Golden, $\sqrt{2}$, and π Flowers: A Spiral Story

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Fibonacci numbers and the golden ratio are ubiquitous in nature. The number $(1 + \sqrt{5})/2$ seems an unlikely candidate for what is arguably the most important ratio in the natural world, yet it possesses a subtle power that drives the arrangements of leaves, seeds, and spirals in many plants from vastly different origins. This story is something like these spirals, twisting and turning in one direction and then another, crisscrossing themes and ideas over and over again. We begin with a mathematical model for making these spirals. Many spirals in nature use the golden ratio, but something beautiful happens when we replace that ratio with some other famous irrational numbers. Another twist takes us to rational approximations and continued fractions. Let us follow these spirals into the beautiful world of irrational numbers.

Seed spirals

When a plant such as a sunflower grows, it produces seeds at the center of the flower and these push the other seeds outward. Each seed settles into a location that turns out to have a specific constant angle of rotation relative to the previous seed. It is this rotating seed placement that creates the spiraling patterns in the seed pod [**7**, p. 176].

These spirals can be very neatly simulated as follows: Let's say there are k seeds in the arrangement, and call the most recent seed 1, the previous seed 2, and so on, so that the farthest seed from the center is seed number k. As an approximation, if each seed has an area of 1, then the area of the circular face is k, and the radius is $\sqrt{k/\pi}$. The distance from the center of the flower to each seed, then, should vary proportionally to the square root of its seed number. If we call the angle α, since the angle between any two seeds is constant, the angle of seed k is simply $k\alpha$. We now have a simple way to describe the location of any seed with polar coordinates: $r = \sqrt{k}$, $\theta = k\alpha$.

