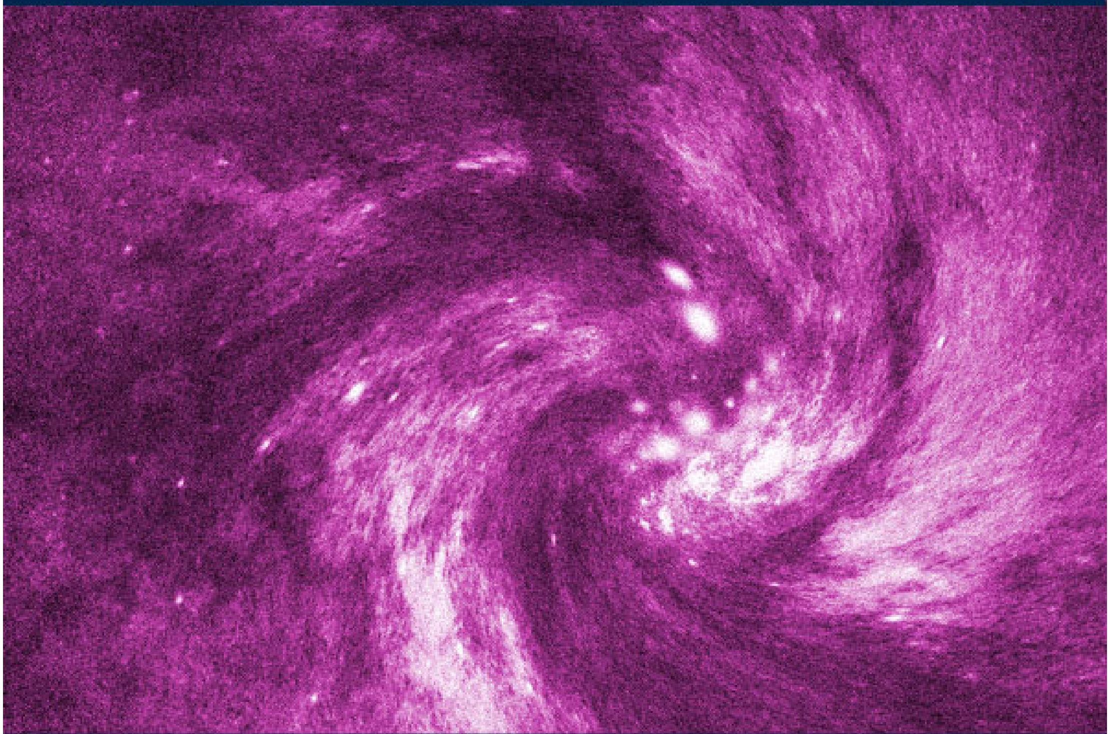
A-LEVEL MATHS TUTOR

Pure Maths



PART SIX
COORDINATE GEOMETRY

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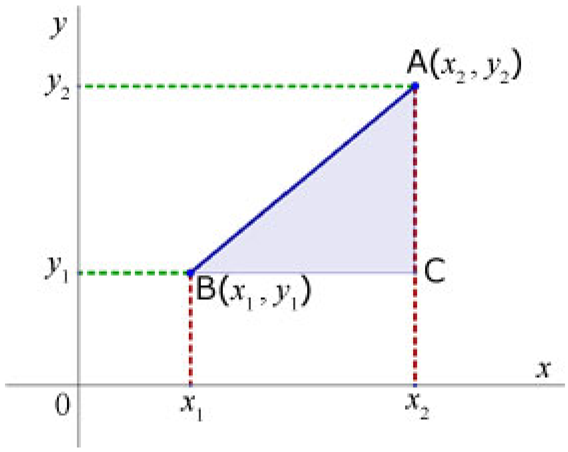
The Line Between Two Points

Distance of a line between two points



Triangle ABC is a right angled triangle.

therefore

$$(AB)^2 = (BC)^2 + (AC)^2$$

hence

$$(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

<u>Example</u>

What is the distance between points A(5,6) and B(-4,-3) correct to 2 d.p.?

$$\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$(x_1, y_1) \equiv (-4, -3)$$
$$\therefore\ x_1 = -4 \qquad y_1 = -3$$

