Counting Lattice Paths and Walks with Several Step Vectors

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***Abstract*—Many famous researchers in computer science, mathematics and other areas have studied enumerative problems in lattice path and walks which could be applied to many fields. We will discuss some new enumerative problems including some pattern avoidance problems in lattice paths and walks with several step vectors. Results on stretches and turns are presented and several open problems are posted. A few approaches are used in this paper such as computational, generating function, closed formula and constructional method. You will observe many interesting integer sequences as well.**

Keywords: Lattice path, prudent self-avoiding walk, generating function, pattern avoidance, stretch

I. Introduction, Notations and Preliminaries

In order to present our problems and results clearly and efficiently, we introduce some notations in the following.

East step: E or $\rightarrow$ or $(1,0)$, x-step

You can see more in the table below:

$(0,1)$	$(1,0)$	$(1,1)$	$(0,-1)$
$\uparrow$	$\rightarrow$	$\nearrow$	$\downarrow$
N	E	NE	S

$(-1,0)$	$(-1,-1)$	$(-1,1)$	$(1,-1)$
$\leftarrow$	$\swarrow$	$\nwarrow$	$\searrow$
W	SW	NW	SE

$\uparrow^{\geq k}$: k or more than k consecutive $\uparrow$ steps

$\uparrow^{=k}$: k consecutive $\uparrow$ steps

avoiding $\uparrow^{\geq k}$: no k or more than k consecutive $\uparrow$ steps

avoiding $\uparrow^{=k}$: no k consecutive $\uparrow$ steps, but can have more than or less than k consecutive $\uparrow$ steps

$\lfloor x \rfloor$: the largest integer not greater than x, $floor(x)$

$\lceil x \rceil$: is the smallest integer not less than x, $ceiling(x)$

$[x^n]f(x)$ denotes the coefficient of x^n in the power series expansion of a function $f(x)$.

$[x^m y^n]f(x,y)$ denotes the coefficient of $x^m y^n$ in the power series expansion of a function $f(x,y)$.

$\binom{n}{r}$, the number of combinations of n things r at a time.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$$

A lattice path is a path from the lattice point (x_1, y_1) to the lattice point (x_2, y_2), $x_1 \leq x_2$, $y_1 \leq y_2$, we mean a directed path from (x_1, y_1) to (x_2, y_2) which passes through lattice points with movements parallel to the positive direction of either axis. Here, we refer to two types steps, viz., $x-$steps and $y-$steps, where an x (y)$-$step is a directed line segment parallel to the x (y) axis going right (up) joining two neighboring points. For counting purposes we may, without loss of generality, consider lattice paths from the origin to (m,n) and observe that each such path is characterized by having exactly m horizontal steps and n vertical steps. If we denote by $f(m,n)$ the number of paths from $(0,0)$ to (m,n), elementary reasoning gives the results

$$f(m,n) = \binom{m+n}{n}.$$

Lattice paths are encountered in a natural way in various problems, e.g., ballot problems, compositions, random walks, fluctuations, queues, and the tennis ball problem.

The number of lattice paths from the origin to (m,n), $m > n+t$, not touching the line $x = y + t$, where t is a nonzero integer satisfies [18]

$$\binom{m+n}{n} - \binom{m+n}{m-t}.$$

When $t = 0$, the paths have to touch the line $x = y$ at the origin, and therefore the number paths from the origin to (m,n) that do not touch the line $x = y$ except at the origin is given by

$$\frac{m-n}{m+n}\binom{m+n}{n}.$$

The number of lattice paths from (r,s) to (m,n) that never rise above the line $y = x$ is [1]

$$\binom{n+m-r-s}{m-r} - \binom{n+m-r-s}{m-s+1}.$$

Then the number of lattice paths from $(0,0)$ to (m,n) that never rise above the line $y = x$ is

$$\binom{n+m}{m} - \binom{n+m}{m+1}.$$

The of n-step lattice paths starting from $(0,0)$ that never rise above the line $y = x$ is

$$\sum_{i=\lceil n/2 \rceil}^{n} \frac{n!(2i+1-n)}{(i+1)!(n-i)!} = \binom{n}{\lfloor n/2 \rfloor}.$$

The number of paths from $(0,0)$ to (n,n) that never rise above the line $y = x$ is the nth Catalan number, denoted by

C_n, and define $C_0 = 1$.

