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CLARIFYING CHAOS III. CHAOTIC AND STOCHASTIC PROCESSES, CHAOTIC RESONANCE, AND NUMBER THEORY

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In this tutorial we continue our program of clarifying chaos by examining the relationship between chaotic and stochastic processes. To do this, we construct chaotic analogs of stochastic processes, stochastic differential equations, and discuss estimation and prediction models. The conclusion of this section is that from the composition of simple nonlinear periodic dynamical systems arise chaotic dynamical systems, and from the time-series of chaotic solutions of finite-difference and differential equations are formed *chaotic processes*, the analogs of stochastic processes. Chaotic processes are formed from chaotic dynamical systems in at least two ways. One is by the superposition of a large class of chaotic time-series. The second is through the compression of the time-scale of a chaotic time-series. As stochastic processes that arise from uniform random variables are not constructable, and chaotic processes are constructable, we conclude that chaotic processes are primary and that stochastic processes are *idealizations* of chaotic processes.

Also, we begin to explore the relationship between the prime numbers and the possible role they may play in the formation of chaos.

1. Introduction

In this tutorial we examine the relationship between chaotic and stochastic processes. In Sec. 1 we discuss the definition of stochastic and chaotic processes and the construction of uniform random variables. In Sec. 2 we show how chaotic variables are constructed and how they are related to the more familiar chaotic time-series. In Sec. 3 we construct the chaotic analog of white noise, and the first form of Brownian motion. In Sec. 4 we construct the chaotic analog of conventional Brownian motion, and in Sec. 5 we construct chaotic analogs of wide-sense stationary processes, Poisson, and other processes. In Sec. 6, we show how chaotic analogs of stochastic differential equations are solved and discuss the role of the Ito and Stratonovich stochastic calculus. In Sec. 7, we look at chaotic processes as driving forces. In Sec. 8 we discuss filtering and estimation briefly. In Sec. 9, we review the *Theory of Chaos*, and in Sec. 10 we touch on the potential role of prime numbers in chaos.

1.1. *Definitions of chaotic and stochastic processes*

A stochastic process is defined as a one-parameter family of random variables and a random variable is defined as a measurable function. The construction of stochastic processes is reducible to the construction of random variables, and the construction of random variables is reducible to the construction of a uniform random variable.

Typically, when a sample of a uniform random variable is needed, one uses a pseudo-random number generator (which, in fact, is a chaotic dynamical system) to make a "random" selection. For example, $f(x) = x$ on the interval $[0, 1]$ is a measurable function that no one is going to want to call a random variable even though it fits the formal definition. However, by sampling this function *randomly*, we are able to work with it as a random variable in the sense of probability theory. This is because when dealing with random variables, we only deal with the range of the function, not its domain. This is due to the fact that it is the frequency with which a function takes on its values that is important to probability theory, not precisely where the time-order of these values are assumed by the function. Because of this last point, the concept of a probability distribution evolved.

A chaotic process may be defined as a one-parameter family of measurable functions which arise from chaotic dynamical systems. Contrary to probability theory, in dynamics we *are* interested in the time-order of the values of the functions we encounter, and so the concept of function is more appropriate than the concept of a probability distribution. In order to discuss both subjects without constantly translating from functions to distributions and back, we will discuss both subjects from the point of view of functions, the language more appropriate to the study of dynamics. To do this, we will need to *construct* a measurable function such that if we were to sample it sequentially, at a very high sampling rate, rather than randomly, we will still obtain a random sample of the function's values.

One more word about random selections. If we are to firmly establish the connection between chaotic and stochastic processes, the role of random selections must be more closely examined. When forming a realization of a stochastic process, for each time, t, one must make a random selection of a value of a measurable function, or present the measurable function explicitly for evaluation. If we were to insist on the measurable function being presented for evaluation, presumably the inverse of its distribution function would be presented and a random evaluation would be made using a pseudo-random number generator. But this introduces a small measure of mystery into the function evaluation process. For example, how is this process formally defined mathematically? It appears to be a function composition between the inverse of the distribution function and a pseudo-random number generator on a computer such as RND in Basic. If f is the inverse of a distribution function in question, then $f(\text{RND}(1))$ is the composition. But the exact definition of the function RND is hidden to most of its users. Since RND is a periodic function, it has only a finite number of values, say N. Successive calls to RND are thus equivalent to functions evaluations at the successive points $1/N$, $2/N$, $3/N, \ldots, 1$. Unless we randomize the timer function. In that case, the evaluation starts at some offset from $1/N$ determined by the computer time function in fractions of seconds of a day. After adding back these details, the first evaluation of the random variable in question can be formally expressed as $f(\text{RND}(1)(1/N + t_0))$ where t_0 is a function of the computer time stamp. But what if we need another random variable with the same distribution as f to get a realization of the process? For example, what if it is a discrete process of independent identically distributed random variables? Now we must have a denumerable set of them. We cannot formally use the same distribution function f since this is not a different measurable function. Typically, it is assumed that if the random selection is independent at each step, then we can use f over again without any harm. But we are trying to be exact and formal, thus we must use another function besides f, otherwise everything is imprecise. At this point it is clear that actually presenting the denumerable set of independent, identically distributed, random variables poses a practical, and perhaps, also a formal problem. If we do not exhibit the denumerable set, then we are only dealing with stochastic processes in some ideal sense while we are burying some interesting formalities in the process of making a random selection. Any formal process that will clear this up will necessarily lead to the use of chaotic dynamical systems to construct random variables and so lead us to the position that stochastic processes are idealizations of what are, in fact, chaotic processes. We defer the details to the following sections.

1.2. *The problem of constructing random variables as measurable functions*

In this section, we review the problem of *constructing* a measurable function on the interval $[0,1]$ which looks like a uniform random variable, i.e. the graph of f on the interval $[0,1]$ must look like a traditional white noise time series. The problem that we discuss in this construction applies to random variables in general, but we restrict our discussion to uniform random variables so that we are not dealing in generalities. To carry out this discussion some measure theory is required. We have kept this to a minimum with the aid of Professor Morris Hirsch who greatly simplified this discussion.

Ideally, we seek a measurable function defined on the interval $[0,1]$ with the following properties:

(1) It is Lesbegue measurable;
(2) It is bounded;
(3) It has a uniform distribution on every subinterval;
(4) It is *not* a constant almost everywhere.

However, no such function exists. The proof is as follows: Suppose, f is such a function. Then by property 1 and 2 it is integrable in the sense of Lesbegue. By property 3 we have

$$s^{-1}\int_t^{t+s} f(x)dx = \int_0^1 f(x)dx = c \tag{1}$$

and so

$$\int_t^{t+s}(f(x)-c)dx = 0 \tag{2}$$

for all s, $s+t \in [0,1]$, which implies[1] $f(x)=c$ almost everywhere, in violation of 4. We must conclude that if a function can be constructed which we would agree is a uniform random variable, it cannot be a measurable function.

While a uniform random variable is not mathematically constructable as a measurable function, a reasonable candidate can be proven to exist as a nonmeasurable function on the interval $[0,1]$. The proof follows a standard argument using the axiom of choice. Partition $[0,1]$ by the equivalence relation $x \sim y$ if $x-y \in \mathbf{Q}$, where $\mathbf{Q}$ is the rational. Select one point from each equivalence class and form the set $\mathbf{P}$. For $x \in \mathbf{P}$, the sets $\{x+\mathbf{Q}\}$ are the mentioned partition. Define $f(x+r)=r$ for $x \in \mathbf{P}$. f is uniform in the sense that for every $x \in \mathbf{P}$, $f(x+\mathbf{Q}) = \mathbf{Q}$, a dense set, while $\{x+\mathbf{Q}\}$ is also a dense set. Thus, f takes on every value of a set of dense subset of $[0,1]$ equally often. However, f is not measurable since $f^{-1}(0)=\mathbf{P}$.[2]

The nonconstructability of a uniform random variable makes better sense than, at first, it seems. If we could construct an idealized white noise process, then Brownian motion could not be the integral of such a process since the integral would be a constant time t. It is the imperfections in white noise models, and nature, that give rise to forms of Brownian motion. These imperfections are the result of our models being chaotic processes rather than stochastic processes. Clearly, by reviewing how stochastic processes are modeled on a computer we conclude that a chaotic process is the mathematical approximation of a stochastic process. Consequently, one form of Brownian motion can be realized as the integral of a chaotic "noise" process, even though it cannot be realized as the integral of stochastic white noise.

2. The Chaotic Analog of a Uniform Random Variable

In this section we construct the chaotic analog of a uniform random variable. To do this, we need a method of function evaluation that will give results similar to making a random selection. Since a pseudo-random selection method can be based on a modulo 1 multiplication scheme such as $x \to 2\cdot x \bmod (1)$, [a chaotic dynamical system with Lyapunov exponent $\log(2)$] we might pursue the following line of development. Given a distribution function F which is invertible, construct its inverse G, fix the number n as a large integer, and form the function $G(T^n(x))$ on the interval $[0,1]$, where $T(x) = 2\cdot x \bmod (1)$. If $F(x)=x$, a uniform distribution, then this results in the construction of a measurable function on $[0,1]$ having a large, but finite, number of discontinuities, and nearly satisfies condition 3. But this is clearly unsatisfactory from a theoretical point of view. In particular, if a

[1]See [Royden, 1988, Chap. 5, Lemma 8].

