16-th Austrian Mathematical Olympiad 1985

Final Round

First Day

1. Determine all quadruples (a,b,c,d) of nonnegative integers satisfying

$$a^2+b^2+c^2+d^2=a^2b^2c^2.$$

2. For $n\in\mathbb{N}$, let $f(n)=1^n+2^{n-1}+3^{n-2}+\cdots+n^1$. Determine the minimum value of $\dfrac{f(n+1)}{f(n)}$.

3. A line meets the lines containing sides BC,CA,AB of a triangle ABC at A_1, B_1, C_1, respectively.The points A_2,B_2,C_2 are symmetric to A_1,B_1,C_1 with respect to the midpoints of BC,CA,AB, respectively. Prove that A_2,B_2, and C_2 are collinear.

Second Day

4. Find all natural numbers n such that the equation

$$a_{n+1}x^2-2x\sqrt{a_1^2+a_2^2+\cdots+a_{n+1}^2}+a_1+a_2+\cdots+a_n=0$$

has real solutions for all real numbers $a_1,a_2,\dots,a_{n+1}$.

5. A sequence (a_n) of positive integers satisfies $a_n=\sqrt{\dfrac{a_{n-1}^2+a_{n+1}^2}{2}}$ for all $n\geq 1$. Prove that this sequence is constant.

6. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$x^2f(x)+f(1-x)=2x-x^4 \quad \text{for all } x\in\mathbb{R}.$$

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