# Topological Chaos for Elementary Cellular Automata*

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**Abstract.** We apply the definition of chaos given by Devaney for discrete time dynamical systems to the case of elementary cellular automata, i.e., 1-dimensional binary cellular automata with radius 1. A discrete time dynamical system is chaotic according to the Devaney's definition of chaos if it is topologically transitive, is sensitive to initial conditions, and has dense periodic orbits. We enucleate an easy-to-check property of the local rule on which a cellular automaton is based which is a necessary condition for chaotic behavior. We prove that this property is also sufficient for a large class of elementary cellular automata. The main contribution of this paper is the formal proof of chaoticity for many non additive elementary cellular automata. Finally, we prove that the above mentioned property does not remain a necessary condition for chaoticity in the case of non elementary cellular automata.

## 1 Introduction

The notion of chaos is very appealing, and it has intrigued many scientists (see [1, 2, 6, 10, 13] for some works on the properties that characterize a chaotic process). There are simple deterministic dynamical systems that exhibit unpredictable behavior. Though counterintuitive, this fact has a very clear explanation. The lack of *infinite precision* in describing the state of the system causes a loss of *information* which is dramatic for some processes which quickly loose their deterministic nature to assume a non deterministic (unpredictable) one.

A chaotic phenomenon can indeed be viewed as a deterministic one, in the presence of infinite precision, and as a nondeterministic one, in the presence of finite precision constraints.

Thus one should look at chaotic processes as at processes merged into time, space, and precision bounds, which are the key resources in the science of computing. A nice way in which one can analyze this finite/infinite dichotomy is by using cellular automata (CA) models. Consider the 1-dimensional CA $(X, \sigma)$, where $X = \{0,1\}^{\mathbf{Z}}$ and $\sigma$ is the shift map on $X$. In order to completely describe the elements of $X$, we need to operate on sequences of binary digits of infinite length. Assume for a moment that this is possible. Then the shift map is

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completely predictable, i.e., one can completely describe $\sigma^n(x)$, for any $x \in X$ and for any integer $n$. In practice, only finite objects can be computationally manipulated. Let $x \in X$. Assume we know a portion of $x$ of length $n$ (the portion between the two vertical lines in Figure 1). One can easily verify that $\sigma^n(x)$ completely depends on the unknown portion of $x$. In other words, if we have finite precision, the shift map becomes unpredictable, as a consequence of the combination of the finite precision representation of $x$ and the *sensitivity* of $\sigma$.

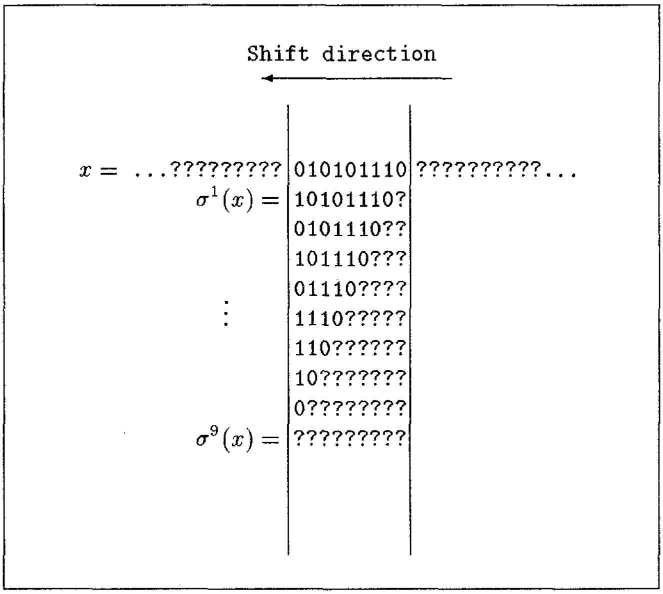


**Fig. 1.** Finite precision combined with sensitivity to initial conditions causes unpredictability after a few iterations ($x$ represents the state of the CA at time step 0, and $\sigma^i(x)$ the state at time step $i$).

In the case of discrete time dynamical systems (DTDS) defined on a metric space, many definitions of chaos are based on the notion of sensitivity (see for example [6, 9]).

We now recall the definition of sensitivity to initial conditions for a DTDS $(X, F)$. Here, we assume that $X$ is equipped with a distance $d$ and that the map $F$ is continuous on $X$ according to the metric topology induced by $d$.

**Definition 1.** A DTDS $(X, F)$ is sensitive to initial conditions if and only if there exists $\delta > 0$ such that for any $x \in X$ and for any neighborhood $N(x)$ of $x$, there is a point $y \in N(x)$ and a natural number $n$, such that $d(F^n(x), F^n(y)) > \delta$. $\delta$ is called the sensitivity constant.