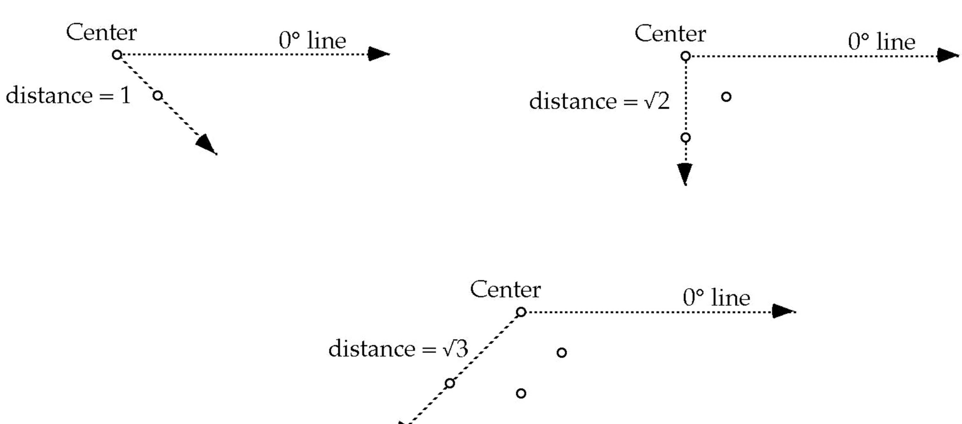

Figure 1 Growing a seed spiral

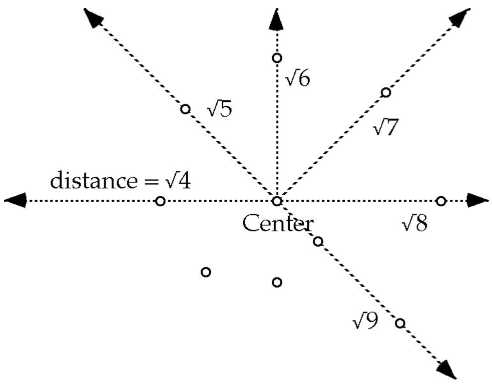

Figure 2 The first 9 seeds

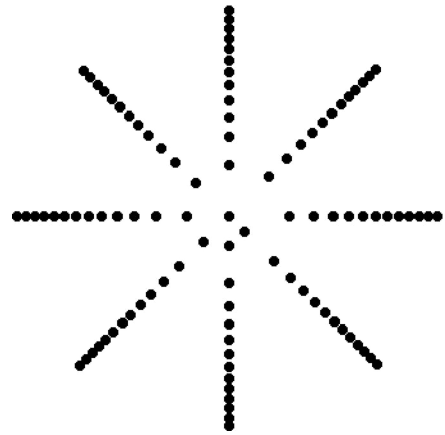

Figure 3 The first 100 seeds

Here's an example of a spiral formed with an angle $\alpha = 45°$, or $1/8$ of a complete rotation. Seed 1 is located at a distance of $\sqrt{1}$ and an angle of 45° (clockwise, in this example). The next seed is located 45° from this seed, or $2 * 45° = 90°$ from the zero line; its distance is $\sqrt{2}$. Seed 3 is located at $3 * 45°$ at a distance of $\sqrt{3}$.

Continuing in this manner, the eighth seed falls on the 0° line, the ninth seed is on the same line as seed 1, and so on (see FIGURE 2). FIGURE 3 shows what the spiral looks like with 100 seeds. It's easy to see the spiral near the center, but the pattern gets lost farther out as the eight radial arms become prominent. Notice how close together the seeds become, and how much space there is between rows of seeds; this is not a very even distribution of seeds. We can get a better distribution of seeds by choosing an angle that keeps the seeds from lining up so readily. If we try an angle of 0.15 revolutions (or 54°), the result is better, especially for the first few dozen seeds, but again we end up with radial arms, 20 this time (see FIGURE 4). Since $0.15 = 3/20$, the 20th seed will be rotated $20 * 3/20$ rotations, or 3 complete rotations to bring it to the 0° line. An angle of 0.48 results in 25 radial arms (see FIGURE 5), since the 25th seed will be positioned at an angle of $25 * 0.48$ rotations, or 12 complete rotations, and the cycle begins anew.

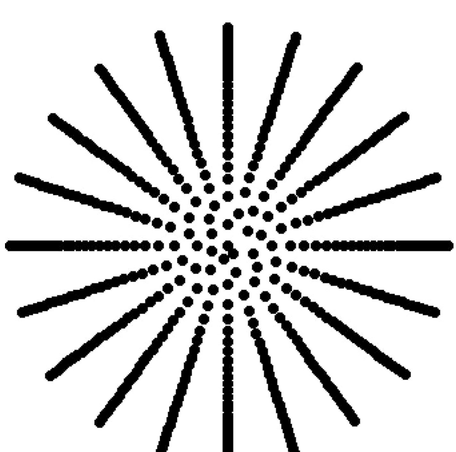

Figure 4 angle = 0.15

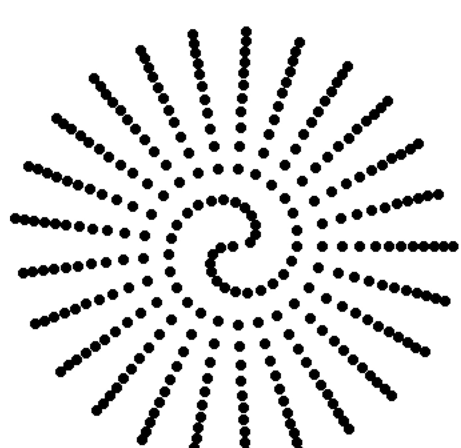

Figure 5 angle = 0.48

Clearly, if the angle is any rational fraction of one revolution, say a/b, seed b will fall on the 0° line, since the angle ab/b is an integral number of complete rotations. Therefore the pattern will repeat, radial arms will be formed, and the distribution be far from ideal. The best choice then, would be an irrational angle—we are then guaranteed that no seed will fall on the same line as any other seed.