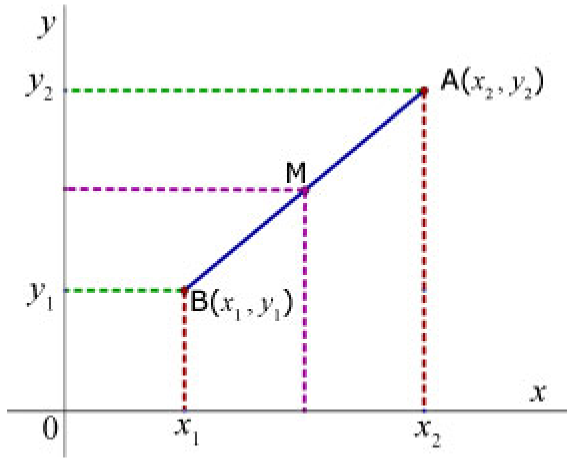
$$(x_2, y_2) \equiv (5, 6)$$
$$\therefore\ x_2 = 5 \qquad y_2 = 6$$

$$\begin{aligned} \Rightarrow \qquad \text{AB} &= \sqrt{(5-(-4))^2 + (6-(-3))^2} \\ &= \sqrt{(5+4)^2 + (6+3)^2} \\ &= \sqrt{(9)^2 + (9)^2} \\ &= \sqrt{81+81} \\ &= \sqrt{162} \\ &= 12.728 \end{aligned}$$

$$\underline{\text{AB} = 12.73} \quad \text{(2 d.p.)}$$

Finding the mid-point of a line between two points

The x-y coordinates of the midpoint M between two point A, B is found by taking the average of the x-coordinates(x_1 , x_2), then repeating for the y-coordinates(y_1 , y_2)



the coordinates of M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example

What are the coordinates of the mid-point of the line joining the coordinates (4,7) and (-8,8)?

$$(x_1, y_1) \equiv (4, 7) \qquad x_1 = 4, \quad y_1 = 7$$
$$(x_2, y_2) \equiv (-8, 8) \qquad x_2 = -8, \quad y_2 = 8$$

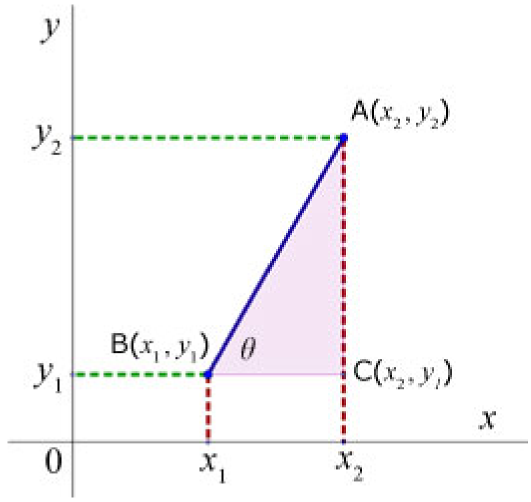
mid-point coodinates(x_M, y_M) are given by

$$x_M = \frac{x_1 + x_2}{2} = \frac{4 + (-8)}{2} = \frac{4 - 8}{2} = \frac{-4}{2} = -2$$

$$y_M = \frac{y_1 + y_2}{2} = \frac{7 + 8}{2} = \frac{15}{2} = 7.5$$

the coordinates of the mid-point are (-2, 7.5)

Finding the gradient of a line



the gradient(m) of line AB is given by:

$$\text{gradient}(m) = \frac{y\text{- step}}{x\text{- step}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

note - the gradient m is equal to $\tan\theta$

$$m = \tan\theta$$

Example

To 2 decimal places, what is the gradient of the line joining the coordinates (-5,6) and (9,-7)?

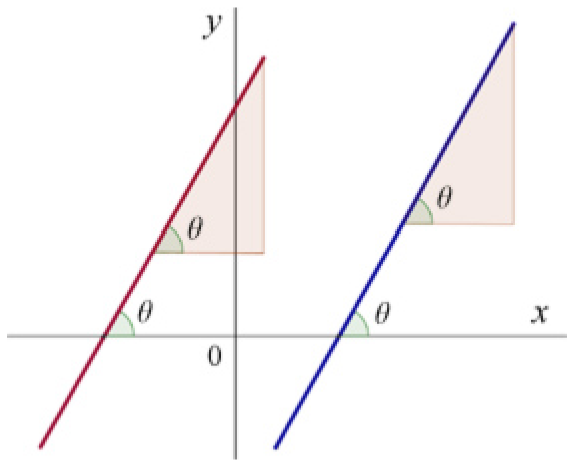
$$(x_1, y_1) \equiv (-5, 6) \qquad x_1 = -5, \quad y_1 = 6$$
$$(x_2, y_2) \equiv (9, -7) \qquad x_2 = 9, \quad y_2 = -7$$

