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Seven Cubes and Ten 24-Cells

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Abstract. The 45 diagonal triangles of the six-dimensional polytope 2_{21} (representing the 45 tritangent planes of the cubic surface) are the vertex figures of 45 cubes $\{4, 3\}$ inscribed in the seven-dimensional polytope 3_{21}, which has 56 vertices. Since $45 \times 56 = 8 \times 315$, there are altogether 315 such cubes. They are the vertex figures of 315 specimens of the four-dimensional polytope $\{3, 4, 3\}$, which has 24 vertices. Since $315 \times 240 = 24 \times 3150$, there are altogether 3150 $\{3, 4, 3\}$'s inscribed in the eight-dimensional polytope 4_{21}. They are the vertex figures of 3150 four-dimensional honeycombs $\{3, 3, 4, 3\}$ inscribed in the eight-dimensional honeycomb 5_{21}. In other words, each point of the $\tilde{E}_8$ lattice belongs to 3150 inscribed $\tilde{D}_4$ lattices of minimal size.

Analogously, in unitary 4-space there are 3150 regular complex polygons $3\{4\}3$ inscribed in the Witting polytope $3\{3\}3\{3\}3\{3\}3$.

1. Introduction

A polygon is said to be *uniform* if it is regular, and a polytope is said to be uniform if all its facets are uniform while its symmetry group is transitive on its vertices [7, p. 41]. It follows that all the vertices of an n-dimensional polytope lie on an $(n-1)$-sphere, the *circumsphere*, of radius ρ, and that all the edges have the same length, which we take to be 1. Thus the farther ends of all the edges at one vertex lie in the hyperplane that contains the intersection of two $(n-1)$-spheres: the circumsphere and the unit sphere whose center is the "one vertex." The section of the polytope by this hyperplane is called the *vertex figure* [11, p. 292]. For instance, the vertex figure of the cube is an equilateral triangle with sides $\sqrt{2}$, and the vertex figure of the uniform triangular prism (with square side-faces) is an isosceles triangle with sides 1, $\sqrt{2}$, $\sqrt{2}$.

2. Successive Vertex Figures

Schläfli discovered two infinite sequences of regular polytopes such that each polytope is the vertex figure of the next: the regular simplices

$$\alpha_1 = \{\ \},\quad \alpha_2 = \{3\},\quad \alpha_3 = \{3,3\},\quad \alpha_4 = \{3,3,3\},\quad \ldots,$$

whose numbers of vertices are

$$2,\qquad 3,\qquad 4,\qquad 5,\qquad \ldots,$$

and the *cross polytopes*

$$\beta_2 = \{4\},\quad \beta_3 = \{3,4\},\quad \beta_4 = \{3,3,4\},\quad \ldots,$$

whose numbers of vertices are

$$4,\qquad 6,\qquad 8,\qquad \ldots$$

[11, p. 37]. It would be natural to allow the latter sequence to begin with β_1, a line segment of length $\sqrt{2}$, since this is the vertex figure of the square, β_2.

There is also a *finite* sequence of the same kind, beginning with an equilateral triangle of side $\sqrt{2}$, continuing with the cube $\gamma_3 = \{4, 3\}$ and the 24-cell $\{3, 4, 3\}$, and ending with the honeycomb of 16-cells $\{3, 3, 4, 3\}$ [11, p. 300]. In this case, the numbers of vertices are

$$8,\qquad 24,\qquad \infty.$$

Abandoning the restriction to *regular* figures, Gosset discovered another such finite sequence of polytopes, beginning with the isosceles triangle mentioned at the end of Section 1, continuing with the uniform triangular prism and "new" polytopes in 4, 5, 6, 7, and 8 dimensions, and ending with a honeycomb of α_8's and β_8's [8, pp. 156, 203; 11, pp. 230, 294, 332]. The numbers of vertices of

$$(-1)_{21} = \alpha_2 \times \alpha_1,\quad 0_{21} = t_1\alpha_4,\quad 1_{21} = h\gamma_5,\quad 2_{21},\quad 3_{21},\quad 4_{21},\quad 5_{21},$$

are

$$6,\qquad 10,\qquad 16,\qquad 27,\quad 56,\quad 240,\quad \infty.$$

By calling the triangular prism $\alpha_2 \times \alpha_1$, we are expressing it as the Cartesian product of the triangle α_2 with the line segment α_1. In four dimensions there is an analogous "prism" $(-1)_{31} = \alpha_3 \times \alpha_1$, based on the regular tetrahedron α_3. This provides a third "finite sequence"

$$(-1)_{31},\quad 0_{31} = t_1\alpha_5,\quad 1_{31} = h\gamma_6,\quad 2_{31},\quad 3_{31}$$