Golden flowers

The irrational angle most often observed in plants is the golden ratio, $\phi = (1 + \sqrt{5})/2$, or approximately 1.618. This angle drives the placement of leaves, stalks, and seeds in pine cones, sunflowers, artichokes, celery, hawthorns, lilies, daisies, and many, many other plants [**5**, pp. 155–66; **2**, pp. 90–105; **1**, pp. 81–113]. With this angle of rotation, each seed is rotated approximately 1.618 revolutions from the previous seed—which is the same as 0.618 revolutions, or about 61.8% of a complete turn (approximately 222.5°). For our purposes, only the fractional part of the angle is significant and the whole number portion can be ignored. FIGURE 6 shows 1000 simulated seeds plotted with this angle of rotation, an arrangement we will call a *golden flower*. Notice how well distributed the seeds appear; there is no clumping of seeds and very little wasted space. Even though the pattern grows quite large, the distances between neighboring seeds appear to stay nearly constant. In the natural world, many plants grow their seeds (or stalks or leaves or thorns) simply where there is the most room [**5**, p. 161]. The resulting golden flower is the most even distribution possible [**1**, pp. 84–88; **6**, pp. 96–99]. (For an excellent discussion of the mechanics of the placement of seeds in a growing plant apex and the inevitability of these golden arrangements, see Mitchison [**3**, pp. 270–75].)

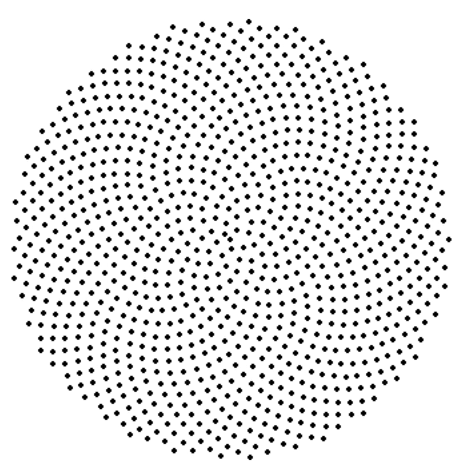

Figure 6 1000 seeds in a golden flower

Notice also the many different spiral arms. Spiral arms seem to fall into certain *families*. In this pattern above, you can see how a group of spirals twist in one direction, only to be taken over by another group of spirals twisting in the opposite direction. The interesting properties of spiral families form the heart of our discussion.

FIGURE 7 shows three families of spirals in the golden spiral. Each set of 300 seeds pictured is identical, but different spirals arms have been drawn on each set. The first set shown consists of 8 spiral arms, the second has 13, and the third 21—all Fibonacci numbers. You may be able to see other spirals not shown in these images, and the size of these groups are Fibonacci numbers as well.

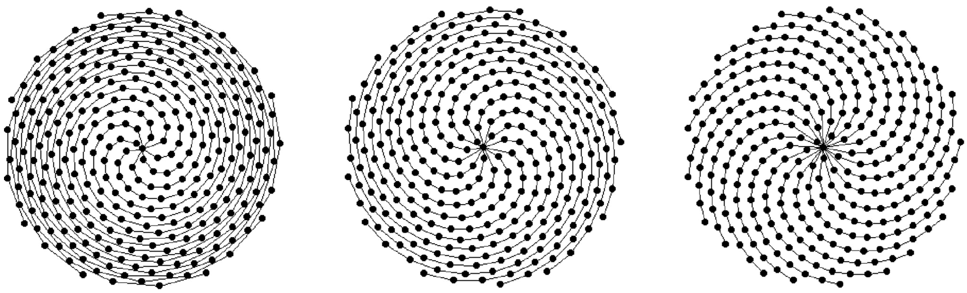

Figure 7 Spiral families 8, 13, and 21

To understand why spirals on a golden flower appear in groups whose size are Fibonacci numbers, it helps to consider placement of individually numbered seeds. In FIGURE 8, the first 144 seeds are numbered and the Fibonacci numbers are enclosed in rectangles. The baseline at 0° has also been added.