$$\begin{aligned} \text{gradient}(m) &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(-7) - 6}{9 - (-5)} \\ &= \frac{-7 - 6}{9 + 5} \\ &= \frac{-13}{14} \\ &= -0.92857 \end{aligned}$$

$$\underline{\text{gradient}(m) \text{ is } -0.93} \quad (2 \text{ d.p.})$$

More About Straight Lines

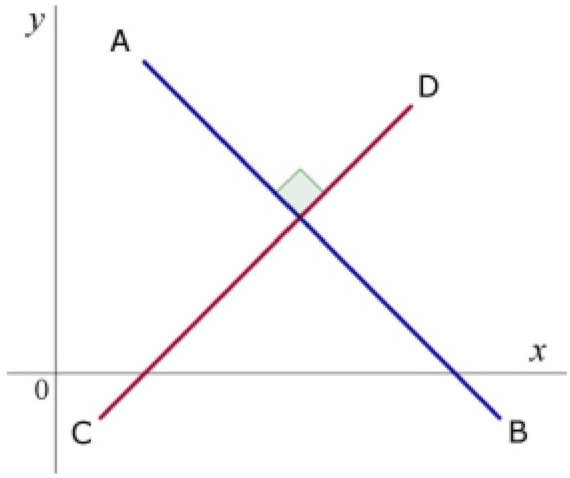
Parallel lines



Parallel lines make equal corresponding angles(θ) with the x-axis.

Therefore their gradients are equal.

Perpendicular lines

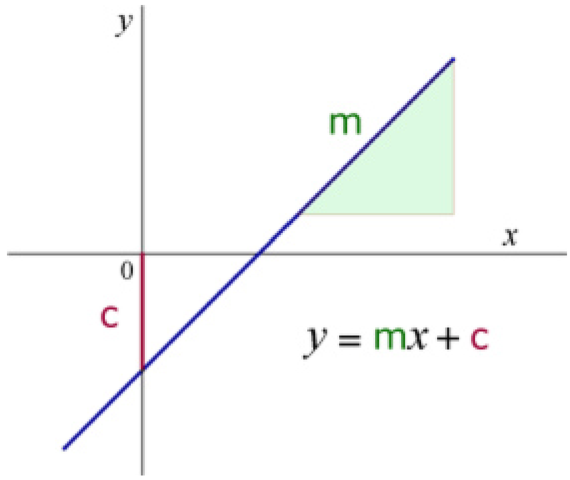


If two lines are perpendicular to eachother, the product of their gradients is -1.

If the gradient of AB is m_1 and the gradient of CD is m_2 , then:

$$m_1 m_2 = -1 \qquad \text{or} \qquad m_1 = -\frac{1}{m_2}$$

Equation of a straight line y = mx + c



The equation of a straight line is given by:

$$y = mx + c$$

- ***m*** is the gradient of the line
- ***c*** is the intercept on the y-axis

Example

What is the equation of the straight line with gradient 3 that crosses the y-axis at y= -3 ?

$$m = 3,\ c = -3\ \therefore\ \text{the equation is}\quad y = 3x - 3$$

Finding the intersection point between two straight lines

There are two types of problem here. One where the lines are not perpendicular to eachother and the other when they are.

To solve the former all that is needed is to solve the equations of the lines simultaneously.

With the later, only one equation is given and the second equation must be worked out from the information supplied. then it is a matter of proceding as before ie to solve the two equations simultaneously.

Example #1

Find the intersection point of the two straight lines:

$$y = 3x + 4 \qquad (i$$
$$y = x + 3 \qquad (ii$$

multiply (ii by 3, subtract (ii from (i

$$\begin{aligned} y &= 3x + 4 \\ -\underline{(3y} &\underline{= 3x + 9)} \\ -2y &= \quad -5 \\ &\therefore \underline{y = 2.5} \end{aligned}$$

substituting for y in equation (ii

$$\begin{aligned} y &= x + 3 \\ 2.5 &= x + 3 \\ x &= 2.5 - 3 \\ &\therefore \underline{x = -0.5} \end{aligned}$$