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \sum_{i=0}^{n}\binom{n}{i}^2$$

with generating function

$$\frac{1-\sqrt{1-4x}}{2x}.$$

Also,

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-1} = \frac{2(2n+1)}{n+2} C_n.$$

The tennis ball problem was presented on pages $304 - 305$ of the book " Sweet Reason: A Field Guide to Modern Logic" by Tom Tymoczko and Jim Henle in 1995. Their presentation deals with adding numbered books to a stack on a table, then removing some, infinitely many times. Motivated by that presentation, Ralph P. Grimaldi and Joseph G. Moser deal with performing the process a finite number of times. Since then more mathematicians have studied the problem, such as C. L. Mallows and L. Shapiro, R.J. Chapman, T.Y. Chow, A. Khetan, D.P. Moulton, R.J. Waters, J. E. Bonin, Anna de Mier, M. Noy, H. Niederhausen, J. Fallon, and S. Gao. [2], [4], [5], [6], [7], [8], [9], [11], [13], [14], [16], [17] However, it is still wildly open and might challenge more people in the future.

The number of ways of putting n like objects into r different cells is $\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$. [19] It is also the number of nonnegative integer solutions to the equation $\sum_{i=1}^{r} x_i = n$. The number of ways of putting n like objects into r different cells with no cell is empty is $\binom{n-1}{r-1}$. It is also the the number of positive integer solutions to the equation $\sum_{i=1}^{r} x_i = n$.

If p is a prime, then $\binom{p}{i}$ is divisible by p for $1 \le i \le p-1$. [21]

Fibonacci number: F_n is defined as $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n \ge 2$.

II. Pattern Avoidance in Lattice Paths and Walks

The number of n-step walks with steps $(0,1)$, $(1,0)$ and $(-1,-1)$ is

$$\frac{(3n)!}{(n!)^3}.$$

Theorem 1. *The number of $3n$-step walks from $(0,0)$ to $(0,0)$, taking steps from {E,N, SW}, and staying above the line $y = x$ (i.e., any point (x, y) along the path satisfies $y \ge x$) is given by*

$$\frac{(3n)!}{(n!)^2(n+1)!}.$$

Example: $n = 1$, three walks: $NE(SW)$, $(SW)NE$, $N(SW)E$.

This is the sequence [22, A007004]:

$$1, 3, 30, 420, 6930, 126126, ...$$

Proof: It is clear that such a $3n$-step walk contains n copies of north, east and southwest step, respectively.
It is also true that the total number of north and east steps is greater or equal to the number of southwest steps at any lattice point on a walk. Now we arrange n north and east steps (total is n) with n southwest steps to get a $2n$-step walk according to: the total number of the chosen steps is greater or equal to the number of southwest steps at any lattice point on a walk, which gives C_n (We do not consider the difference of north steps and east steps at this moment). Next, we have $2n + 1$ positions to insert the remaining n steps of the north steps and east steps into the $2n$-step walk, giving $\binom{3n}{n}$ ways. Now combine them:

$$C_n \binom{3n}{n} = \frac{(3n)!}{(n!)^2(n+1)!}.$$

■

This theorem also could be proved by using *André's Reflection Method*:

$$\binom{3n}{n, n, n} - \binom{3n}{n+1, n-1, n} = \frac{(3n)!}{(n!)^2(n+1)!}.$$

Theorem 2. *The number of $3n$-step walks from $(0,0)$ to $(0,0)$, taking steps from {W,S,NE}, and staying within the first quadrant (i.e., any point (x, y) along the walk satisfies x, $y \ge 0$) is given by [12]*

$$\frac{4^n(3n)!}{(n+1)!(2n+1)!}.$$

Example: $n = 1$, two walks: $(NE)SW$, $(NE)WS$.
This is the sequence [22, A006335]:

$$1, 2, 16, 192, 2816, 46592, 835584, ...$$

Theorem 3. *The number of lattice paths avoiding $\uparrow^{\ge 2}$, from $(0,0)$ to (m, n) is*

$$\binom{m+1}{n}.$$

Proof: The m East steps provide $m+1$ positions (we can say $m+1$ different cells) for n North steps to be inserted with each cell containing at most one element (North step). Then there are $\binom{m+1}{n}$ ways to choose n cells. ■

Corollary 4. *The number of lattice paths from $(0,0)$ to $(ns+1, nt-1)$, avoiding $\uparrow^{\ge 2}$is*

$$\binom{ns+2}{nt-1}.$$

Corollary 5. *The number of n-step paths with east and north steps and with two consecutive north steps forbidden is equal to [3]*

$$\sum_{i=0}^{\lceil n/2 \rceil} \binom{n+1-i}{i} = F_{n+2}.$$

Theorem 6. *The number of walks from* $(0,0)$ *to* (m,n) *(* $m \geq n$*) take steps from {E,N,NE} is*

$$\sum_{k=0}^{n}\binom{m+n-2k}{n-k}\binom{m+n-k}{k}.$$

Proof: Without loss of generality, we assume that there are k Northwest steps, $m-k$ East steps and $n-k$ North steps in a walk from $(0,0)$ to (m,n). It is clear that $0 \leq k \leq n$.