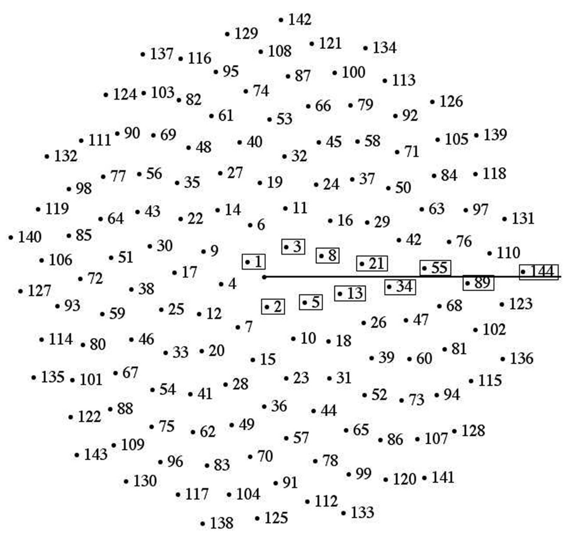

Figure 8 Fibonacci seeds

The Fibonacci numbered seeds converge on the 0° line, alternating above and below, just as the ratios of pairs of consecutive Fibonacci numbers converge to ϕ, alternately greater and less than ϕ. A seed that is numbered with a Fibonacci number will fall close to the zero degree line, since its angle (a Fibonacci number times ϕ) is approximately an integer. For example, since $55/34$ is approximately ϕ, seed 34 will be located at an angle of about $34 * (55/34)$, or very nearly 55 complete rotations (actually ~ 55.013 rotations, a slight over-rotation). The larger the Fibonacci numbers involved, the closer their ratio is to ϕ and therefore the closer the seeds lie to the zero degree line.

It is for this reason that seeds in each spiral arm in a golden flower differ by multiples of a Fibonacci number. For example, seed 34 is slightly over-rotated past the 0° line, seed 68 is rotated by the same angle from seed 34, as are seeds 102, 136, 170, and every other multiple of 34. These seeds form one spiral arm in family 34. Another arm in this family is 1, 35, 69, 103 . . . , and another is 2, 36, 70, 104, . . . , etc. Members of an arm in family 34 are seeds with numbers $34m + n$, where m and n are nonnegative integers and n is constant for that arm. Trace any spiral arm in the golden flower and you will find that its seed numbers are in arithmetic progression, since all share a common difference—a Fibonacci number.

π flowers

Why should the golden ratio be the preferred irrational number in nature? Shouldn't any irrational number work just as well? Let's take a look at a simulated seed pod generated with an angle of π rotations, or $\pi * 360°$. This angle is ~ 3.14159 revolutions, which is the same as ~ 0.14159 revolutions, or $\sim 50.97°$. FIGURE 9 shows the first 500 seeds—not a very even distribution at all! Seven spiral arms dominate the pattern with no new spirals apparent. With 10,000 seeds (FIGURE 10), a new set of spirals become visible, 113 arms in this family with so little curvature that the next set of spirals doesn't show until about a million seeds have been grown.