[2]The inverse image of a measureable set, in this case the point 0, is not a measureable set, i.e. $\mathbf{P}$ is not measurable, thus f is not measurable by the definition of measurable function.

measurable function on $[0,1]$ is to be considered a random variable it should not be possible to sample this function at a uniform sampling rate and obtain a constant value. In fact, the average of every large uniform sample should be the average of the random variable. This is not true of $G(T^n(x))$. We overcome this difficulty with the following construction:

$$g(x) = \sum_{n=0}^{\infty} a_n f(T^n(x)) \tag{3}$$

where a_n are the terms of an absolutely convergent series, f is a function on $[0,1]$ which is not constant almost everywhere, and T is a measure preserving multiplication mod 1 mapping of $[0,1]$ onto itself. This series is uniformly and absolutely convergent. If $f(x) = x$, we have the chaotic analog of a uniform random variable. In the case where $T(x) = 2x \bmod (1)$, the number of discontinuities of the nth term of this series is greater than 2^n. Figure 1 is the graph for this case where $a_n = 0.165^{(19-n)}$, and $T(x) = 16807.123 \bmod (1)$ and only 20 terms are used. We use 16807.123 in this example so that we do not have to resort to double precision to get a similar result using a large number of terms of $x \to 2 \cdot x \bmod (1)$. The choice of 16807 arises from the fact that $x \to 16807 \cdot x \bmod (2147483647)$ is a good choice for a simple pseudo-random number generator, see [Brown & Chua, 1996]. The red, nearly straight line across the figure is the integral of g as a function of x. As g becomes more uniform, this integral must approach a straight line whose slope is the mean value of g on $[0,1]$.

By the following example, we conclude that a random variable results from the superposition of the dynamics of a large number of chaotic processes.

Example. Superposition of Chaotic Time-series. The superposition of a discrete set of chaotic time-series can be written (where we assume they are evaluated at the time step $l \cdot h$, for h a small increment of time) as

$$g(l \cdot h) = \sum_{k=-\infty}^{\infty} a_k T^l(x_k) \tag{4}$$

where the x_k are a dense set of initial conditions, T is a chaotic map, and a_k are arbitrary constants. Assuming, ergodicity this could be considered to be of the form

$$g(l \cdot h) = \sum_{k=-\infty}^{\infty} a_k T^l(T^k(x)) \tag{5}$$

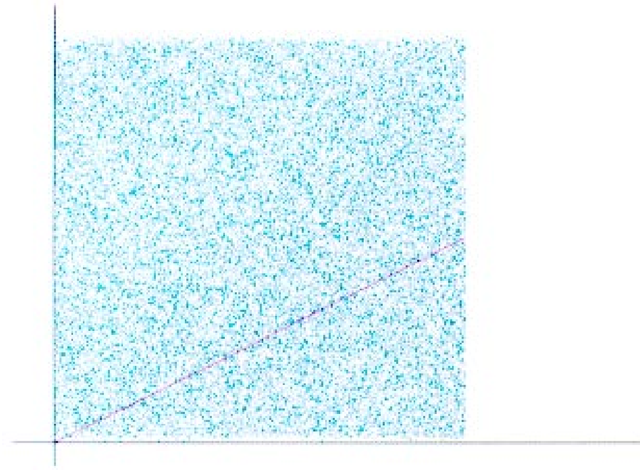


Fig. 1. The graph of $g(x)$ of Eq. (4) where $a_n = 0.165^{|n-19|}$ and $T(x) = 16807.123 \cdot x \bmod (1)$, and only 20 terms are used. Graphing g gives a resulting image similar to making a random selection of g.

for some fixed x. Rearranging this we get

$$g(l \cdot h) = \sum_{k=-\infty}^{\infty} a_k T^{k+l}(x)) \tag{6}$$

or

$$g(l \cdot h) = \sum_{k=-\infty}^{\infty} a_k T^k(T^l(x)) \tag{7}$$

By fixing l, this can be considered a function of the initial condition x,

$$g(x) = \sum_{k=-\infty}^{\infty} a_k T^k(x) \tag{8}$$

Another discrete form is

$$g(x) = \sum_{k=1}^{\infty} a_k T^k(x) \tag{9}$$

If we choose the logistic map as T we get

$$g(x) = \sum_{k=1}^{\infty} a_k \sin^2(2^k 2\pi x) \tag{10}$$

With the left unilateral binary shift we get

$$g(x) = \sum_{k=1}^{\infty} a_k \{2^k x\} \tag{11}$$

and for the map $x_{n+1} = 2x_n^2 - 1$ we get

$$g(x) = \sum_{k=1}^{\infty} a_k \cos(2^k 2\pi x) \tag{12}$$

The coefficients, a_k have the role of *arbitrary constants* as occurring in the theory of ordinary differential equations to be determined by measurements. For chaotic maps with positive Lyapunov exponents, as these example have, such series can define functions which are C^∞, continuous, continuous and nowhere differentiable, and nowhere continuous.

Another variation of Eq. (6) is

$$g(x,\, l \cdot h) = \sum_{k=-\infty}^{\infty} a_{k-l} T^k(x) \tag{13}$$

which gives a moving average representation to be encountered later.

As implied by the above examples, Eq. (3) cannot be the solution of any finite system of ODEs since the arbitrary constants, a_n form an infinite set.

2.1. *Generalized chaotic variables*

The construction of Eq. (3) can be generalized:

$$g(x) = \sum_{n=0}^{\infty} a_n f(T_n(x)) \tag{14}$$

where $T_n : [0,\, 1] \to [0,\, 1]$, are different dynamical systems, at least some, but not necessarily all, of which are chaotic, and the a_n are the terms of an abolutely convergent series. If f is the inverse of a Gaussian distribution function, then this gives a generalization of a Gaussianly distributed chaotic variable. A continuous version of this construction is

$$g(x) = \int_0^\infty a(s) f(T_s(x)) ds$$

2.2. *Chaotic variables, chaotic time-series, and realizations of chaotic or stochastic processes*

One difference between a random variable and a realization of a independent, identically distributed (iid) stochastic process is that a random variable is defined formally on a measure space, such as $[0, 1]$, of measure 1, and a realization of a iid stochastic process is defined for all time. The stochastic properties of a random variable must also exist for realizations of iid stochastic processes. Similar distinctions hold for chaotic variables and realizations of chaotic processes. With these distinctions made, we proceed to the main discussion of this section.

An important distinction must be made between chaotic variables, and later chaotic processes, and solutions of chaotic difference and differential equations. We conventionally refer to the latter as chaotic time-series. Chaotic variables arise from the superposition of a large number of chaotic time-series. We recognize that this distinction is not yet precise, but it is adequately descriptive.

Chaotic time-series are the building blocks, and thus the source, of chaotic variables, processes, and, we argue, stochastic processes. Chaotic time-series lead to chaotic variables and random variables in at least two ways. The preceding constructions of chaotic variables is one route. A second route is a result of the ratio of the sampling rate to the time-scale of the series. To make this point clear, we graph a chaotic time-series using three sampling rates and compare the integral of the squares of the sampled points for each of the three techniques. Figure 2 shows the three graphs. In each graph, the red curve is the plot of the average of the sum of square of the time-series samples. This is essentially a simple integration scheme. For a highly oscillating function we have the approximate relationship:

$$\int_0^t g(x)^k dx \approx c_k \cdot t \tag{15}$$

for $k = 2$, as was found in Fig. 1. In Fig. 2(a), the time-series is sampled at a rate sufficient to reveal its true form, i.e. sampling at any higher rate provides no new information. The integral of the square is, therefore, highly variable. In Fig. 2(b) the time-scale is compressed and the sampling rate held constant so that the total number of points in the graphs are constant. The integral of the square reflects the increasing degree of complexity in the time-series. In Fig. 2(c), the time-series appears almost as complex as Fig. 1, and the integral reflects this increase in variability of the time-series as it converges toward a straight line.