Firstly, we only consider the number of arrangements of $m-k$ East steps and $n-k$ North steps, which give us $\binom{m+n-2k}{n-k}$ ways.

Secondly, $m-k$ East steps and $n-k$ North steps provide $m+n-2k+1$ positions (we can say $m+n-2k+1$ different cells) for k Northwest steps to be inserted, which give $\binom{m+n-k}{k}$.

Therefore, we get the number:

$$\sum_{k=0}^{n}\binom{m+n-2k}{n-k}\binom{m+n-k}{k}.$$

■

We obtain sequence $[22, A001850]$ for $m = n$:

$$1, 3, 13, 63, 321, 1683, 8989, 48639, ...$$

Sequence $[22, A002002]$ for $m = n+1$:

$$1, 5, 25, 129, 681, 3653, 19825, ...$$

The number of walks from $(0,0)$ to $(n, n-1)$ ($m \geq n$) take steps from {E,N,NE}.

Sequence $[22, A026002]$ for $m = n+2$:

$$1, 7, 41, 231, 1289, 7183, 40081, ...$$

Sequence $[22, A190666]$ for $m = n+3$:

$$9, 61, 377, 2241, 13073, 75517, 433905, ...$$

Example 7. *There are* 13 *walks in the above theorem for* $m = n = 2$*:* 6 *walks with* 2 *East steps and* 2 *North steps,* 1 *walk with two Northeast steps,* 6 *walks with* 1 *Northeast step,* 1 *East step and* 1 *North step: (NE)NE, (NE)EN, NE(NE), EN(NE),E(NE)N, N(NE)E.*

Corollary 8. *The number of walks from* $(0,0)$ *to* (n,n) *take steps from {E,N,NE} is*

$$\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!(k!)^2}.$$

Theorem 9. *The number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 3}$*is*

$$\sum_{i=0}^{\lfloor n/2\rfloor}\binom{m+1}{n-i}\binom{n-i}{i}.$$

Proof: Without loss of generality, we assume that there are i copies of double North Steps, $n-2i$ copies of single North step in a lattice path from $(0,0)$ to (m,n) and avoiding $\uparrow^{\geq 3}$. It is clear that $0 \leq i \leq \lfloor n/2 \rfloor$. The m East steps provide $m+1$ positions (we can say $m+1$ different cells) for n North steps to be inserted with each cell containing at most one element ($\uparrow$ or $\uparrow^2$). There are $\binom{m+1}{n-i}$ ways to choose n cells for the i copies $\uparrow^2$ and $n-2i$ copies of $\uparrow$. We have $\binom{n-i}{i}$ ways to distribute the i copies $\uparrow^2$. Therefore, we can get the number. ■

Corollary 10. *The number of lattice paths from* $(0,0)$ *to* $(ns+1, nt-1)$ *avoiding* $\uparrow^{\geq 3}$*is*

$$\sum_{i=0}^{\lfloor (nt-1)/2\rfloor}\binom{ns+2}{nt-1-i}\binom{nt-1-i}{i}.$$

Theorem 11. *The number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 2}$ *and* $\rightarrow^{\geq 3}$*is*

$$\binom{n+1}{m-n-1}+2\binom{n}{m-n}+\binom{n-1}{m-n+1}.$$

Theorem 12. *The number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*is*

$$2\sum_{i=m-n}^{\lfloor m/2\rfloor}\binom{m-i}{i}\binom{m-i}{n-m+i} + \sum_{i=m-n-1}^{\lfloor m/2\rfloor}\binom{m-i}{i}\binom{m-i-1}{n-m+i+1} + \sum_{i=m-n+1}^{\lfloor m/2\rfloor}\binom{m-i}{i}\binom{m-i+1}{n-m+i-1}.$$

The generating function of the above numbers is

$$[x^m y^n]\big(\left(1+x+x^2+x^3\right)\left(1+y+y^2+y^3\right) (1-xy-xy^2-xy^3-x^2y-x^2y^2 - x^2y^3-x^3y-x^3y^2-x^3y^3)^{-1}\big).$$

Corollary 13. *The number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*is*

$$2\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}\binom{n-i}{i}+2\sum_{j=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-j}{j}\binom{n-j+1}{j-1}.$$

Theorem 14. *The generating function for the number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*, is*

$$\frac{(1-t)^2\sqrt{(1+t+t^2)\left(1-3t+t^2\right)}-(1-3t+t^2)(1+t^2)}{t^2(1-3t+t^2)} = 2t+6t^2+14t^3+34t^4+84t^5+208t^6+518t^7+... .$$

A proof of this theorem involved finite operator calculus is in [10].