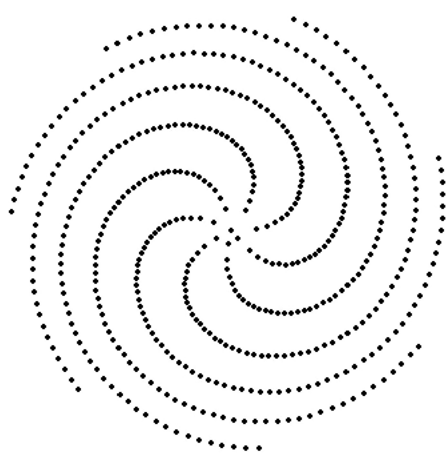

Figure 9 500 seeds, angle = π

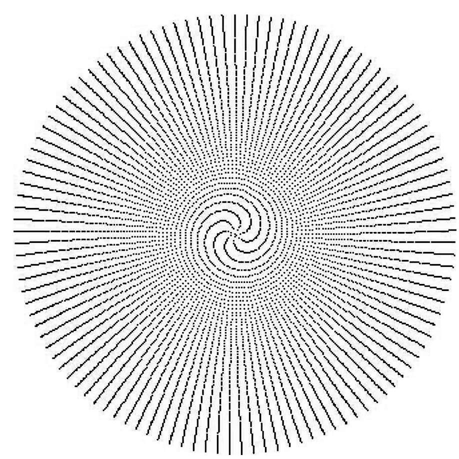

Figure 10 10,000 seeds, angle = π

Why should there be 7 spiral arms so prominently displayed in the center, and 113 arms in the next set of spirals? Perhaps you recognize these numbers as denominators in well-known rational approximations of π. An excellent approximation of π is 22/7. The decimal expansion of 1/7 is 0.142857 . . . and the angle of rotation in a π flower is 0.14159 . . .—a close match! Another great approximation of π is 355/113, accurate to 6 decimal places, and for this reason the next set of spirals has 113 arms.

The gap between these spiral families (7 and 113) in a π flower is huge compared to that of a golden flower. No other sets of spirals are apparent between family 7 and family 113—does this mean that there are no better rational approximations of π with denominators between 7 and 113? Plotting and numbering the seeds in a π flower suggests an answer. In FIGURE 11, seed 7 in the first spiral arm falls near the 0° line as expected, as does seed 113. Since seed 113 is part of the second spiral arm to cross this line, there is no seed less than 113 that lies closer to the 0° line than seed 7, and thus there is no better rational approximation of π than 22/7 with a denominator less than 113. The approximation 355/113 is so accurate that the spirals in family 113 have very little curvature and their members dominate the 0° line for generations. The nearest seed in the third arm to cross is seed 226—part of the same arm as seed 113. In fact, we need to check tens of thousands of seeds before we find one that falls closer to the 0° line than any multiple of 113—a topic we will visit again later.

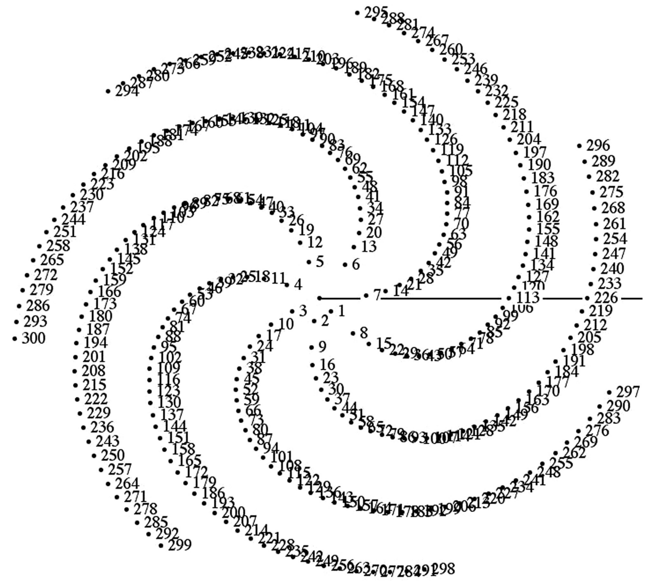

Figure 11 A numbered π flower

$\sqrt{2}$ flowers

An angle of rotation of $\sqrt{2}$ produces a very even distribution of seeds, rivaling that of the golden ratio. Five hundred seeds are shown in FIGURE 12; families of spirals are again readily apparent in this arrangement. A study of these $\sqrt{2}$ spirals is worthwhile, as their structure illuminates many properties of algebraic numbers and seed spirals in general.