This example shows that a chaotic time-series may be sampled in such a way as to begin to appear stochastic. Alternatively, if a chaotic time-series has a very high oscillation rate it can approach a realization of a chaotic or stochastic variable in its appearance. For example, this can happen when

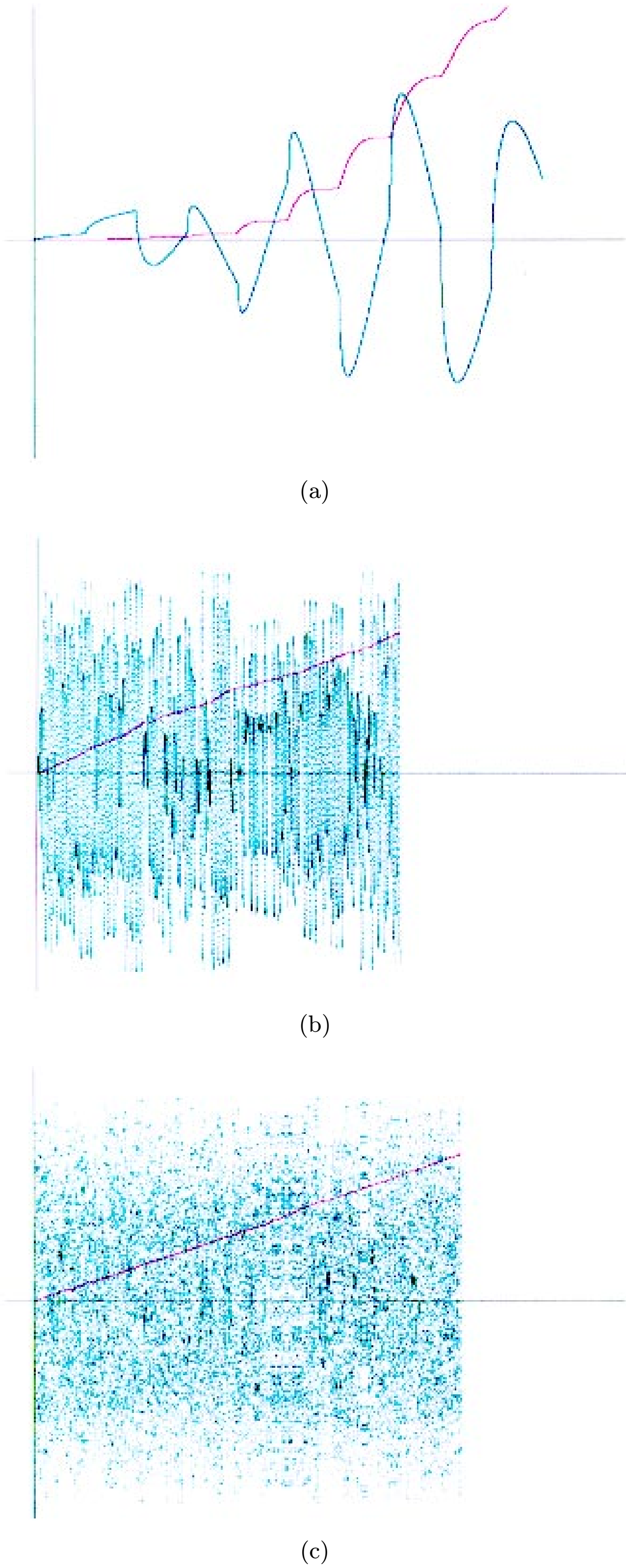


Fig. 2. (a) In this figure we graph the time-series of one component of the solution of a twist-and-flip map equation having gyration conductance function $\log(0.69 \cdot \exp(\sin(2 \cdot \pi \cdot r))/r)$ for $r \leq 1$ and 5 for $r > 1$, but in (b) we have compressed the time-scale by a factor of 150, and in (c) the time-scale is now compressed by a factor of 700.

the system is time compressed as when one samples the time-one map of a chaotic time-series in place of the time-series itself. This point-of-view provides a physical interpretation of the Smale–Birkhoff theorem and illustrates the difficulty in distinguishing between chaotic time-series and realizations of chaotic or stochastic processes. It also explains why if the function $\sin(t)$ is sampled in exponential time, it will produce a chaotic sequence as in the logistic equation.

We note that Eq. (15), for $k = 2$, is not a sufficient condition to identify a realization of a chaotic or stochastic process as demonstrated by the following example:

$$\int_0^x \sin^2(\lambda s) ds \approx c \cdot x \tag{16}$$

for large λ. This function has only one very high frequency and a uniform sampling scheme will reveal that this function is not a realization of a chaotic or stochastic process.

A function having a large number of frequencies and which satisfies Eq. (15) for $k = 2$ is also not necessarily a realization of a chaotic or stochastic process. What is needed is a chaotic range of frequencies and/or a chaotic range of amplitudes combined with satisfying Eq. (15) for $k = 2$ for a measurable function of x to be a realization of a chaotic or stochastic process. Thus, while not defining chaotic or stochastic phenomena, Eq. (15) for $k = 2$ can be used to distinguish between chaotic time-series and realizations of chaotic and stochastic processes. However, if Eq. (15) holds for a large number of integer values of k, then it may be sufficient to define a chaotic or stochastic process. This is still an open question.

A conclusion of this example is that the complete distinction between realizations of chaotic or stochastic processes and chaotic time-series is reduced to the ratio of the sampling rate to the time scales: The limit of a chaotic time-series, as we compress the time-scale and hold the sampling rate constant, appears to be a realization of a stochastic or chaotic process.

Also, Fig. 1 illustrates that the limit of a superposition of chaotic time-series as the number of time-series approaches infinity, where the time-scale and sampling rates are held constant, is a chaotic variable which cannot be distinguished from a random variable by any measurement process.

This suggests that there is a duality, when considering chaotic dynamical systems, between compression of the time-scale and an increase in the number of systems: The dynamical features of a single chaotic system under time-scale compression can be equivalent to the combined dynamics of a large ensemble of chaotic dynamical systems considered in uncompressed time.

This duality concept can be further illustrated. Consider the logistic equation $x \to 4x(1-x)$. Its solution is $x_n = \sin^2(2\pi 2^n \theta_0)$. Thus the logistic time-series in real time is equivalent to sampling a periodic process in exponential time, $t = 2^n$, or the superposition of an uncountable number of periodic time-series, a Fourier transform sampled in real time. A chaotic variable can be equivalent to a chaotic time-series sampled in exponential time or to the superposition of an infinite number of chaotic time-series sampled in real time. Thus the nature of exponential time sampling is to complexify dynamics in the same way as multiplying the number of dynamical systems in real time.

A real-world example provides an interesting consequence of this duality. Using a uniform sampling rate to measure a periodic signal source that is initially accelerating at an exponential rate results in obtaining a chaotic sample of the signal. However, as the velocity of the signal source approaches the speed of light, order is restored.

3. Chaotic Analog of Stochastic Processes

Starting from Eq. (3) we may form an analog of a stochastic process as follows:

$$g(x, t) = g_t(x) = \sum_{n=0}^{\infty} a_n f(T^n(x+t)) \qquad (17)$$

If f is the inverse of a Gaussian distribution, then the process is the chaotic analog of a Gaussian stochastic process. For any choice of x, this gives a Gaussian process in t, and vice versa, for T, a measure preserving chaotic map of the interval $[0, 1]$. For any choice of the sequence a_n, so long as the sum of a_n converges, we can obtain a mean zero and variance 1 process by normalizing $g(x, t)$ for $t = 0$.

As a result, the entire process has these features. This works because g is a function of $x + t$.

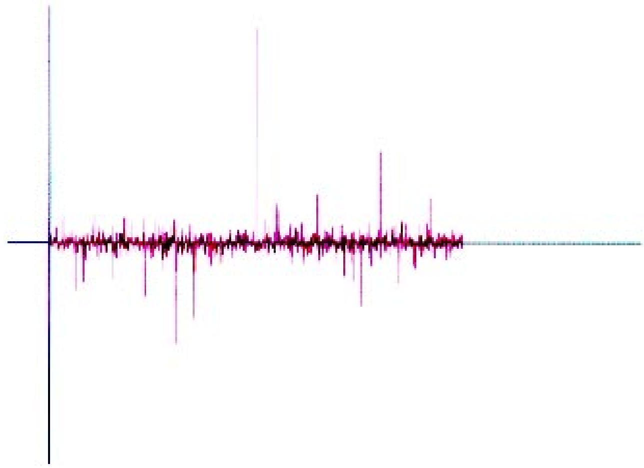


Fig. 3. The covariance of $g_t(x)$ and $g_s(x)$ of Eq. (9), where $a_n = 0.165^n$, $t = 0.5$, $0 < s < 1$, and $T(x) = \{229.1 \cdot x\}$. 60 terms are used, and $f(x) = \tan(2 \cdot x - 1)$. The deviations from white noise covariance is an important feature in chaotic processes.

By choosing

$$g(x, t) = g_t(x) = \sum_{n=0}^{\infty} a_n f(T^n(x \exp(t))) \qquad (18)$$

we obtain another chaotic process. Figure 3 is a graph representing the degree of independence of g_{s+t} and g_s, for Eq. (18). The lack of correlation between the process at s and t is due to the sensitive dependence on initial conditions of the chaotic dynamical system used in the construction. In particular the vertical axis is the covariance:

$$h(s) = \int_0^1 g_{0.5}(x) g_s(x) dx \qquad (19)$$

where s ranges from 0 to 1. If we could graph an ideal white noise *stochastic* process, we would not see the small irregularities that we see in Fig. 3. However, the irregularities in Fig. 3 are to be expected in chaotic processes. They represent chaotic resonance that can appear when chaotic dynamics reinforce one another in unpredictable ways.