$\underline{\text{the point of intersection is } (-0.5, 2.5)}$

Example #2

A straight line $y = 2x + 4.5$ intersects another perpendicularly. If the second straight line has an intercept of -0.5 on the y-axis, what are the coodinates of the point of intersection of the two lines? (answer to 1 d.p.)

because the lines are perpendicular,
gradient m of second line is given by

$$(m)(2) = -1$$

$$\therefore \quad m = -\frac{1}{2}$$

since the intercept of the 2nd. equation is -0.5
using the form $y = mx + c$ its equation is:

$$y = -0.5x - 0.5$$

treating the two equations simultaneously

$$y = 2x + 4.5 \qquad \text{(i}$$

$$y = -0.5x - 0.5 \qquad \text{(ii}$$

multiplying (ii by 4, add (ii & (i

$$y = 2x + 4.5$$

$$\underline{y = -2x - 2}$$

$$4y = 2.5$$

$$\underline{y = 0.625}$$

substituting for y in equation (i

$$y = 2x + 4.5$$

$$0.625 = 2x + 4.5$$

$$2x = 0.625 - 4.5$$

$$x = \frac{-3.875}{2}$$

$$\underline{x = -1.938}$$

$\underline{\text{the point of intersection is } (-1.9, 0.6)}$

Finding the eq. of a straight line from one point + gradient

Solution is by using the expression for gradient(m) for an actual point(x_1,y_1) and a generalized point(x,y).

$$\text{gradient}(m) = \frac{y-\text{step}}{x-\text{step}} = \frac{y-y_1}{x-x_1}$$

The straight line equation is found by substituting values of x_1, y_1 and m into the above.

Example

A line of gradient 3 passes through a point (2,5). What is the equation of the line?

$$m = 3$$
$$x_1 = 2, \quad y_1 = 5$$

$$\text{using} \qquad m = \frac{y-y_1}{x-x_1}$$
$$\Rightarrow \qquad 3 = \frac{y-5}{x-2}$$
$$3(x-2) = y-5$$
$$3x-6 = y-5$$
$$3x-1 = y$$

$$\underline{y = 3x-1}$$

Finding the equation of a straight line from two points

Solution is by first finding the gradient m from the x and y values from the points (x_1,y_1) and (x_2,y_2)

$$\text{gradient}(m) = \frac{y-\text{step}}{x-\text{step}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Then we use the expression again, but this time with one actual point and a generalized point(x,y).

$$\text{gradient}(m) = \frac{y-\text{step}}{x-\text{step}} = \frac{y - y_1}{x - x_1}$$

The straight line equation is found by substituting for x_1, y_1 and m.

Example

Find the equation of the line between the two points (2,3) and (-5,7).

$$x_1 = 2, \quad y_1 = 3$$
$$x_2 = -5, \quad y_2 = 7$$

using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{7-3}{-5-2}$$
$$m = -\frac{4}{7}$$

using $m = \dfrac{y - y_1}{x - x_1}$

substituting for x_1 and y_1

$$-\frac{4}{7} = \frac{y-3}{x-2}$$
$$-4(x-2) = 7(y-3)$$
$$-4x + 8 = 7y - 21$$
$$-4x + 29 = 7y$$
$$y = -\tfrac{4}{7}x + \tfrac{29}{7}$$

$$\underline{y = -\tfrac{4}{7}x + 4\tfrac{1}{7}}$$

Parametric Equations

Introduction

There is another way of writing y as a function of x and that is to use two separate equations.

One equation has x as a function of t (or θ) eg $x = 2t$

and the other equation has y as function of t (or θ) eg $y = t^2$

The variables t, θ are called **parameters** and the two equalities **parametric equations**.

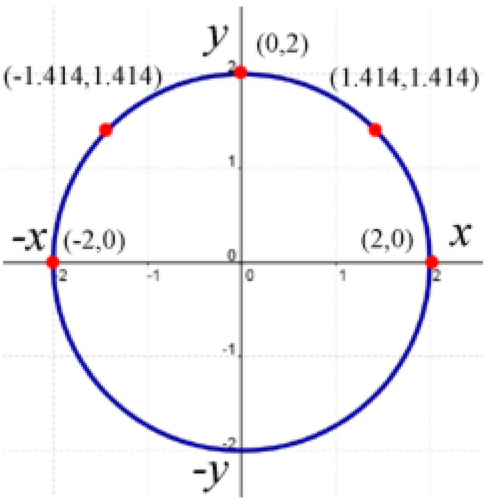
Common questions on this topic are the plotting of parametric equations and their conversion to a single Cartesian equation.