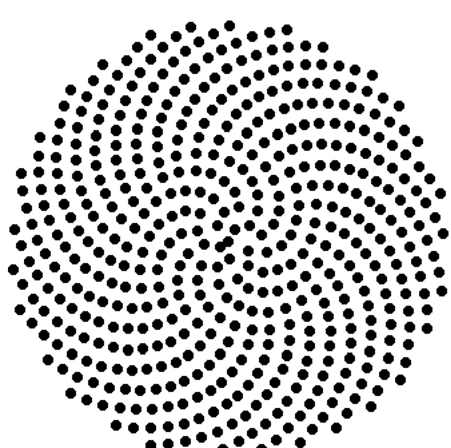

Figure 12 A root-two spiral

FIGURE 13 shows the results of a brute-force analysis—the first 12 families of spirals in a $\sqrt{2}$ spiral. Family 1 is made by connecting the seeds in order, family 2 is made by connecting seeds whose numbers differ by 2, family 3 by connecting seeds whose numbers differ by 3, etc. Study these spiral families for a moment. Notice that

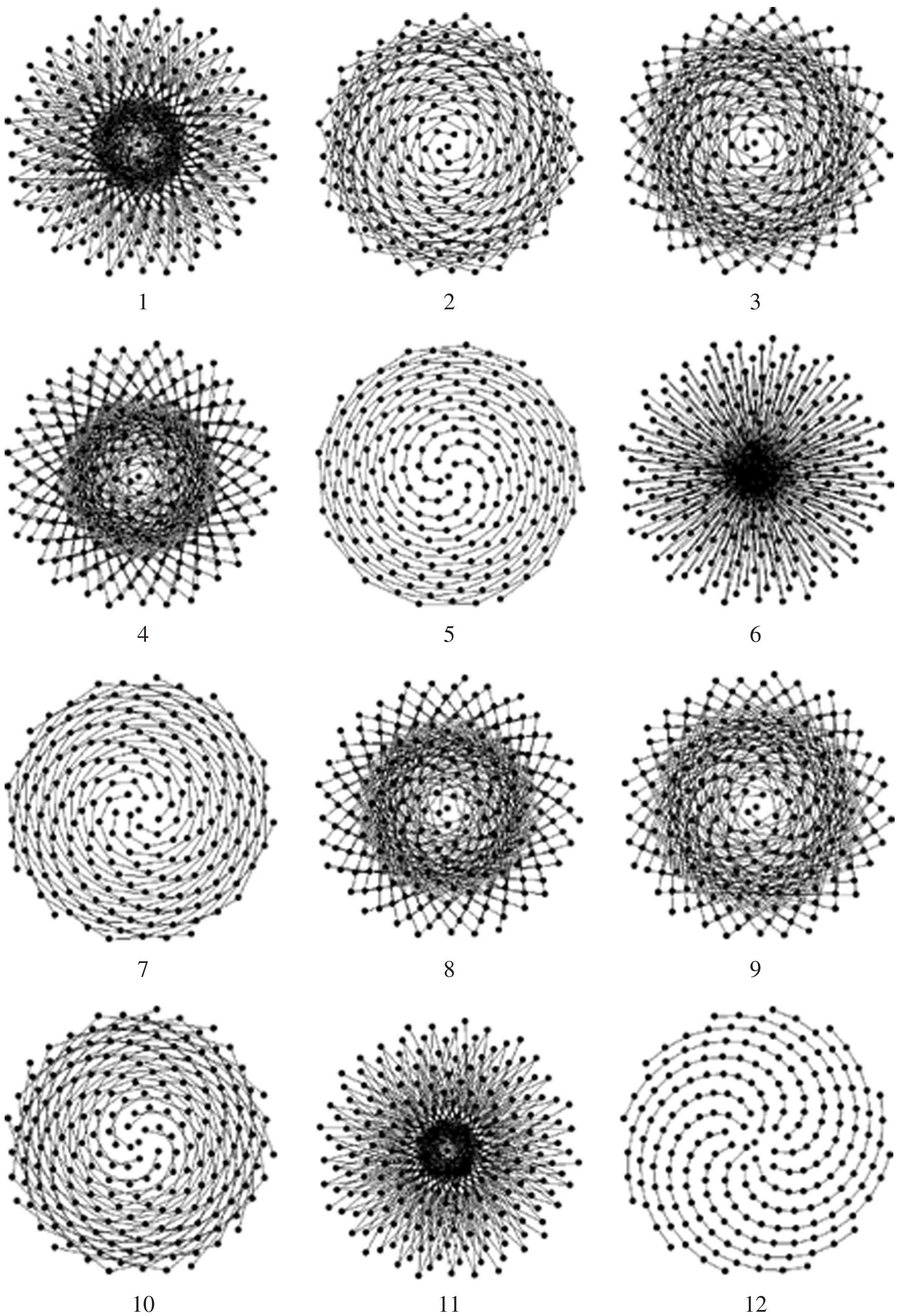

Figure 13 Spiral families 1–12 of a $\sqrt{2}$ spiral

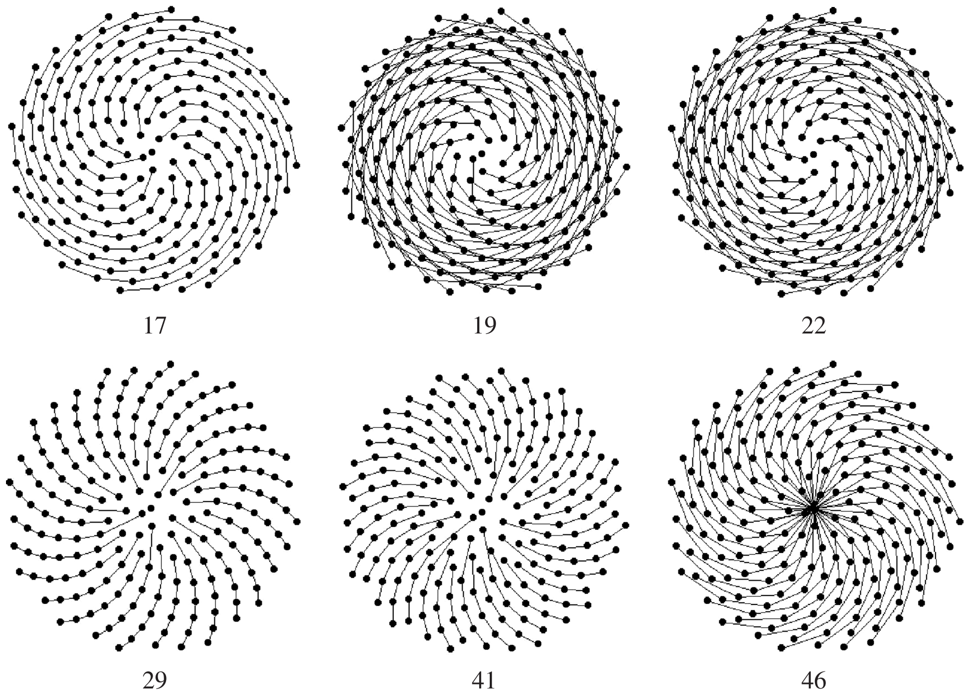

Figure 14 Selected families of the $\sqrt{2}$ spiral

some of the families produce very clean spiral arms, while others do not appear to be spirals at all, crossing themselves in a star-like or even scribble-like pattern.

Families 2 and 3 start well but quickly die, that is, they cross themselves after a small number of iterations. Families 5 and 7 are smooth. Family 10 looks like smooth spirals, but on closer examination it is seen that it crosses itself immediately—since 10 is a multiple of 5, this is the same as family 5 but with alternate seeds on the arms connected. Family 12 has the best looking spirals among the first 12.

The numbers of the families that produce nice spirals look suspicious: 2, 3, 5, 7, 12 . . . could there be a Fibonacci-like relationship between spiral families in a $\sqrt{2}$ spiral as well? More spiral families are shown in FIGURE 14, but the next spiral family after 12 is not 19 as we might expect by adding 7 and 12, but rather 17, and the next family better than 17 is 29. The sequence is in fact: 1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, . . . Before reading further, can you find the pattern in this sequence and extend it? The numbers in this sequence are the numbers in the Columns of Pythagoras. The Columns of Pythagoras are a pair of columns of integers. The top entry in each column is 1. Given a row with numbers A and B in that order, the next row is generated by summing A and B and writing this number, C, in the first column underneath A, then summing A and C and writing it in the second column underneath B. This process generates all of the spiral families of the $\sqrt{2}$ flower (see FIGURE 15). Further, the ratio of the numbers in each row converges to $\sqrt{2}$: $1/1 = 1$, $3/2 = 1.5$, $7/5 = 1.4$, $17/12 = 1.41666\ldots$, $41/29 = 1.41379\ldots$, etc.