[11, p. 337], whose numbers of vertices are

$$8,\qquad 15,\qquad 32,\qquad 126,\qquad \infty.$$

Another Cartesian product is the four-dimensional “double-prism” $(-1)_{22} = \alpha_2 \times \alpha_2$, which provides a fourth finite sequence

$$(-1)_{22}, \quad 0_{22} = \mathrm{t}_2\alpha_5, \quad 1_{22}, \quad 2_{22}$$

[11, pp. 332, 333], whose numbers of vertices are

$$9, \qquad 20, \qquad 72, \quad \infty.$$

3. Circumradii

The circumradius ρ of a Cartesian product is related to the circumradii ρ' and ρ'' of its factors by the Pythagorean relation

$$\rho^2 = \rho'^2 + \rho''^2$$

[3, p. 351].

A simple geometric construction [2, p. 31] shows that the circumradius ρ of any uniform polytope is related to the circumradius σ of its vertex figure by the formula

$$2\rho^2 = \frac{1}{2 - 2\sigma^2},$$

which makes it easy to write down the value of $2\rho^2$ for each member of a “sequence” of polytopes as soon as this property is known for the initial member: we just subtract from 2 and reciprocate.

For the sequence of regular simplices α_n, beginning with the unit line-segment α_1, we have

$$\begin{array}{llllll} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \ldots, & \alpha_n: \\ \frac{1}{2}, & \frac{2}{3}, & \frac{3}{4}, & \frac{4}{5}, & \ldots, & \frac{n}{n+1}. \end{array}$$

For the sequence of cross polytopes β_n, beginning with the longer line-segment β_1, we have

$$\begin{array}{llllll} \beta_1, & \beta_2, & \beta_3, & \beta_4, & \ldots, & \beta_n: \\ 1, & 1, & 1, & 1, & \ldots, & 1. \end{array}$$

For the four finite sequences, beginning with $\alpha_2 \times \alpha_1$, we have

$$\begin{array}{lllllll} (-1)_{21}, & 0_{21}, & 1_{21}, & 2_{21}, & 3_{21}, & 4_{21}, & 5_{21}: \\ \frac{2}{3}+\frac{1}{2}=\frac{7}{6}, & \frac{6}{5}, & \frac{5}{4}, & \frac{4}{3}, & \frac{3}{2}, & 2, & \infty. \\ & & (-1)_{31}, & 0_{31}, & 1_{31}, & 2_{31}, & 3_{31}: \\ & & \frac{3}{4}+\frac{1}{2}=\frac{5}{4}, & \frac{4}{3}, & \frac{3}{2}, & 2, & \infty. \\ & & & (-1)_{22}, & 0_{22}, & 1_{22}, & 2_{22}: \\ & & & \frac{2}{3}+\frac{2}{3}=\frac{4}{3}, & \frac{3}{2}, & 2, & \infty. \end{array}$$

For instance, the value $2\rho^2 = \frac{3}{2}$ for 3_{21} shows that this seven-dimensional polytope has the same circumradius $\frac{1}{2}\sqrt{3}$ as the ordinary cube.

These computations provide the easiest way to establish infinite order for each of the groups $[3^{2,2,2}]$, $[3^{3,3,1}]$, and $[3^{5,2,1}]$.

4. The Six-Dimensional Polytope 2_{21}

The 27 vertices

$$a_1, \ldots, a_6;\ \ b_1, \ldots, b_6;\ \ c_{12}, \ldots, c_{56}$$

of 2_{21} form a 2-distance set [6, p. 465]: any 2 of them form either an edge (distance 1) or a diagonal (distance $\sqrt{2}$). In fact, the $\binom{27}{2}$ pairs of vertices form $15+15+6+60+60+60 = 216$ *edges*

$$a_1a_2, \ldots;\ \ b_1b_2, \ldots;\ \ a_1b_1, \ldots;\ \ a_1c_{23}, \ldots;\ \ b_1c_{23}, \ldots;\ \ c_{12}c_{13}, \ldots,$$

and $30 + 30 + 30 + 45 = 135$ *diagonals*

$$a_1b_2, \ldots;\ \ a_1c_{12}, \ldots;\ \ b_1c_{12}, \ldots;\ \ c_{12}c_{34}, \ldots.$$