Example 15. *For* $m \leq 7$ *and* $n \leq 8$*, the number of lattice paths from* $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*is* *as follows:*

n=8				1	15	87
n=7				4	30	114
n=6			1	10	43	113
n=5			3	16	45	**84**
n=4		1	6	18	**34**	45
n=3		2	7	**14**	18	16
n=2	1	3	**6**	7	6	2
n=1	1	**2**	3	2	1	
n=0		1	1			
	m=0	m=1	m=2	m=3	m=4	m=5

from $(0,0)$ *to* (m,n) *avoiding* $\uparrow^{\geq 3}$ *and* $\longrightarrow^{\geq 3}$

Theorem 16. *Let* $f(m,n)$ *be the number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{k}$ *and* $\longrightarrow^{k}$*,taking steps from* $\{\uparrow, \longrightarrow\}$*. Then*

$$f(m,n) = f(m-1,n) + f(m,n-1) - f(m-k,n-1) - f(m-1,n-k) + f(m-k,n-k).$$

Corollary 17. *The number of lattice paths from* $(0,0)$ *to* $((ns+1), nt-1)$ *avoiding* $\uparrow^{\geq 3}$*and* $\rightarrow^{\geq 3}$*is*

$$2\sum_{i=(ns+1)-n}^{\lfloor (ns+1)/2 \rfloor} \binom{m-i}{i}\binom{m-i}{n-m+i} + \sum_{i=m-n-1}^{\lfloor m/2 \rfloor} \binom{m-i}{i}\binom{m-i-1}{n-m+i+1} + \sum_{i=m-n+1}^{\lfloor m/2 \rfloor} \binom{m-i}{i}\binom{m-i+1}{n-m+i-1}.$$

Theorem 18. *The number of lattice path from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq 4}, \rightarrow^{\geq 4}$*is*

$$2\sum_{i=0}^{\lfloor n/3 \rfloor}\sum_{j=0}^{\lfloor (n-3i)/2 \rfloor} \binom{n-2i-j}{i}\binom{n-3i-j}{j} \sum_{s=0}^{\min\{\lfloor n/3 \rfloor, \lfloor \frac{2i+j}{2} \rfloor\}} \binom{n-2i-j}{s}\binom{n-2i-j-s}{2i+j-2s} +2\sum_{i=0}^{\lfloor n/3 \rfloor}\sum_{j=0}^{\lfloor (n-3i)/2 \rfloor} \binom{n-2i-j}{i}\binom{n-3i-j}{j} \sum_{s=0}^{\min\{\lfloor n/3 \rfloor, \lfloor \frac{2i+j+1}{2} \rfloor\}} \binom{n-2i-j-1}{s}\binom{n-2i-j-s-1}{2i+j+1-2s}.$$

The above numbers equal

$$[x^n y^n]\left(\frac{(1+x+x^2+x^3)\,(1+y+y^2+y^3)}{1-xy\,(1+y+y^2)\,(1+x^2+x)}\right).$$

Theorem 19. *The generating function of the number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq i}, \rightarrow^{\geq j}$*satisfies*

$$\frac{(\sum_{k=1}^{i} x^{k-1})(\sum_{k=1}^{j} y^{k-1})}{1-(\sum_{k=2}^{i} x^{k-1})(\sum_{k=2}^{j} y^{k-1})}.$$

Problem 20. *How to find the number of lattice paths from* $(0,0)$ *to* (n,n) *avoiding* $\uparrow^{\geq i}, \rightarrow^{\geq j}$*, and weakly above the the diagonal* $y=x$ *. And how to find a good generating function for this problem?*

III. Stretches and Turns [15]

A. Lattice Path with East, North Steps:

Consider the paths from $(0,0)$ to (m,n) with s level-stretches (a stretch is one or some unextendable continues level steps),k right turns and h up-stretches.