If in Eq. (17), f is the inverse of a Gaussian distribution function, we get the analog of a Gaussian white-noise chaotic process. For any t, and for fixed x, this is a function whose absolute value is measurable, and is thus integrable. The integral

$$\int_0^t g_s(x) ds \qquad (20)$$

could be considered the chaotic analog of Brownian motion, if Brownian motion were actually the integral of white noise. As such, it is smoother than stochastic Brownian motion in that it is differentiable.

In analogy with stochastic processes we will refer to Eq. (17) as $W(x, t)$ and Eq. (20) as $B_0(x, t)$ when f is the inverse of a Gaussian distribution function. The dependence on x, the "random" variable, is shown explicitly to distinguish it from its stochastic analog.

We do not consider Eq. (20) to be the final word on the chaotic analog of Brownian motion, hence we label it $B_0(x, t)$ and have reserved the notation $B(x, t)$ for a form of Brownian motion to be discussed in the next section. There we specifically examine Brownian motion as it is used in stochastic processes and, while Brownian motion models can be constructed by using coarse approximations of the integral of white noise, this approximation does not converge to Brownian motion in the limit. Hence, there is something missing when we start with the discrete theory of Brownian motion and try to pass to the limit to obtain continuous Brownian motion.

We conclude by noting that there are as many chaotic processes as one may imagine. For example, in analogy with Eqs. (17) and (18) we may consider the processes,

$$g(x, t) = g_t(x) = \sum_{n=0}^{\infty} a_n f(T^n(x(1+t^2))) \qquad (21)$$

or

$$g(x, t) = g_t(x) = \sum_{n=0}^{\infty} a_n h_n(t) f(T^n(x)) \qquad (22)$$

4. Chaotic Analog of Brownian Motion

Constructing Brownian motion directly without constructing white noise still requires constructing discrete white noise, and thus a uniform random variable, hence it is not constructable as a measurable process. However, if we assume that white noise can be constructed, then, as we have seen, Brownian motion cannot be the integral of white noise. In order to understand the conventionally understood relationship of Brownian motion to white noise, we examine a typical construction, which assumes the existence of a uniform random variable, such as that found in [Karlin & Taylor, 1975, Chap. 7]. This construction depends on the construction of a continuous function that is nowhere differentiable. Such constructions are a subject of classical analysis that go back to Weierstrass. The construction of Karlin and Taylor starts with the Haar functions, derives the Schauder functions, $S(t)$, and then Brownian motion. Given $S(t)$, the representation from Karlin and Taylor of $B(t)$ is

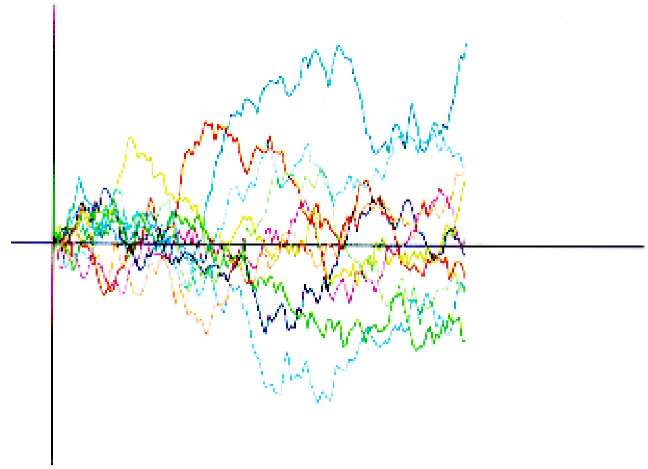


Fig. 4. Ten sample paths for Brownian motion model constructed from the Schauder functions following [Karlin & Taylor, 1973] where a random number generator is used to obtain the Gaussian "random" samples.

$$B(t) = \sum_{1}^{\infty} \xi_k S_k(t) \qquad (23)$$

where the ξ_k are independent mean zero, variance one, Gaussian random variables. In Fig. 4 we show the paths of ten sample functions using the Schauder function construction of Brownian motion.

We clarify the exposition of Karlin and Taylor with the observation that all of the Schauder functions can be obtained from the single function:

$$h(t) = 1 - |2\{t\} - 1|$$

by iteration of a chaotic map. Here $\{t\}$ is the fractional part of t. Specifically, by considering the iterates of h under the unilateral shift $T : t \to \{2 \cdot t\}$ we clarify their presentation to get

$$B(x, t) = W(x, t_0) \cdot t + \sum_{k=1}^{\infty} W_k(x, t) \cdot \frac{h(T^{k-1}(t))}{2^{(k/2)-1}} \qquad \text{(BRN)}$$

where

$$W_k(x, t) = \sum_{j=1}^{2^k-1} W(x, t_j)\chi_j(t)$$

$W(x, t)$ is a Gaussian white noise process, and $\chi_j(t)$ is the indicator function for the interval $[2^{j-1}, 2^j]$. We note that in the construction found in [Karlin & Taylor, 1975], $W_k \to W$ in the L_1 sense, and that W, in their presentation, must be thought of as ideal white noise.

Our rearrangement of the classical Brownian motion formula suggests that it is possible to draw a closer relationship to chaos if we can construct the indicator functions from dynamical systems. This can be done as follows: By the formula $f \vee g$ we will mean the maximum of f and g. Next, let $f(t) = \text{sgn}(\sin(2\pi t))$, and $f_j(t) = f(T^j(t)) = f_{j-1}(T(t))$, where $T(t) = \{2 \cdot t\}$. Finally, define iteratively

$$f_{j+1,k}(t) = (-1)^{(k-1)} \cdot f_{j+1}(t) \cdot (f_{j,1+k \bmod (2^j)}(t) \vee 0) . \tag{24}$$

With this dynamical system formulation we have

$$W_k(x, t) = \sum_{j=1}^{2^k-1} W(x, T^j(t_0)) f_{j,k}(t) . \tag{25}$$

This rearrangement of B shows that the relationship between W and B is neither one of integral or derivative.

If we define W in Eq. (25) by Eq. (17), Brownian motion is realized as an infinite dimensional chaotic process.

In summary, Fig. 1 shows that as g in Eqs. (3) and (17) becomes more uniform, the integral of g approaches a straight line whose slope is the mean value of g. If we use Eq. (17) to construct a mean zero Gaussian process, then this integral must be nearly zero. The more perfectly we construct our mean zero Gaussian chaotic variable, the more closely its integral must approach zero. In the limit of this process, Brownian motion, as the integral of white noise, must cease altogether. If we choose to think of Brownian motion as the integral of a Gaussian chaotic white noise processes, then we obtain the function $B_0(x, t)$, which is differentiable. As the chaotic processes become more uniform, $B_0(x, t) \to 0$ and to the extent they are less uniform, $B_0(x, t)$ models conventional Brownian motion. We also find that the process $B(x, t)$ is constructable as a chaotic process and is thus derivable from chaotic dynamics as well. However, $B(x, t)$ and $B_0(x, t)$ only coincide in appearance for a limited range of chaotic processes.

Thus we have the chaotic analog of Brownian motion in Eq. (BRN) and Eq. (20) if we use Eq. (17) for our white noise process.

We emphasize at this point that we are led to this model of chaotic white noise by imposing the restriction that its integral, $B_0(x, t)$, must provide a credible candidate for Brownian motion if it is approximated with a coarse numerical integration technique, and that it coincides in appearance with $B(x, t)$ in accordance with conventional practice. Once we have made a commitment to this model, we convey with it the implications that can be derived from using the model. Among these implications is found, see Fig. 3, the phenomena of chaotic resonance, which may be the chaotic analog of stochastic resonance, and will be discussed in the next paragraph. A second phenomena, perhaps more unexpected, is the phenomena we are calling *chaotic catastrophe* which is unrelated to what is commonly known as catastrophe theory. This is discussed in the section on financial markets. Chaotic catastrophe occurs when Brownian motion transitions from B_0 to B. A third consequence of this theory is that if we start with a coarse model of a chaotic analog of white noise, $B_0(x, t)$ is not constant. If we then add more chaotic noise to our model to force it to approach a uniform Gaussian white noise, then $B_0(x, t) \to 0$ and the effects of noise are cancelled out, essentially by adding noise rather than removing it. Fourth, the determination of whether the Ito or Stratonovich theory applies can be made by determining whether the forces involved in a process are primarily frequency chaotic, leading to $B_0(x, t)$, or amplitude chaotic, leading to $B(x, t)$. When both are involved, a hybrid theory may be appropriate.