Continued fractions

Let us follow one more twist on this spiraling journey. The golden ratio may be written as the following continued fraction:

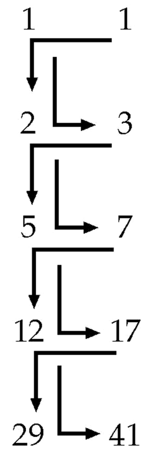

Figure 15 The Columns of Pythagoras

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

This result is easily verified by setting the continued fraction equal to some variable, say x, and then recognizing that x is repeated in the denominator of the fractional part, that is, $x = 1 + 1/x$. This expresses the continued fraction perfectly as one root of an easily evaluated quadratic.

Partial evaluations of this continued fraction, called *convergents*, result in ratios of Fibonacci numbers, that is,

$$1+\frac{1}{1}=\frac{3}{2}, \quad 1+\cfrac{1}{1+\frac{1}{1}}=\frac{5}{3}, \quad 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}=\frac{8}{5}, \quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}=\frac{13}{8}.$$

The reader may enjoy checking that the following continued fraction gives an expression for $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cdots}}}.$$

Partial evaluations of this continued fraction yield the following ratios:

$$1+\frac{1}{2}=\frac{3}{2}1+\cfrac{1}{2+\frac{1}{2}}=\frac{7}{5}, \quad 1+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}=\frac{17}{12}, \quad 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}=\frac{41}{29}.$$

These are the same numbers in the Columns of Pythagoras and the same ratios found in the $\sqrt{2}$ flower!

Let us examine the continued fraction for the other irrational number we have used to build flowers, π. The continued fraction begins $3 + 1/7\ldots$ and the values of numbers leading the expressions under the denominators at each level, starting with the 7, are: 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, ...

Looking at the partial evaluations yields rational approximations to π that reflect the number of spiral arms in the π flower:

$$\pi = 3 + \frac{1}{7} = \frac{22}{7},$$

$$\pi = 3 + \frac{1}{7 + \frac{1}{15}} = 3 + \frac{1}{\frac{106}{15}} = 3 + \frac{15}{106} = \frac{333}{106},$$

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = 3 + \frac{1}{7 + \frac{1}{16}} 3 + \frac{1}{\frac{113}{16}} = 3 + \frac{16}{113} = \frac{355}{113}.$$

Remember the 113 arms in the π spiral? It would take a lot of seeds to begin to find the next series of spirals—the next partial evaluation explains why:

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292}}}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{292}{293}}} = 3 + \frac{1}{7 + \frac{293}{4687}} = 3 + \frac{4687}{33102} = \frac{103993}{33102}.$$

The next family of spirals past family 113 is family 33102. We would need 33,102 seeds just to get one seed in each spiral arm! If we plot about a million seeds we may be able to starting seeing these spirals; however, there would be nearly 92 spirals packed into each degree arc of the circular face. An illustration 10 cm in diameter would have over 1000 spiral arms in each cm of the circumference—the illustration would appear to be nothing other than a black circle! (Note that 333/106 is also an approximation of π. However, due to the closeness of a better approximation, 355/113, the set of 106 spiral arms is immediately obscured by the set of 113 spiral arms.)

The continued fraction expansion for the golden ratio uses the smallest possible numbers in the expansion, namely 1s. Therefore, it converges to a rational number the least quickly. In this sense, the golden ratio is the most irrational number and therefore gives the best possible distribution [**6**, pp. 96–99].

More... Given that seed spirals are easily plotted using polar coordinates, you may wish to create your own irrational flowers using mathematics software. Software (Mac OS) used to create many of the images seen here is also available to download for free from the author's web page at http://www.wwu.edu/~mnaylor.

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