Let $f_1(m,n,k)$ denote number of walks from $(0,0)$ to (m,n) with k right turns, then

$$f_1(m,n,k) = \binom{m}{k}\binom{n}{k}.$$

Example: $f_1(1,1,1) = 1,\ f_1(1,1,0) = 1,\ f_1(2,2,1) = 4.$

Let $f_2(m,n,s)$ denote the number of walks from $(0,0)$ to (m,n) with s level-stretches, then

$$f_2(m,n,s) = \binom{m-1}{s-1}\binom{n+1}{s}.$$

Example: $f_2(1,1,1) = 2, f_2(1,2,1) = 3, f_2(3,2,2)$=6.

Let $f_3(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$f_3(m,n,h) = \binom{m+1}{h}\binom{n+1}{h-1}.$$

Example: $f_3(1,1,1) = 2, f_3(1,2,2) = 3, f_3(3,3,2) = 24.$

Let $f_4(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t stretches, then $f_4(m,n,t) =$

$$2\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ when } t \text{ is even,}$$

$$\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1} + \binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1} \text{ when } t \text{ is odd.}$$

Example:$f_4(1,1,2) = 2, f_4(3,3,4) = 8, f_4(2,1,3) = 1, f_4(3,4,5) = 9.$

Let $f_5(m,n)$ denote the number of walks from $(0,0)$ to (m,n). It is well known that

$$f_5(m,n) = \binom{m+n}{n} = \binom{m+n}{m}.$$

We also have

$$f_5(m,n)=\sum_{i\geq 1}\binom{m+1}{i}\binom{n-1}{i-1}.$$

Example: $f_5(1,1)=2, f_5(1,2)=3, f_5(2,2)=6$.

B. ENW Walks

Counting walks which start at the origin $(0,0)$ and take unit steps $(1,0)$, $(0,1)$, and $(-1,0)$ with the restriction that no E step immediately follows a W step and vice verse. The restriction has the effect of making the walks self-avoiding. It is a major unsolved problem to enumerate all self-avoiding walks. We start by counting walks which start at the origin $(0,0)$ and take unit steps $(1,0)$,and $(0,1)$. Let $p(m,n)$ denote the number of ENW walks from $(0,0)$ the (m,n). We have

$$p(m,n)=p(m,n-1)+2\sum_{i>0}p(m-j,n-1)$$

$$p(m+n)=\sum_{i=0}p(m+i,n-i).$$

Let $p_1(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$p_1(m,n,h)=2^{h+1}\binom{m}{h}\binom{n-1}{h-1}+2^{h-1}\binom{m-1}{h-2}\binom{n-1}{h-1}.$$

Example:$p_1(1,1,1)=4, p_1(3,4,2)=78$.

Let $p_2(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t stretches, then

$$p_2(m,n,t)=$$

$$2^{(t/2+1)}\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ when } t \text{ is even;}$$

$$2^{(t+1)/2}\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1}+$$
$$2^{(t-1)/2}\binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$
when t is odd.

Example:$p_2(1,1,2)=4, p_2(4,3,4)=48, p_2(1,2,3)=2$,$p_2(3,4,5)=48$.

Let $p_2(N,t)$ denote the number of walks with length N and t stretches, then $p_2(N,t)=$

$$\sum_{n=t/2}^{N-t/2}2^{t/2+1}\binom{N-n+1}{t/2-1}\binom{n}{t/2-1} \text{ for even } t$$

$$\sum_{n=(t-1)/2}^{N-(t+1)/2}2^{(t+1)/2}\binom{N-n-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1}$$

$$+\sum_{n=(t+1)/2}^{N-(t-1)/2}2^{(t-1)/2}\binom{N-n-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$

for odd t.

Example: $p_2(2,2)=4, p_2(3,2)=8$, $p_2(1,1)=3$, $p_2(3,3)=6$.

Let $p(N)$ be the number of walks of length N, then

$$p(N)=3+\sum_{t=1}^{N}p_2(N,t).$$

C. ENW Walks without Ending With a W Step

We now consider ENW walks from $(0,0)$ to (m,n) with the additional restriction no walk ends with a W step. Let $q_1(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$q_1(m,n,h)=2^{h-1}\binom{n-1}{h-1}(\binom{m}{h-1}+2\binom{m}{h}).$$

Example:$q_1(1,1,1)=3, q_1(2,2,1)=5$,$q_1(3,4,2)=54$.