Stochastic resonance is described as a phenomena whereby the presence of noise actually enhances a signal rather than obscures it. Chaotic resonance is the coincidental correlation that appear to be random in occurrence, but which enable complex phenomena to reinforce one another in a manner that might enable the emergence of new species or other dynamical events which have an order to them, i.e. are not formless and "random". The vanishing of B_0 as noise becomes more uniform may connect these two ideas. When B_0 exists, it is due to a lack of uniformity in a noise process, thus there must exist chaotic resonance. As $B_0 \to 0$ during a period of increasing noise, or decreasing chaotic resonance, a signal hidden in the noise will also appear just as the integral of a uniform process in Fig. 1 converges to a constant multiplied times time. Thus increasing noise, i.e. making it more uniform, looks like stochastic resonance as described earlier.

Thus we see that chaos provides two mechanisms for what might be called *emergence*. First, through the presence of some level of chaotic resonance complex order appears or emerges; second from the total absence of chaotic resonance, $B_0 \to 0$ and a hidden dynamical system may emerge from an increase in the level of uniformity of noise.

The second form of emergence may have a bearing on the natural neural noise in the human brain. For example, particular signals may selectively appear as indications of specific thought or other cognitive processes when the level of neural noise is made more uniform according to some deriving force. Thus, cognitive processes may appear as a result of an increase in the uniformity of neural noise. Also, cognitive processes may appear as a result of a decrease in the level of uniformity of neural noise as this will give rise to chaotic resonance. In both cases, a robust source of neural noise may be the ideal medium for the generation of cognitive processes. Each of these processes is linked to the other through B_0. As $B_0 \to 0$ the first process is engaged, as $B_0 \to B$, the second process is engaged.

4.1. *Amplitude chaos and Brownian motion*

The graphs of the Schauder functions, $S_k(t)$, in Eq. (23) are little tents. Each function used is nonzero on an interval of length $1/2^k$ and zero elsewhere. By grouping these terms by the length of the domain where the function is nonzero we obtain sums of the form

$$\sum_{j=1}^{2^k} h(t_j)S_j(t)\,. \qquad (26)$$

If, instead of tent shaped graphs, we used functions whose graphs were like $1 - \cos^2(2\pi \cdot 2^k t)$ over the same interval we would obtain a segment of a solution to a differential equation which has amplitude chaos much like Example 2, Sec. 5 of [Brown & Chua, 1998]. Solutions to such chaotic amplitude equations all have the same frequency but with each segment of the solution over one period of the forcing function having a different amplitude, the sequence of amplitudes being chaotic. The situation is much like the solutions of the chaotic Duffing equation, except that only the amplitude changes chaotically instead of both amplitude and frequency changing.

The net result of this discussion and that of white noise is that classical Brownian motion and white noise can be realized as the superposition of an infinite set of time-series solutions from chaotic differential equations having amplitude chaos, in the case of Brownian motion, and frequency chaos in the case of white noise.

5. Other Stochastic Processes

In this section we construct analogs of continuous, wide-sense stationary, Poisson, and other processes.

5.1. *Continuous chaotic processes*

To this point we have constructed chaotic analogs of stochastic processes which contain the assumed discontinuity features of stochastic processes. In this section, we construct continuous analogs and note that from a measurement point of view, it may be impossible to detect the difference between continuous and nowhere continuous stochastic or chaotic processes.

As a special case of Eq. (17) we have

$$g_t(x) = \sum_{n=0}^{\infty} a_n \sin(2^n \cdot 2\pi(x+t)) \qquad (27)$$

where we choose $a_n \cdot 2^n > 1$. This series converges uniformly and is thus continuous, but is not differentiable term by term. If we only use a finite number of terms, the function has all derivatives and is periodic. However, if we use a very large number of terms, then it begins to resemble white noise. Also, when we use only a finite number of terms, it must be the solution of a homogeneous linear differential equation of very high order. If we use an infinite number of terms, it must solve, in a formal sense, a second-order hyperbolic linear partial differential equation having as a boundary condition a chaotic variable. All of these features demonstrate that there may be a very fine line between periodic and nonchaotic phenomena and chaotic or stochastic processes. If we are given a sampling rate, it is always possible to use a large enough number of terms to assure that the sampling rate will fail to reveal that the equation is periodic and not classical white noise.

With a small modification, the periodicity can be abolished. For example we may use:

$$g_t(x) = \sum_{n=0}^{\infty} a_n \sin(2.1^n \cdot 2\pi(x+t)) \qquad (28)$$

A further property of these processes is something analogous to sensitive dependence on initial conditions. Ideally this property should read: There exists a number τ such that given a point x_0 and a neighborhood of the point, U_{x_0}, there is another point $x \in U_{x_0}$ with $|g_t(x) - g_t(x_0)| > \tau$, for almost all t. While this cannot be true for a continuous function, it can be true in practical circumstances when the sampling rate, for a fixed choice of the a_n, cannot confirm or deny the continuity of the function. Thus, relative to a sampling rate, it can happen that the function may as well not be continuous. We conclude that relative to a given measuring frequency it may be impossible to decide if a process is stochastic and discontinuous or chaotic and continuous. Further, given any measurement process, we may always construct a finite dimensional, infinitely differentiable chaotic process for which it is impossible to distinguish said process from an infinite dimensional, totally discontinuous stochastic process. For example, it is possible to construct a twist-and-flip map where the integral curves are diamond shaped, and only the amplitude is chaotic, thus allowing us to mimic Brownian motion exactly with a superposition of a finite number of twist-and-flip chaotic processes.

5.2. *Wide-sense stationary processes and chaos*

Every measure-preserving mapping of a measure space defines a stationary stochastic process. Conversely, every stationary processes can be formally understood as arising from measure-preserving dynamical systems. This is the subject of ergodic theory. A close review of the proof of this fact in [Doob, 1953], reveals that this line of thought, while accurate, could benefit from a more direct example of how chaotic and stationary processes overlap.

The most general wide-sense real-valued stationary process can be put into the form

$$x(t) = \sum_{j=1}^{k} u_j \cos(2\pi\lambda_j t) + v_j \sin(2\pi\lambda_j t)$$

where u_j, v_j are mutually orthogonal real random variables, or can be approximated arbitrarily close by a process of this form, [Doob, 1953], where the λ_j depend on the process.

If we take $u_j = a_j \sin(2\pi 2^j x)$, $v_j = a_j \cos(2\pi 2^j x)$, and $\lambda_j = 2^j$, then we have the chaotic process

$$g(x, t) = \sum_{j=1}^{k} a_n \sin(2\pi 2^j (x + t))$$

discussed earlier. By our choice of the sequence a_n, and the frequencies λ_j we can make this process as chaotic as we wish.

In general, we may construct the wide-sense stationary process

$$\sum_{j=-\infty}^{\infty} a_j f_j(x) \exp(2\pi i \lambda_j t)$$

as a chaotic process by choosing the f_j appropriately.

5.3. *Poisson processes*

The basic construction of a Poisson process from chaotic processes is illustrated by the following example:

$$f(x, t) = \sum_{n=0}^{\infty} \alpha_n f_n(x, t) \tag{29}$$

where

$$f_n(x, t) = 0.5 \cdot (1 + \mathrm{sgn}(t - g_n(x)))$$

and

$$g_n(x) = \sum_{k=1}^{n} \beta_k h(T^k(x))$$

T is a chaotic map of the unit interval, α_n, β_n are constants chosen to fit the data, and h is a properly chosen function. For example, one choice for h is $\exp(-\lambda \cdot u)$

5.4. *Martingale and Markov processes*

In this section, we will be using conventional terminology found in leading textbooks on stochastic processes, particularly [Doob, 1953].

In order to define a Martingale we need the notion of conditional expectation. Given two measurable functions, f, g on $[0, 1]$, we define the conditional expectation of f, given that $g(x) \in [a, b]$ by the average value of f over the interval $g^{-1}[a, b]$, and we write this as

$$\mathbf{E}(f | g(x) \in [a, b]) = \frac{1}{\mu(g^{-1}([a, b]))} \int_{g^{-1}([a,b])} f(x) dx$$

where $\mu(\cdot)$ may be thought of as Lesbegue measure.

In ergodic theory, conditional expectation is defined with respect to a partition of the domain of a measurable function as follows: Let $\mathcal{P} = \{E_i\}$ be a partition of $[0, 1]$. Then

$$\mathbf{E}(f|\mathcal{P}) = \sum_i \frac{1}{\mu(E_i)} \cdot \int_{E_i} f(x)dx \cdot \chi_{E_i}(x) .$$

The conditional expectation of f takes on the average value of f over each element of the partition, E_i. If we start with a partition of the range of g and take its inverse under g, let the granularity of the partition increase indefinitely and apply this definition, we obtain the familiar definition of conditional expectation found in elementary texts on probability theory.