The 135 diagonals are the sides of $30 + 15 = 45$ *diagonal triangles*

$$a_1b_2c_{12}, \ldots;\ \ c_{12}c_{34}c_{56}, \ldots.$$

It is remarkable [1, p. 15] that nine of these 45 triangles can be chosen so as to use each of the 27 vertices just once. The enneagonal projection of 2_{21} [6, p. 460, Fig. 3] instantly provides two examples of such a set of nine diagonal triangles. One, namely

$$a_1b_6c_{16}, a_6b_5c_{56}, a_2b_3c_{23}, a_3b_4c_{34}, a_4b_2c_{24}, a_5b_1c_{15}, c_{12}c_{35}c_{46}, c_{13}c_{26}c_{45}, c_{14}c_{25}c_{36},$$

comes from the three sets of three equilateral triangles inscribed in the three concentric enneagons. The other, namely,

$$a_1b_4c_{14}, a_2b_1c_{12}, c_{13}c_{24}c_{56}, c_{16}c_{25}c_{34}, a_5b_3c_{35}, a_4b_5c_{45}, a_3b_6c_{36}, c_{15}c_{23}c_{46}, a_6b_2c_{26},$$

comes from the nine diameters.

5. The Seven-Dimensional Polytope 3_{21}

When 2_{21} is considered as the vertex figure of 3_{21}, its 45 diagonal triangles (of side $\sqrt{2}$) are the vertex figures of 45 cubes {4, 3}, all sharing two antipodal vertices of 3_{21} [11, p. 340]. Since 3_{21} has 56 vertices, while $45 \times 56 = 8 \times 315$, there are altogether 315 such inscribed cubes. When the $28 + 28$ vertices $(u_1, u_2, \ldots, u_8)$ are given, in the 7-flat $\sum u_\nu = 0$, as the permutations of

$$(3, 3, -1, -1, -1, -1, -1, -1) \qquad \text{and} \qquad (1, 1, 1, 1, 1, 1, -3, -3),$$

the 3-flats containing the 315 cubes consist of 105 such as

$$u_1 = u_2, \qquad u_3 = u_4, \qquad u_5 = u_6, \qquad u_7 = u_8, \tag{5.1}$$

and 210 such as

$$u_1 - u_2 + u_3 - u_4 = 0, \qquad u_5 = u_6 = u_7 = u_8. \tag{5.2}$$

Different coordinates were described in *Kaleidoscopes* [11, p. 339], but an unfortunate misprint mars the table of 14-gons at the top of page 341: the row beginning with 013 repeats alternate entries of the row beginning with $12\overline{4}$; instead, it should have been

$$013\ \ \overline{450}\ 124\ \overline{561}\ 235\ \overline{602}\ 346\ \ \overline{013}\ 450\ \overline{124}\ 561\ \overline{235}\ 602\ \ \overline{346}.$$

In terms of the plane quartic curve of genus 3, whose 28 bitangents correspond to the 28 diameters of 3_{21} [10, p. 354], each of the 315 cubes represents one of Dickson's 315 conics, whose eight points of intersection with the quartic curve are the points of contact of four bitangents. It was observed by Patrick DuVal [5, p. 186] that seven of these 315 conics can be chosen so as to use each of the 28 bitangents just once, in agreement with the fact that the 56 vertices of 3_{21} can be distributed into seven sets of eight belonging to inscribed cubes [11, p. 340].

6. The Eight-Dimensional Polytope 4_{21}

When 3_{21} is considered as the vertex figure of 4_{21}, its 315 inscribed cubes $\{4, 3\}$ are the vertex figures of 315 24-cells $\{3, 4, 3\}$, all sharing two antipodal vertices of 4_{21}. The $240 = 112 + 128$ vertices of a 4_{21} (of edge $2\sqrt{2}$) are conveniently given by the permutations of

$$(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0)$$

along with

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1),$$

where the number of minus signs is even.

Since the 24 vertices of a $\{3, 4, 3\}$ of edge 2 are given by the permutations of $(\pm 2, 0, 0, 0)$ along with $(\pm 1, \pm 1, \pm 1, \pm 1)$, one such inscribed 24-cell lies in the 4-flat (5.1): we simply use the coordinates

$$(u_2, u_4, u_6, u_8).$$

It is not too laborious to verify that another is given by (5.2). And, of course, there are many more. The most obvious is the $\{3, 4, 3\}$ that lies in the 4-flat

$$u_5 = u_6 = u_7 = u_8 = 0. \tag{6.1}$$

Since 4_{21} has 240 vertices, while $315 \times 240 = 24 \times 3150$, there are altogether 3150 such inscribed 24-cells. Among them:

Ten can be chosen so as to use each of the 240 *vertices just once*.