Let $q_2(m,n,t)$ denote the number of walks from $(0,0)$ to (m,n) with t stretches, then $q_2(m,n,t)=$

$$3\times 2^{t/2-1}\binom{m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ for even } t$$
$$2^{(t-1)/2}\binom{m-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$
$$+2^{(t-1)/2}\binom{m-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1} \text{ for odd } t.$$

Example:$q_2(1,1,2)=3, q_2(4,3,4)=36, q_2(1,2,3)=2$,$q_2(3,4,5)=36$.

Let $q_2(N,t)$ denote the number of walks with length N and t stretches, then $q_2(N,t)=$

$$\sum_{n=t/2}^{N-t/2}3\times 2^{t/2-1}\binom{N-m-1}{t/2-1}\binom{n-1}{t/2-1} \text{ for even } t$$

$$\sum_{n=(t-1)/2}^{N-(t-1)/2}(2^{(t-1)/2}\binom{N-n-1}{(t+1)/2-2}\binom{n-1}{(t+1)/2-1}$$
$$+2^{(t-1)/2}\binom{N-n-1}{(t-1)/2}\binom{n-1}{(t-1)/2-1})$$

for odd t.

Example: $q_2(2,2)=3$,$q_2(3,2)=6, q_2(3,3)=4, q_2(5,3)=24$.

Let $q(N)$ be the number of walks of length N, then

$$q(N)=\sum_{t=1}^{N}p_2(N,t).$$

Let $q(m,n)$ denote the number of walks from $(0,0)$ to (m,n), then

$$q(m,n)=\sum_{h=1}^{n}2^{h-1}\binom{n-1}{h-1}(\binom{m}{h-1}+2\binom{m}{h}).$$

Example:$q(1,1)=3, q(1,2)=5, q(3,4)=129$.

$$q(N)=\sum_{n=1}^{N}(\sum_{h=1}^{n}2^{h-1}\binom{n-1}{h-1}(\binom{N-n}{h-1}+2\binom{N-n}{h}))+1.$$

Example: $q(0)=1, q(1)=2, q(2)=5, q(3)=12$,$q(4)=29, q(5)=70, q(6)=169$.

D. END Walks

We now consider walks from $(0,0)$ to (m,n) and taking unit steps $(1,0)=E$(east) and $(0,1)=N$(north) and double east steps of length 2 denoted by D.

Let $c(m,n,h,d)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches and d copies of D, then

$$c(m,n,h,d)=\binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1,1,0)=2, c(2,1,1,1)=2$,$c(6,6,4,2)=300, c(6,6,4,3)=10$.

Let $c(m,n,h)$ denote the number of walks from $(0,0)$ to (m,n) with h up-stretches, then

$$c(m,n,h)=\sum_{d=0}^{m}\binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1,1)=2, c(2,1,1)=5, c(4,3,2)=62$.

Let $c(m,n)$ denote the number of walks from $(0,0)$ to (m,n),then

$$c(m,n)=\sum_{h=0}^{n}\sum_{d=0}^{m}\binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}.$$

Example: $c(1,1)=2, c(1,2)=3, c(3,3)=40$.

Also,

$$\begin{aligned}c(m,n)&=\sum_{h=0}^{n}\sum_{d=0}^{m}\binom{n-1}{h-1}\binom{m-d}{d}\binom{m-d+1}{h}\\&=\sum_{d=0}^{m}\binom{m-d}{d}\sum_{h=0}^{n}\binom{n-1}{h-1}\binom{m-d+1}{h}\\&=\sum_{d=0}^{m}\binom{m-d}{d}\binom{n+m-d}{m-d}.\end{aligned}$$

Let $c(N)$ denote the number of walks of length N, then

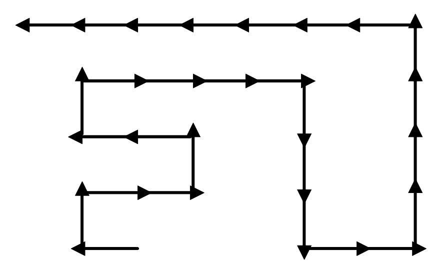
$$\begin{aligned}c(N)&=\sum_{n=0}^{N}\sum_{d=0}^{N-n}\binom{N-n-d}{d}\binom{N-d}{N-n-d}\\&=\sum_{n=0}^{N}\sum_{d=0}^{N-n}\binom{N-n-d}{d}\binom{N-d}{n}.\end{aligned}$$

Example:$c(1)=2, c(2)=5, c(3)=12, c(4)=29$.