Given the definition of conditional expectation, a stochastic process ξ_t, is a martingale, if each random variable has a finite mean value, and if $t_1 < t_2 < \cdots < t_n < t$ then

$$\mathbf{E}(\xi_t|\xi_{t_1}, \ldots, \xi_{t_n}) = \xi_{t_n} . \tag{30}$$

In our terminology, the random variables ξ_t are measurable functions, such as $g(x, t)$ of Eq. (17), on $[0, 1]$, and the implications of this condition is just a routine statement about a series of nested partitions of $[0, 1]$. The most common example of a Martingale is the sum of a sequence of independent random variables having mean zero. This equates to a sum of chaotic functions of the form $g(x, t_n)$ of Eq. (17).

The definition of a Markov process is parallel to that of a Martingale with the notion of conditional probability replacing that of conditional expectation. The conditional probability of a set B given a partition, $\mathcal{P}$, is defined as $\mathbf{E}(\chi_B|\mathcal{P})$. As with a Martingale, there is nothing special about this process that is not already discussed. A sum of independent random variables, or chaotic functions, defines a Markov process.

5.5. *Levy processes*

A Levy process can be decomposed into a sum of a composite Poisson, a translation, and a Brownian motion process. As these processes are constructable from chaotic processes, there is nothing further needed to construct a Levy process from chaotic processes than has already been discussed.

6. Stochastic Differential Equations and the Stochastic Calculus

In this section we examine chaotic analogs of stochastic differential equations and the role of the stochastic calculus.

6.1. *Chaotic analog of stochastic differential equations*

In direct analogy with the theory of stochastic dynamical systems, we have the systems, see [Karlin & Taylor, 1981],

$$\frac{dY(t)}{dt} = h(t, Y(t), W(t, x)) \tag{31}$$

$$\frac{dY(t)}{dt} = h(t, Y(t)) + W(t, x) \tag{32}$$

$$\frac{dY(t)}{dt} = -\beta Y(t) + W(t, x) , \quad \beta > 0 \tag{33}$$

$$\frac{dY(t)}{dt} = -\mu(Y(t), t) + \sigma(Y(t), t) \frac{dB_0(t, x)}{dt} \tag{34}$$

$$dY(t) = -\mu(Y(t), t) + \sigma(Y(t), t)dB(t, x) \tag{35}$$

Equation (33) is the analog of the Ornstein–Uhlenbeck equation, and Eq. (34) is the analog of the standard stochastic differential equation model for a diffusion processes. Equation (35) is needed in the theory of chaotic processes, just as in the case of stochastic processes, to be able to handle the chaotic process $B(x, t)$, which is nowhere differentiable.

For chaotic processes, the integral

$$\int_0^t f(s)dB_0(s, x) \tag{36}$$

exists. This equation has the representation

$$\int_0^t f(s)W(s, x)ds \tag{37}$$

This implies that Eq. (34), as a chaotic process, can be written as

$$\frac{dY(t)}{dt} = -\mu(Y(t), t) + \sigma(Y(t), t)W(t, x) . \tag{38}$$

We may construct g of Eq. (17) so that, to a very close approximation,

$$\int_0^1 W(t, x)W(s, x)dx = \int_0^1 W(t, x)^2 dx\,, \quad \text{for } s = t\,, \quad (39)$$

and nearly zero elsewhere, and

$$\int_0^1 W(t, x)^2 dx \approx 1\,. \quad (40)$$

Also,

$$\int_0^t W(s, x)^2 ds \approx c \cdot t\,. \quad (41)$$

The existence and uniqueness theory for Eqs. (31)–(34) can be found in [Coddington & Levinson, 1955]. In general, there exists solutions of these equations. When the right-hand sides are Lipschitz, these solutions are unique.

Integration of stochastic differential equations may proceed through two separate paths, resulting in very different solutions. The theory of Stratonovich requires that the rules of calculus must hold for these equations. The theory of Ito holds that Martingales should be preserved through integration even if it means that the rules of calculus fail. See [Karlin & Taylor, 1981] for a complete discussion.

The rules of calculus hold for chaotic processes that involve B_0, thus the solution of Eqs. (31)–(34) are in agreement with the Stratonovich stochastic calculus. However, processes involving $B(x, t)$ are solved using the Ito theory. Thus, in the theory of chaotic processes, the Ito theory and the Stratonovich theory are both required.

6.2. *Example: Solving chaotic analogs of stochastic differential equations*

With the above results, Eq. (33), the Ornstein–Uhlenbeck process, has the same solution as if it were treated as a stochastic ODE, with the exception that the solution involves $B_0(x, t)$ instead of $B(x, t)$. All chaotic process equations which do not involve the Ito calculus, and thus involve B_0 instead of B, are solved in the same way as their stochastic analogs. For Eq. (33) we have the solution,

$$Y(x, t) = B_0(x, t) - \beta \int_0^t \exp(-\beta(t-\tau))B_0(x, \tau)d\tau\,. \quad (42)$$

Expectations and variances are easily computed by using the formulae

$$E[Y(x, t)] = \int_0^1 Y(x, t)dx \quad (43)$$

$$E[Y(x, t)Y(x, s)] = \int_0^1 Y(x, t)Y(x, s)dx \quad (44)$$

The Growth Equation is a good example of the differences between the Ito solution and the S-solution of a stochastic differential equation. Our construction must necessarily agree with the S-solution if we use B_0 instead of B. We proceed as follows: We rewrite the growth equation

$$dY = Y dB_0 \quad (45)$$

as

$$\frac{dY}{dt} = Y(x, t)\,\frac{dB_0}{dt} = Y(x, t)W(x, t)\,. \quad (46)$$

By applying the usual rules of calculus we have

$$Y(x, t) = C(x) \cdot \exp(B_0(x, t))\,. \quad (47)$$

The Ito solution is

$$Y(x, t) = C(x) \cdot \exp\left(\frac{B(x, t) - t}{2}\right)\,. \quad (48)$$

These two solutions of Eq. (45) are quite different depending on the underlying dynamics.

6.3. *Chaos and the financial markets*

The existence of the chaotic processes $B_0(x, t)$ and $B(x, t)$ lead to the possibility of processes being formed which coincide with B_0 over some time spans and with B over others. For example, the solution of the following equation

$$h(x, t) = 0.5(1 + \text{sgn}(\sin(\omega t))) \cdot B_0(x, t) + 0.5(1 - \text{sgn}(\sin(\omega t))) \cdot B(x, t) \quad (49)$$

transitions between these two processes on a periodic basis when ω is an integer.

This possibility may have significant consequences for the financial analysis of derivative securities. Presently, market analysts use the Black–Scholes option pricing model for derivative pricing. Black–Scholes in turn assumes that the underlying processes are B and not B_0. This leads to the use of the Ito transformation law in the treatment of derivative security pricing equations. However, if the relevant processes involve B_0, then errors in speculating on derivative securities may result that lead to disastrous consequences. The prospect of this happening leads us to define chaotic catastrophe.

Definition. A chaotic catastrophe occurs when a chaotic process involving B_0 transitions to one that also involves B, or vice versa.

This definition allows for the situation where B_0 and B may be simultaneously present although this may seem ambiguous. This is called a *catastrophe* because the solution of a given problem such as the growth equation, Eq. (45), will transition from Eq. (47) to Eq. (48) without warning. The solution given by Eq. (47) is differentiable almost everywhere whereas the solution given by Eq. (48) is nowhere differentiable as-well-as having a factor of $\exp(-t/2)$ not present in Eq. (47). Betting on the solution of Eq. (45) being Eq. (48), when in fact the solution, over a short period of time, is actually Eq. (47) could result in a significant loss.

This observation underscores the need for methods of analysis to distinguish between dynamical systems involving B_0 and those involving B. This need is made more clear by the fact that over certain approximation ranges, using coarse approximation methods, B and B_0 appear to be the same.

7. Stochastic Chaos as a Driving Force

Consider the chaotic process given by

$$s(x,\, t) = \sum_{n=0}^{\infty} a_n \sin(b^n \cdot (x+t)) \tag{50}$$

where $b > 1$, and the a_n chosen so that

$$\int_0^y s(x,\, t)^2 dx \approx c(t) \cdot y\,. \tag{51}$$

If we truncate this series after any finite number of terms, it is infinitely differentiable, and, if b is an integer, the truncated series is periodic while Eq. (51) is still true. That is,

