theorem 1, there are only finitely many positive integers x and y satisfying (3). Therefore, there are only finitely many magic constants in the sequence of s-gonal numbers. ∎

Corollary 2 *Let $s > 2$ be a fixed positive integer, and $M_2(n)$ be the magic constant associated with $n \times n$ magic square. Then there are finitely many magic constants in the sequence of centered s-gonal numbers.*

Proof. The proof is similar to Corollary 1 and follows directly by Definition 2 and Theorem 1. ∎

Using Corollary 1, Corollary 2, and the properties of magic constant discussed in Proposition 1, we deduce some more properties of polygonal and centered polygonal numbers. In Theorem 2, we establish results that relate s-gonal numbers with triangular numbers and centered triangular numbers.

Theorem 2 *Let $P_s(y)$ be the yth s-gonal number. Then*

(1) *there are finitely many positive integers x and y such that $xP_s(y) = P_3(P_4(x))$.*

(2) *there are finitely many positive integers x and y such that*

$$P_s(y) = (P_3(x-1)+1) + (P_3(x-1)+2) + \cdots + P_3(x).$$

(3) *there are finitely many positive integers x and y such that*

$$P_s(y) = CP_3(1) + CP_3(2) + \cdots + CP_3(x).$$

Proof.

(1) By Proposition 1(i), for every positive integer x, we have $xM_2(x) = P_3(P_4(x))$. However, using Corollary 1, we get finitely many positive integer solutions (x, y) of $P_s(y) = M_2(x)$. With the above two results, the proof follows directly.

(2) By Proposition 1(ii), for every positive integer x, we have

$$M_2(x) = (P_3(x-1)+1) + (P_3(x-1)+2) + \cdots + P_3(x).$$

By Corollary 1, there are finitely many positive integer solutions (x, y) of $P_s(y) = M_2(x)$. Hence, the result follows.

(3) By Proposition 1(iii), for every positive integer x,

$$M_2(x) = CP_3(1) + CP_3(2) + \cdots + CP_3(x).$$

However, using Corollary 1, we get finitely many positive integer solutions (x, y) of $P_s(y) = M_2(x)$. Hence, from the above two arguments, the result follows.

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In Theorem 3, we establish results that relate centered s-gonal numbers with triangular numbers and centered triangular numbers.

Theorem 3 *Let $CP_s(y)$ be the yth centered s-gonal number. Then*

(1) *there are only finitely many positive integers x and y satisfying $xCP_s(y) = P_3(P_4(x))$.*

(2) *there are only finitely many positive integers x and y satisfying*

$$CP_s(y) = (P_3(x-1)+1) + (P_3(x-1)+2) + \cdots + P_3(x).$$

(3) *there are only finitely many positive integers x and y satisfying*

$$CP_s(y) = CP_3(1) + CP_3(2) + \cdots + CP_3(n).$$

Proof. The proof follows a similar approach to Theorem 2, using Corollary 2 and Proposition 1. Therefore, we omit the proof. ∎

Remark 2 *We conclude this note with a few comments on Theorem 1. Using Siegel's result, we transformed the Diophantine equation problem into the problem of finding integer points on an elliptic curve. Furthermore, we provide an upper bound for the values of integer points using Baker's result. However, in general, finding all integer points on elliptic curves is itself a computationally challenging problem. Table 1 lists the magic constants in the sequence of some polygonal numbers.*

s	s-gonal number	magic constant
3	triangular number	$1, 15$
4	square number	1
5	pentagonal number	1
6	hexagonal number	$1, 15$
7	heptagonal number	$1, 19669$

Table 1: Magic constants in polygonal numbers.

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