AN INFINITE FAMILY OF OVERPARTITION CONGRUENCES MODULO 12

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Abstract

A number of arithmetic properties of overpartitions have been proven recently. However, all such results have involved moduli which are powers of 2. In this brief note, we prove the first infinite family of congruences with a modulus that is not a power of 2 by proving that, for all $n \geq 0$ and all $\alpha \geq 0$, $\overline{p}(9^\alpha(27n + 18)) \equiv 0 \pmod{12}$.

1. Introduction

In this brief note, we let $\overline{p}(n)$ be the number of overpartitions of the integer n. An **overpartition** of the nonnegative integer n is a partition of n where the first occurrence of parts of each size may be overlined. For example, the overpartitions of the integer 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

From this example, we see that $\overline{p}(3) = 8$.

The function $\overline{p}(n)$ has been considered recently by a number of mathematicians; please see [1, 2, 3, 4, 5, 6, 7, 8]. In [4] and [8], several Ramanujan-like congruences modulo small powers of two were proven for $\overline{p}(n)$. But as of this writing, no one has proven a family of congruences satisfied by $\overline{p}(n)$ for a modulus that is not a power of 2. Our goal in this note is to prove such a family.

Theorem 1.1. *For all $n \geq 0$ and all $\alpha \geq 0$,*

$$\overline{p}(9^\alpha(27n + 18)) \equiv 0 \pmod{3}.$$

This theorem settles a conjecture stated by the authors in [4] (which was the $\alpha = 0$ case of Theorem 1.1).

2. The Machinery

Throughout this note, all power series in q will be viewed as formal power series, so questions of convergence will not be considered. Indeed, such questions are not of importance as we do not evaluate any of the series at a particular value of q. With this said, we focus our attention on proving Theorem 1.1, which is a corollary of the following:

Theorem 2.1. *For all $n \geq 0$,*

$$\overline{p}(27n+18) \equiv 0 \pmod 3 \quad \textit{and}$$
$$\overline{p}(27n) \equiv \overline{p}(3n) \pmod 3.$$

We utilize generating function manipulations to prove Theorem 2.1. We begin with the generating function for $\overline{p}(n)$:

$$\sum_{n\geq 0} \overline{p}(n) q^n = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = \frac{1}{D(q)},$$

where

$$D(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

In the proof of Theorem 2.1, we will make use of the following straightforward lemmata:

Lemma 2.2.

$$D(q)^3 \equiv D(q^3) \pmod 3.$$

Proof. This follows from the fact that

$$(1-q^n)^3 \equiv 1 - q^{3n} \pmod 3$$

together with the fact that

$$D(q) = \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^{2n})}.$$

□

Lemma 2.3.

$$D(q) \equiv D(q^9) + qY(q^3) \pmod 3$$

where

$$Y(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}.$$

Proof. We have

$$\begin{aligned} D(q) &= \sum_{n\equiv 0 \pmod 3} (-1)^n q^{n^2} + \sum_{n\equiv \pm 1 \pmod 3} (-1)^n q^{n^2} \\ &= D(q^9) - 2qY(q^3) \\ &\equiv D(q^9) + qY(q^3) \pmod 3. \end{aligned}$$

□

Proof. (of Theorem 2.1) Our first goal is to rewrite the generating function for $\overline{p}(n)$ in such a way that we can dissect it to obtain the generating function for $\overline{p}(3n)$ in a straightforward manner. We have, after multiple applications of the above lemmata,

$$\begin{aligned} \sum_{n\geq 0} \overline{p}(n) q^n &= \frac{1}{D(q)} \\ &= \frac{D(q)^2}{D(q)^3} \\ &\equiv \frac{D(q)^2}{D(q^3)} \pmod 3 \quad \text{using Lemma 2.2} \\ &\equiv \frac{(D(q^9)+qY(q^3))^2}{D(q^3)} \pmod 3 \quad \text{using Lemma 2.3} \\ &= \frac{D(q^9)^2 + 2qD(q^9)Y(q^3) + q^2Y(q^3)^2}{D(q^3)}. \end{aligned}$$

From this we see that

$$\begin{aligned} \sum_{n\geq 0} \overline{p}(3n) q^n &\equiv \frac{D(q^3)^2}{D(q)} \pmod 3 \\ &= \frac{D(q^3)^2 D(q)^2}{D(q)^3} \\ &\equiv \frac{D(q^3)^2 D(q)^2}{D(q^3)} \quad \text{using Lemma 2.2} \\ &= D(q^3)D(q)^2 \\ &\equiv D(q^3)(D(q^9)+qY(q^3))^2 \pmod 3 \quad \text{using Lemma 2.3} \\ &= D(q^3)(D(q^9)^2 + 2qD(q^9)Y(q^3) + q^2Y(q^3)^2). \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{n\geq 0} \overline{p}(9n) q^n &\equiv D(q)D(q^3)^2 \pmod 3 \\ &\equiv D(q^3)^2(D(q^9)+qY(q^3)) \pmod 3 \quad \text{using Lemma 2.3.} \end{aligned}$$

We can now prove both congruence properties stated in Theorem 2.1 from the last fact above. First, note that we now have

$$\sum_{n\geq 0} \overline{p}(27n+18) q^n \equiv 0 \pmod 3.$$

This is clear because there are no powers of q in the last line above that are congruent to 2 modulo 3. Secondly,

$$\begin{aligned}\sum_{n\geq 0}\overline{p}(27n)q^n &\equiv D(q)^2D(q^3) \pmod 3\\ &\equiv \sum_{n\geq 0}\overline{p}(3n)q^n \pmod 3\end{aligned}$$

from the work above. This completes the proof of Theorem 2.1. □

3. Closing Thoughts

We close this note with two brief comments.

First, we note that since the sequence $9^\alpha(27n+18)$ does not contain squares and since $\overline{p}(n) \equiv 0 \pmod 4$ unless n is a square, we can actually strengthen Theorem 1.1 as follows:

Theorem 3.1. *For all $n \geq 0$ and all $\alpha \geq 0$,*

$$\overline{p}(9^\alpha(27n+18)) \equiv 0 \pmod{12}.$$

Second, it is apparent (computationally) that other congruences with moduli which are not powers of 2 are satisfied by $\overline{p}(n)$. We hope in the future to prove some of these as well.

References

[1] S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), 1623-1635.

[2] S. Corteel, J. Lovejoy, and A. Yee, Overpartitions and generating functions for generalized Frobenius partitions, *Mathematics and Computer Science III: Algorithms, Trees, Combinatorics, and Probabilities* (2004) 15-24.

[3] J.-F. Fortin, P. Jacob, and P. Mathieu, Jagged partitions, to appear in *The Ramanujan Journal*.

[4] M. D. Hirschhorn and J. A. Sellers, Arithmetic Relations for Overpartitions, *J. Comb. Math. Comb. Comp.* **53** (2005), 65–73.

[5] J. Lovejoy, Gordon's theorem for overpartitions, *J. Comb. Theory Ser. A* **103**, no. 2 (2003), 393–401.

[6] J. Lovejoy, Overpartition theorems of the Rogers-Ramaujan type, *J. London Math. Soc.* **69** (2004), 562-574.

[7] J. Lovejoy, Overpartitions and real quadratic fields, *J. Number Theory* **106** (2004), 178-186.

[8] K. Mahlburg, The Overpartition Function Modulo Small Powers of 2, *Discrete Mathematics* **286**, no. 3 (2004), 263-267.