$$s(x,\, t) = \sum_{n=0}^{\mathrm{N}} a_n \sin(b^n \cdot (x+t)) \tag{52}$$

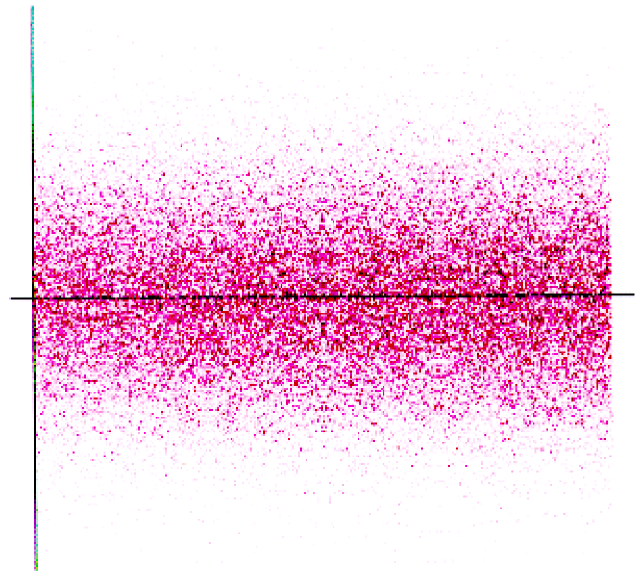


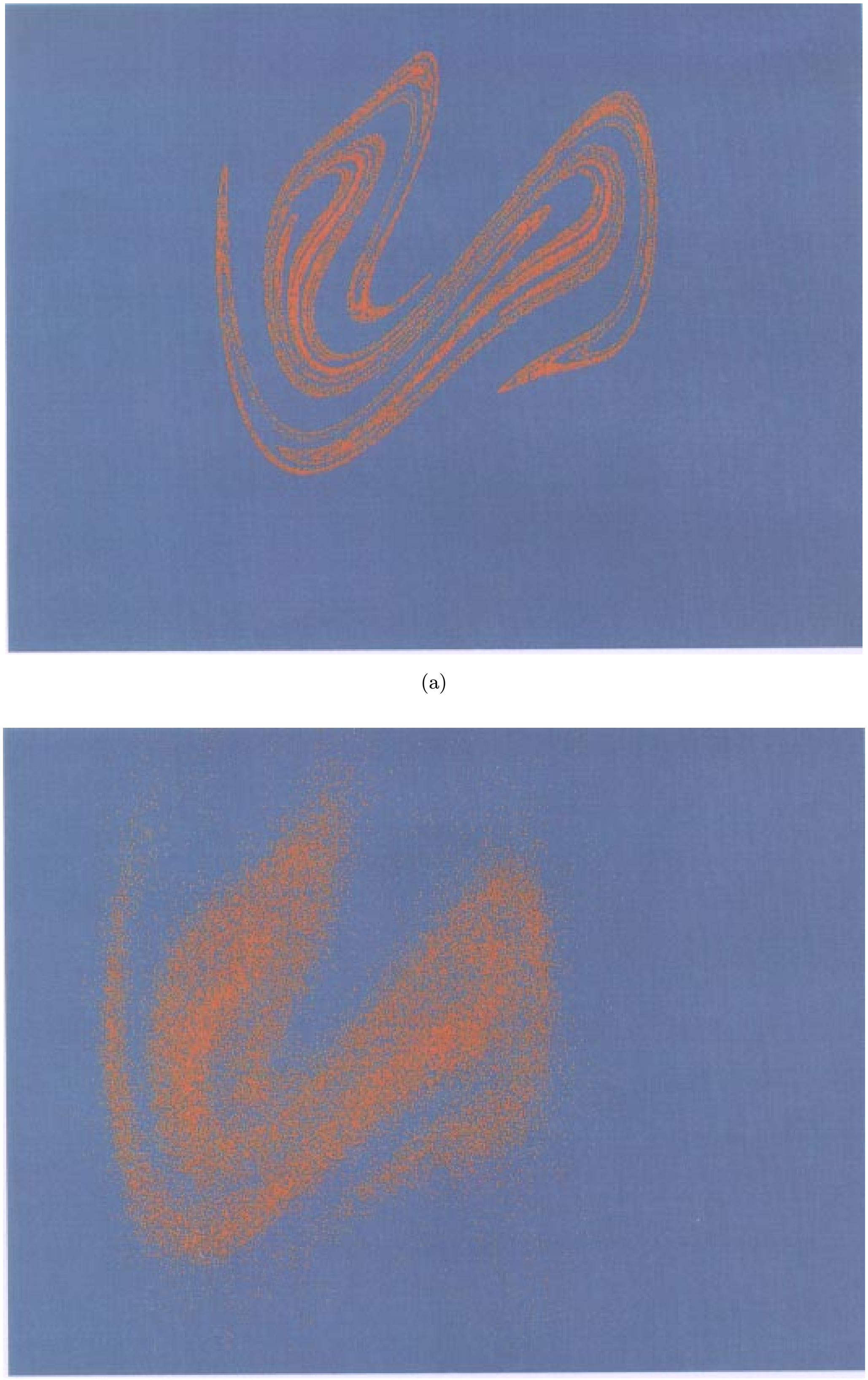
Fig. 5. The graph of $s(0.5,\, t)$, Eq. (42), as a function of t, where $T(x) = \{16807 \cdot x\}$, $a_n = 0.995^n$, and ten terms are used.

is not distinguishable from a stochastic process for large N, say $N > 10$. This is due to the sensitive dependence on sampling that characterized chaotic processes, see Fig. 5. We may use this as a driving force in a second-order ODE such as the Duffing equation:

$$\ddot{y} + \alpha\dot{y} + y^3 = \sum_{n=0}^{N} a_n \sin(b^n \cdot (x+t)) \tag{53}$$

which can be expressed as an autonomous equation in three dimensions. If $\alpha = 0.05$, $a_0 = 7.5$, and $a_n = 0$ for $n > 0$ this equation is the well-known Duffing equation having the Ueda Japanese attractor shown in Fig. 6(a).

When $a_n \neq 0$, b an integer, this is in a practical sense equivalent to the Duffing equation having a stochastic forcing term. But clearly, this is just a very complex periodic forcing term that resembles a noise process, as Fig. 5 shows. The first return map for the solution of Eq. (53) exists on $\mathbf{R}^2 \times S^1$. A hyperbolic fixed point can be located near (1.04, 0.58) and the first return map generates a strange attractor, Fig. 6(b). Due to monitor resolution limitations, this attractor does not show the fine detail seen in Fig. 6(a). However, the attractor is what we would expect to see from a system having a stochastic forcing term in that the attractor is diffusing under the influence of the chaotic forcing process.



(a)

(b)

Fig. 6. The strange attractor for Eq. (53) where the forcing term is (a) 7.5 $\sin(t)$. A hyperbolic fixed point can be found near (1.04, 0.595); (b) that of Fig. 5, Eq. (42). A hyperbolic fixed point can be found near (1.00067, 0.50099).

8. Filtering of Chaotic Processes

The chaotic analog of the conventional Kalman filter setting is

$$Y(x, t_{n+1}) = \Phi Y(x, t_n) + W_1(x, t_n) \tag{54}$$

$$\psi_n = \theta Y(x, t_n) + W_2(x, t_n) \tag{55}$$

where W_i are chaotic processes that are analogs of Gaussian white noise, Φ, θ are matrices, and $Y(x, t_n)$ is a vector. Clearly, when these equations are modeled on a computer, the relevant noise processes are chaotic, rather than stochastic. The advantage of approaching this problem as a chaotic process problem is that, in reality, white noise does not exist. Hence, the noise processes must deviate from white noise in some fashion. These deviations may be taken into consideration in the estimation process, thus providing additional information about the dynamics that will lead to a better estimation formula. By assuming Gaussian white noise and proceeding with the use of conditional expectations along conventional lines, valuable information may be omitted. A better approach is as follows. First, perform a spectral analysis of the noise. From this analysis, and the facts of the physical environment, the likely chaotic dynamics involved may be determined. As we have noted, white noise can be constructed from superposition of a wide range of chaotic dynamics, as was the case from Brownian motion and Poisson processes. Each of these constructions will have a deviation from actual white noise that is peculiar to the situation. If we derive the chaotic dynamics involved and use a superposition of these dynamics to formulate our dynamical system estimation method, we will likely obtain a better estimation process as it will be much closer to reality than the assumption of white noise will provide.

This theory does not diminish the significance of stochastic processes. Stochastic process theory is appropriate when the number and variety of chaotic dynamical systems that are present in a superposition processes are so large that there is no practical value to be gained from considering the chaotic dynamics separately. It is likely however, that there are many problems that have been traditionally treated with stochastic process methods that are likely better treated with chaotic process methods.

9. Theory of Chaos

9.1. *Stochastic processes arise from chaos*

Every chaotic process we have discussed *formally* fits the definition of a stochastic process, i.e. a one-parameter family of measurable functions. Every chaotic variable is, likewise, a *random* variable. However, without the aid of the definition of random selection, the reverse is not true in any useful sense. The problem still lies with the fact that there is no universally accepted way of defining *random selection* within the framework of the foundations of mathematics. Chaotic processes need not be sampled randomly, and so we avoid this issue when we are dealing with chaotic processes.

We now define every chaotic process we have discussed as a stochastic process, and retain a single formalism, stochastic processes, within which to discusses these processes. This requires extending the boundaries of stochastic processes to include these processes more directly. Doing this does not limit our formalism. Rather, it provides new avenues for exploration. Traditional stochastic processes is bound to consider only the evolution of processes which are *idealizations* such as white noise, whereas the extension to chaotic processes calls for an examination of imprefect processes; our objective being to exploit these imprefections to improve interpolation, filtering, and prediction of complex phenomona.