To verify this statement, consider any one of the 3150 {3, 4, 3}'s, say (6.1), and transform it by the cyclic group $\mathfrak{C}_{10}$ generated by the isometry $Q = R^3$, where

$$R = R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8$$

is the product (in any order) of the eight reflections which generate the symmetry group $[3^{4,2,1}]$ of 4_{21} [11, pp. 159, 207–221]. The result is a set of five {3, 4, 3}'s: five, not ten, because the isometry $Q^5 = R^{15}$ is the central inversion [11, p. 163], which is a symmetry operation of both 4_{21} and {3, 4, 3}. Now choose another inscribed {3, 4, 3} that has no common vertex with any of those five, and apply the same group $\mathfrak{C}_{10}$. Clearly, the new set of five must complete the desired set of ten.

An intriguing question remains: What kind of arrangement of 120 points is formed by the 120 vertices of either set of *five* {3, 4, 3}'s? We recall that a different set of five {3, 4, 3}'s have together the same 120 vertices as the regular 600-cell {3, 3, 5} [8, p. 270]; and the ten-dimensional coordinates for 4_{21} reveal that its 240 vertices fall into two congruent sets of 120, each set projecting orthogonally into the 120 vertices of a 600-cell [11, p. 349]. (In *Kaleidoscopes*, page 350, line 5, the page number 578 should be 298.) Clearly, then, the desired 120 points have the coordinates

$$\begin{array}{lr}
(\sqrt{5},\ 0,\ 0,\ 0,\ -\sqrt{5};\ \tau\sqrt{5},\ 0,\ 0,\ 0,\ -\tau\sqrt{5}), & 20\\
(-\tau^{-2},\ -\tau^{2},\ 1,\ 1,\ 1;\ \tau^{3},\ \tau^{-1},\ -\tau,\ -\tau,\ -\tau), & 20\\
(\tau^{-1},\ \tau^{-1},\ -\tau,\ -\tau,\ 2;\ \tau^{2},\ \tau^{2},\ -1,\ -1,\ -2\tau), & 30\\
(-2,\ \tau,\ \tau,\ -\tau^{-1},\ -\tau^{-1};\ 2\tau,\ 1,\ 1,\ -\tau^{2},\ -\tau^{2}), & 30\\
(-1,\ -1,\ -1,\ \tau^{2},\ \tau^{-2};\ \tau,\ \tau,\ \tau,\ -\tau^{-1},\ -\tau^{3}), & 20\\
\hline
 & 120
\end{array}$$

where $\tau = (\sqrt{5}+1)/2$ and it is understood that the twenty or thirty permutations of the five coordinates before and after each semicolon are always the same.

7. The Eight-Dimensional Honeycomb 5_{21}

We recall that the 24-cell {3, 4, 3} is the vertex figure of Schläfli's honeycomb {3, 3, 4, 3} of 16-cells, whose vertices constitute the $\tilde{D}_4$ lattice [11, p. 287]. These points have coordinates which are all the sets of four integers whose sum is even [3, pp. 9, 117–119]. (Conway and Sloane call this "the D_4 lattice," but our tilde avoids confusion with the use elsewhere of D_4 as a symbol for the group $[3^{1,1,1}]$.)

Analogously, the polytope 4_{21} is the vertex figure of Gosset's eight-dimensional honeycomb 5_{21} of α_8's and β_8's, whose vertices constitute the famous $\tilde{E}_8$ lattice [3, pp. v, 10, 48, 90, 119, 123]. These points have coordinates which are all sets of eight integers, mutually congruent modulo 2, whose sum is a multiple of 4 [4, pp. 385, 393] or, equally well, all sets of nine integers, mutually congruent modulo 3, whose sum is 0 [11, p. 344].

The 3150 {3, 4, 3}'s inscribed in 4_{21} are the vertex figures of 3150 {3, 3, 4, 3}'s sharing one vertex of 5_{21} and having for all their vertices a subset of its vertices. In other words:

Each point of the $\tilde{E}_8$ lattice belongs to 3150 *inscribed $\tilde{D}_4$ lattices of minimal size.*

There are, of course, infinitely many such {3, 3, 4, 3}'s inscribed in 5_{21}. Among them, an infinite subset, using each vertex just once, can be obtained by taking one {3, 3, 4, 3} and translating it by integer distances along the $120 - 12$ edge directions that occur in 5_{21}, but not in the chosen {3, 3, 4, 3}.

8. Concluding Remark

For regular 24-cells 0_{111} inscribed in the eight-dimensional uniform polytope 4_{21}, the complex counterpart is regular polygons 3{4}3 inscribed in the four-dimensional regular Witting polytope 3{3}3{3}3{3}3 [9, pp. 47, 133, and Frontispiece].

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