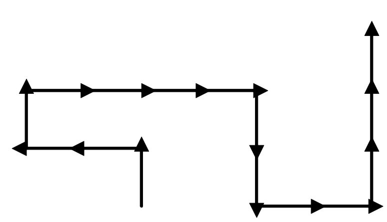
IV. OPEN PROBLEMS IN PRUDENT SELF-AVOIDING WALKS

We will talk about two-sided prudent self-avoiding walks (PSAWs) in the following:

A well-known long standing problem in combinatorics and statistical mechanics is to find the generating function for self-avoiding walks (SAW) on a two-dimensional lattice, enumerated by perimeter. A SAW is a sequence of moves on a square lattice which does not visit the same point more than once. It has been considered by more than one hundred researchers in the pass one hundred years, including George Polya, Tony Guttmann, Laszlo Lovasz, Donald Knuth, Richard Stanley, Doron Zeilberger, Mireille Bousquet-Mélou, Thomas Prellberg, Neal Madras, Gordon Slade, Agnes Dittel, E.J. Janse van Rensburg, Harry Kesten, Stuart G. Whittington, Lincoln Chayes, Iwan Jensen, Arthur T. Benjamin, and others. More than three hundred papers and a few volumes of books were published in this area. A SAW is interesting for simulations because its properties cannot be calculated analytically. Calculating the number of self-avoiding walks is a common computational problem. A PSAW is a proper subset of SAWs on the square lattice. The walk starts at $(0,0)$, and the empty walk is a PSAW. A PSAW grows by adding a step to the end point of a PSAW such that the extension of this step - by any distance - never intersects the walk. Hence the name prudent. The walk is so careful to be self-avoiding that it refuses to take a single step in any direction where it can see - no matter how far away - an occupied vertex. The following walk is a PSAW.



In a *one-sided* PSAW, the endpoint lies always on the top side of the box. The walk is *partially directed*. A prudent walk is *two-sided* if its endpoint lies always on the top side, or on the right side of the box. The walk in the following figure is a two-sided PSAW.



We studied the pattern avoidance problems on one-sided and two-sided PSAWs and obtained:

(1) The generating function of the number, say $f(n,k)$, of one-sided n-step prudent walks exactly avoiding $\leftarrow^{=k}$ and $\uparrow^{=k}$

(both at the same time) satisfies

$$\frac{1+t-2t^k+2t^{k+1}}{1-2t-t^2+2t^{k+1}-2t^{k+2}}, \text{ and}$$

$$f(n,k) = g(n,k) + g(n-1,k) - 2g(n-k,k) + 2g(n-k-1,k), \text{ where}$$

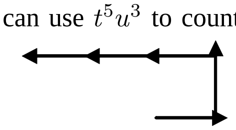
$$g(n,k) = \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\times \binom{j}{l}\binom{l}{-i-j+l-kl+n}(-1)^{i+j+kl-n}\,2^{i-j+l}.$$

(2) The generating function of the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq k}$, and $\downarrow^{\geq k}$ $(k > 2)$ taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$satisfies:

$$\left(1-t^2u\frac{1-t^ku^k}{1-tu}-\frac{tu}{u-t}\right)T(t,u) = 1+tu\frac{1-t^ku^k}{1-tu}+\frac{u-2t}{u-t}tT(t,t)$$

where u counts the distance between the endpoint and the north-east corner of the box.

For instance, in the following figure, a walk takes 5 steps, and the distance between the endpoint and the north-east corner is 3. So we can use t^5u^3 to count this walk.



(3) The generating function of the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^{=2}$, $\downarrow^{=2}$, taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, equals

$$(1-\frac{t^2u}{1-tu}-\frac{tu}{u-t}+u^2t^3)T(t,u) = \frac{1}{1-tu}-u^2t^2+\frac{u-2t}{u-t}tT(t,t).$$

Then we come to two open problems and two new results here:

Problem 21. *How to enumerate the number of two-sided n-step prudent walks, ending on the top side of their box, avoiding both $\leftarrow^{\geq i}$, and $\downarrow^{\geq j}$ $(i > j > 2)$ taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?*

Problem 22. *How to enumerate the number of two-sided n-step prudent walks, ending on the top side of their box, exactly avoiding both $\leftarrow^{=i}$, $\downarrow^{=j}$$(i > j > 2)$ taking steps from $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$?*

Theorem 23. *The number of one-sided n-step prudent walks in the first quadrant, starting from $(0,0)$ and ending on the y-axis, taking steps from $\{\uparrow, \leftarrow, \rightarrow\}$, avoiding $\leftarrow^{\geq 2}$ and $\rightarrow^{\geq 2}$is*

$$\sum_{j=0}^{\lfloor\frac{n+1}{4}\rfloor}\frac{1}{j+1}\binom{2j}{j}\binom{n-2j+1}{2j}.$$

Proof: We suppose that there are j copies of East steps, j copies of West steps and $n-2j$ copies of North steps.