As noted in Sec. 2, a superposition of time-series of chaotic dynamical systems leads us to chaotic processes, which, based on our discussion, we will now refer to as stochastic processes. An interesting conclusion of our constructions is that every chaotic dynamical system leads us to a stochastic process in precisely the manner of our example. In fact, each chaotic dynamical system leads us to an array of stochastic processes that range from noise processes to diffusion processes. Our example using the logistic map illustrated this. In an upcoming paper we will explore the formation of stochastic processes from a wide range of chaotic systems further. For now, we observe that given a chaotic dynamical system such as is defined by the Hénon map, there is a noise process and a diffussion process that arises from this system. Likewise we have processes based on the Chua circuit, and the Chirikov map. Appropriately enough, we will call these stochastic processes, Hénon processes, Chua processes, etc. As these systems define an array of processes, we will single out two for which we will have special names: *Noise* processes and *Diffusion* processes. Thus, for the Chua processes, we have Chua noise, and Chua diffusion. For the Hénon processes, we have Hénon noise and Hénon diffusion, and so forth.

The significance of these processes is that if while examining a complex phenomona in a noisy environment, it can be determined that the underlying noise is Chua noise, for example, there may be some advantages obtained in the filtering and prediction of the phenomona.

9.2. *Six conjectures*

Whether or not a *Theory of Chaos* is possible is unanswerable at present. However, we set forth six conjectures, based on our examples and counterexamples in the hope that these conjectures may help lead to a theory eventually.

Conjecture 1. *The spatial and temporal complexity which arises from chaotic dynamics has a special significance in that it is both the source of creative processes and the source of disorder.*

Conjecture 2. *Stochastic processes arise in nature from chaotic time-series in at least two ways: One is when the time-series is sampled in exponential time, or is time-compressed exponentially. The formation of stochastic processes also proceeds through the superposition of a large class chaotic time-series.*

Conjecture 3. *The imprefect stochastic processes formed by chaos can correlate with noncomplex processes to create stochastic resonances.*

Conjecture 4. *Chaotic time-series arise from nonlinear interactions which can be as simple as the composition of two period-two nonlinear dynamical systems. This is demonstrated by the Henon map and many other systems.*

Conjecture 5. *Pure chaotic systems may be classified into three categories: Demiurgic systems, Bernoulli systems and regenerative systems. We refer to the system illustrated by Example 2 of [Brown & Chua, 1997] as regenerative since, while it does not have a positive Lyapunov exponent, it does have a negative Lyapunov exponent that is generating complexity in a dimension separate from the subspace of the negative Lyapunov exponent. Hybrid systems are combinations of these, or have subseries of these embedded in their time series. These subseries may be spread very thin through the seires so long as they form a true infinite subseries. All such systems are examples of* chaos.

Conjecture 6. *A chaotic time-series may appear as a stochastic process depending on the sampling frequency used to measure the time-series. Further, periodic processes may appear chaotic depending on the sampling frequency.*

10. Prime Numbers in Dynamics

This section is unrelated to the first and is included to show that there may be a link between prime numbers and chaos.

In this section we prove three theorems showing that the prime numbers can be used to approximate complex sequences defined by symbolic dynamics. The first theorem is proven in detail and provides an example of how to prove the later theorems which are only sketched. All theorems are sufficiently simple and straight-forward that they are likely already known, at least, informally. We hope that we have not inadvertently omitted credit to the proper authors.

Theorem 1. *The set of rational numbers of the form p/q, where p and q are primes, $p > q$, is dense in the interval* $[0, 1]$.

Proof. We show that these rationals are dense in the rationals in $[0, 1]$.

Let p_n be the sequence of prime numbers. From number theory, see [Hasse, 1980], we know that $p_{n+1} - p_n < p_n^\gamma$, where $0.5 < \gamma < 1$. Define the function $p_+(n)$ as the first prime above n, and $p_-(n)$ as the first prime below n. Let a/b be a rational number in $[0, 1]$.

We have the estimate:

$$\left|\frac{a \cdot n}{b \cdot n} - \frac{p_+(a \cdot n)}{p_+(b \cdot n)}\right| = \left|\frac{a \cdot n \cdot p_+(b \cdot n) - b \cdot n \cdot p_+(a \cdot n)}{n \cdot b \cdot p_+(b \cdot n)}\right|$$

$$\leq \left|\frac{\begin{array}{c} a \cdot n \cdot p_+(b \cdot n) - p_+(a \cdot n)p_+(b \cdot n) \\ +p_+(a \cdot n)p_+(b \cdot n) - p_+(a \cdot n)b \cdot n \end{array}}{n \cdot b \cdot p_+(b \cdot n)}\right|$$

$$\leq \left|\frac{a\cdot n\cdot p_+(b\cdot n) - p_+(a\cdot n)p_+(b\cdot n)}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$+ \left|\frac{p_+(a\cdot n)p_+(b\cdot n) - p_+(a\cdot n)b\cdot n}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$\leq \left|p_+(b\cdot n)\,\frac{a\cdot n - p_+(a\cdot n)}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$+ \left|p_+(a\cdot n)\,\frac{p_+(b\cdot n) - b\cdot n}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$\leq \left|\frac{a\cdot n - p_+(a\cdot n)}{n\cdot b}\right|$$

$$+ \left|p_+(a\cdot n)\,\frac{p_+(b\cdot n) - b\cdot n}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$\leq \left|\frac{a\cdot n - p_+(a\cdot n)}{n\cdot b}\right|$$

$$+ \left|p_+(b\cdot n)\,\frac{p_+(b\cdot n) - b\cdot n}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

since $p_+(b\cdot n) \geq p_+(a\cdot n)$.

$$\left|\frac{a\cdot n - p_+(a\cdot n)}{n\cdot b}\right| + \left|p_+(b\cdot n)\,\frac{p_+(b\cdot n) - b\cdot n}{n\cdot b\cdot p_+(b\cdot n)}\right|$$

$$= \left|\frac{a\cdot n - p_+(a\cdot n)}{n\cdot b}\right| + \left|\frac{p_+(b\cdot n) - b\cdot n}{n\cdot b}\right|$$

$$\leq \left|\frac{p_+(a\cdot n) - p_-(a\cdot n)}{n\cdot b}\right| + \left|\frac{p_+(b\cdot n) - p_-(b\cdot n)}{n\cdot b}\right|$$

$$\leq \left|\frac{p_-(a\cdot n)^\gamma}{n\cdot b}\right| + \left|\frac{p_-(b\cdot n)^\gamma}{n\cdot b}\right|$$

$$\leq \left|\frac{(a\cdot n)^\gamma}{n\cdot b}\right| + \left|\frac{(b\cdot n)^\gamma}{n\cdot b}\right|$$

$$\leq \left|\frac{(b\cdot n)^\gamma}{n\cdot b}\right| + \left|\frac{(b\cdot n)^\gamma}{n\cdot b}\right|$$

$$= 2\left|\frac{(b\cdot n)^\gamma}{n\cdot b}\right|$$

$$= 2(b\cdot n)^{\gamma-1} \to 0$$

as $n \to \infty$, since $\gamma < 1$. ■

If n is an integer and we place a decimal point to the left of n, we obtain a number in the interval $[0,1]$. For example, if $n = 76532$ then the number in $[0,1]$ we get is 0.76532. This number defines many more fractions by considering all of its right shifts, i.e. $0.076532, 0.0076532, 0.00076532, \ldots$. The set of all such numbers formed from integers in this way is clearly dense in $[0,1]$. We may formally describe this set, $\mathcal{N}$, as follows:

For some integer n, and some integer $k > 2$,

$$\mathcal{N} = \left\{x \middle| x = \frac{n}{10^{[(\log_{10} n)]+k}}\right\}$$

where $[y]$ is the integer part of y.

Using this fact we have the following less obvious theorem:

Theorem 2. *The set*

$$\mathcal{P} = \left\{x \middle| x = \frac{p}{10^{[(\log_{10} p)]+k}}\right\}$$

where p is prime, and $k \geq 2$ is dense in $[0, 1]$.

The proof of this theorem follows from the following result:

Theorem 3. *Let n be any integer. Then there is a prime number whose leading digits are the same as those of n.*

Proof of Theorem 3. Let $k = n\,10^{3[\log_{10} n]}$. There exist a prime no further away from k than $y = [k^{2/3}]$. Any number less than $y + k$ and greater than k will have the digits of n as its first $10^{[\log_{10} n]+1}$ digits. Since the set $\mathcal{N}$ is dense in $[0,1]$, so is $\mathcal{P}$. ■

These results show that prime numbers may possibly play a role in dynamics. First, since the ratios of primes are dense in the rational numbers, demiurgic chaotic systems, [Brown & Chua, 1997], can generate complex orbits from the prime ratios. Second, since any irrational number having positive algorithmic complexity can be approximated by the elements of $\mathcal{P}$ to any desired degree of accuracy, the complexity inherent in any finite sequence of digits is already present in some prime number.

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