Example #1

Plot the graph of the curve given parametrically by the equations:

$$x = 2\cos\theta \qquad y = 2\sin\theta$$

θ°	0	45	90	135	180
x = 2cosθ	2	1.414	0	-1.414	-2
y = 2sinθ	0	1.414	2	1.414	0



Example #2

What is the Cartesian equation given parametrically by:

$x = t^2+3 \qquad y = t^3 + 3t$

$$x = t^2 + 3 \qquad (i$$
$$y = t^3 + 3t \qquad (ii$$

factorising (ii $\qquad y = t(t^2 + 3)$

but $\quad x = t^2 + 3$

$\therefore \quad y = tx, \quad \Rightarrow \quad t = \dfrac{y}{x}$

substituting for t in (i

$$x = \left(\frac{y}{x}\right)^2 + 3$$
$$x = \frac{y^2}{x^2} + 3$$

multiplying both sides by x^2

$$x^3 = y^2 + 3x^2$$
$$x^3 - 3x^2 = y^2$$
$$y^2 = x^3 - 3x^2$$

$$\underline{y^2 = x^2(x - 3)}$$

Example #3

What is the Cartesian equation given parametrically by:

x = 2sinθ y = 2sin2θ

$y = 2\sin 2\theta$ but $\sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \quad y = 2(2\sin\theta\cos\theta)$

$y = 4\sin\theta\cos\theta$

$\therefore \quad y^2 = 16\sin^2\theta\cos^2\theta \qquad$ (i

but $x = 2\sin\theta \quad \Rightarrow \quad \dfrac{x}{2} = \sin\theta \qquad$ (ii

using the identity $\cos^2\theta = 1 - \sin^2\theta$

and substiuting for $\sin\theta$ from (ii

$$\Rightarrow \quad \cos^2\theta = 1 - \frac{x^2}{4}$$

now substituting in (i for $\sin\theta$ and $\cos^2\theta$

$$y^2 = 16\sin^2\theta\cos^2\theta$$

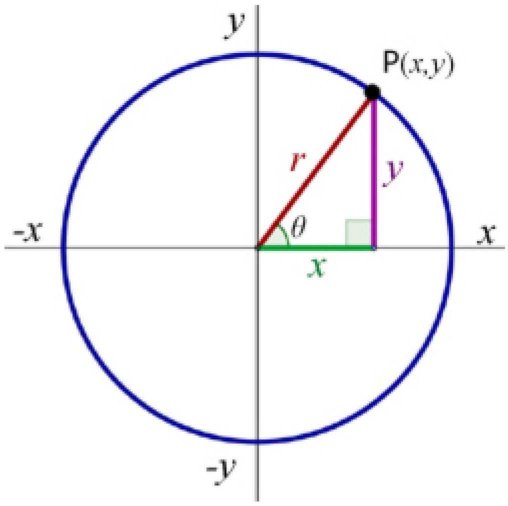
$$= 16\left(\frac{x}{2}\right)^2\left(1 - \frac{x^2}{4}\right)$$

$$= 16\frac{x^2}{4}\left(1 - \frac{x^2}{4}\right)$$

$$\underline{y^2 = 4x^2\left(1 - \frac{x^2}{4}\right)}$$

Circles & Ellipses

Circles

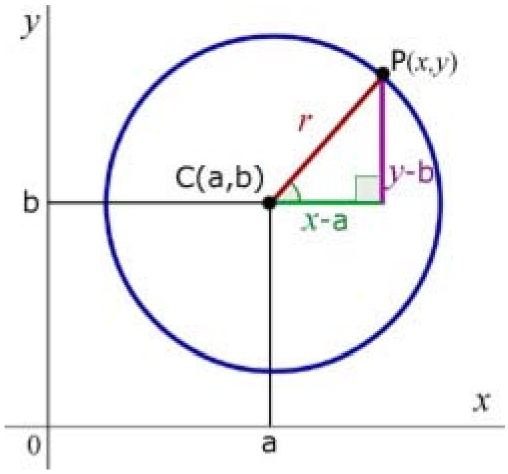


Any point P is described by Pythagoras' Theorem. So the equation of a circle with centre (0,0)and radius *r* is given by:

$$x^2 + y^2 = r^2$$

or in terms of parameters,

$$x = r\cos\theta \qquad y = r\sin\theta$$



For a circle with its centre off-set from the origin at a point C(a,b), again, by Pythagoras, the equation is given by:

$$(x-a)^2 + (y-b)^2 = r^2$$

Circle equation expanded(usual form)