Now we arrange j copies of East steps and j copies of West steps according to: the total number of the East steps is greater or equal to the total number of West steps from $(0,0)$ to any lattice point on a walk, which gives C_j, the jth Catalan number.

The $n-2j$ North steps in a walk provide $n-2j+1$ positions (i.e., cells) for j East steps and j West steps to be distributed, with each cell containing at most 1 East step or 1 West step. Then $0 \leq j \leq \lfloor\frac{n+1}{4}\rfloor$. There are $\binom{n-2j+1}{2j}$ way to choose $2j$ cells for the j East steps and j West steps.

Therefore,

$$\sum_{j=0}^{\lfloor\frac{n+1}{4}\rfloor}\frac{1}{j+1}\binom{2j}{j}\binom{n-2j+1}{2j}.$$

■

References

[1] B. Bollobás, The Art of Mathematics: Coffee Time in Memphis, Cambridge University Press, 2006.

[2] J. E.Bonin, A. de Mier, and M. Noy, Lattice path matroids: Enumerative aspects and Tutte polynomials, J. Combin. Theory Ser. A **104** (2003), 63-94.

[3] L. Carlitz, Zero-one sequences and Fibonacci numbers of higher order, The Fibonacci Quarterly, Vol. **12** (1974) 1–10.

[4] R. Chapman, T. Chow, A. Khetan, D. P. Moulton, and R. J. Waters, Simple formulas for lattice paths avoiding certain periodic staircase boundaries, arXiv:0705.2888v1 (2007).

[5] A. de Mier, A natural family of flag matroids, SIAM Journal on Discrete Mathematics, **21** (2007),130-140.

[6] A. de Mier, and M. Noy, A solution to the tennis ball problem, Theoret. Comput. Sci. **346** (2005), 254-264.

[7] J. Fallon, S. Gao, and H. Niederhausen, A finite operator approach to the tennis ball problem, Congr. Numer. **184** (2007)**,** 5-10.

[8] J. Fallon, S. Gao, and H. Niederhausen, Proof of a lattice paths conjecture connected to the tennis ball problem, Journal of Statistical Planning and Inference, **140** (2010), 2227–2229.

[9] R. Grimaldi, and J. Moser, The Catalan numbers and a tennis ball problem, Congr. Numer. **125** (1997), 65-71.

[10] S. Gao, Using finite operator calculus solving pattern avoidance problems in lattice paths and walks (in preparation)

[11] M. Jani, and M. Zeleke, A bijective proof of a tennis ball problem, Bulletin of the ICA, **41 (2004)**, 89-95.

[12] G. Kreweras, Sur une class de problèmes de dénombrement liés au treillis des partitions d'entiers." Cahiers Buro 6, 2-107(1965).

[13] C. Mallows, and L. Shapiro, Balls on the lawn, J. Integer Seq. **2** (1999).

[14] D. Merlini, R. Sprugnoli, and M.C. Verri, The tennis ball problem, J. Combin. Theory Ser. A **99** (2002), 307-344.

[15] A. Nkwanta, and L. Shapiro, Pell walks and Riordan matrices,The Fibonacci Quarterly,Vol **43**.(2005),170-180.

[16] L. Shapiro, Catalan trigonometry, Congr. Numer. **156** (2002), 129-136.

[17] T. Tymoczko, and J. Henle, Sweet Reason: A Field Guide to Modern Logic, New York: W.H.Freeman and Company (1995).

[18] S. G. Mohanty, Lattice Path Counting and Applications, New York: Academic Press(1979).

[19] J. Riordan, An Introduction to Combinatorial Analysis (originally published: New York: John Wiley 1958), Dover Publications (2002).

[20] G. C. Rota, D. Kahaner, and A. Odlyzko, On the foundations of combinatorial theory VIII: finite operator calculus. *J*. Math. Anal. Appl. **42** (1973) 684-760.

[21] H.J. Ryser, Combinatorial Mathematics, Carus mathematical monographs **14** (Math. Assoc. Am.) 1963.

[22] The Online Encyclopedia of Integer Sequences (2014), published electronically at http://oeis.org.