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = r^2$$
$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = r^2$$
$$x^2 + y^2 - 2(a)x - 2(b)y + (a^2 + b^2 - r^2) = 0$$

$$f = a \qquad g = b \qquad c = a^2 + b^2 - r^2$$

$$x^2 + y^2 - 2(f)x - 2(g)y + (c) = 0$$
$$\underline{x^2 + y^2 - 2fx - 2gy + c = 0}$$

$$\Rightarrow \qquad (x + f)^2 + (y + g)^2 = f^2 + g^2 + c$$

the centre of the circle is at $(-f, -g)$

the radius of the circle is $\sqrt{g^2 + f^2 + c}$

Example

What is the radius and the coodinates of the centre of the circle with equation:

$$x^2 + y^2 - 8x - 10y + 5 = 0$$

rearranging

$$(x^2 - 8x) + (y^2 - 10y) + 5 = 0 \qquad \text{(i}$$

using 'completing the square'

$$x^2 - 8x + 16 = (x-4)^2$$

$$\Rightarrow \qquad x^2 - 8x = (x-4)^2 - 16$$

and $\quad y^2 - 10y + 25 = (y-5)^2$

$$\Rightarrow \qquad y^2 - 10y = (y-5)^2 - 25$$

substituting into (i for $(x^2 - 8x)$, $(y^2 - 10y)$

$$\therefore \quad (x-4)^2 - 16 + (y-5)^2 - 25 + 5 = 0$$

$$(x-4)^2 + (y-5)^2 - 25 - 16 + 5 = 0$$

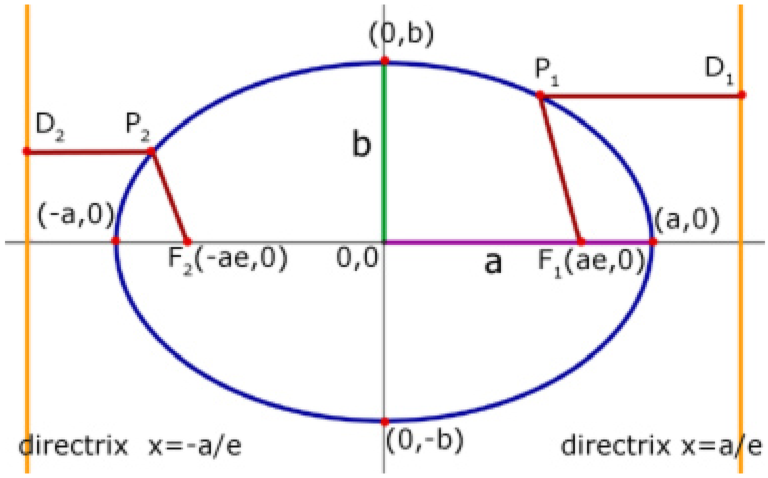
$$(x-4)^2 + (y-5)^2 - 36 = 0$$

$$(x-4)^2 + (y-5)^2 = 36$$

comparing with circle $\quad (x-a)^2 + (y-b)^2 = r^2$
with centre (a, b) and radius r

$\Rightarrow$ <u>centre is at $(4,5)$ and radius is 6</u>

<u>Ellipses</u>



The maximum displacement (**a**) along the x-axis is called the **semi-major axis**, while the maximum displacement along the y-axis (**b**) is called the **semi-minor** axis. This is when a>b. When b>a the names are interchanged.

The eccentricity(e) of an ellipse is defined as:

$$e = \sqrt{\left(1 - \frac{b^2}{a^2}\right)}$$

From the eccentricity we can define the points of focus (plural foci):

$\mathbf{F_1}$(ae,0) and $\mathbf{F_2}$(-ae,0)

and the **directrices**(directrix lines) at x=a/e and x=-a/e.

The directrices are two special lines parallel to the y-axis and either side of it(when the ellipse is centred at the origin).

Their unique property concerns the ratio of the distance between a point(P_1)on the curve to a focal point(F_1) and a line from the point to the directrix.

The ratio gives the eccentricity 'e' .

$$\text{eccentricity(e)}=\frac{F_1P_1}{P_1D_1}=\frac{F_2P_2}{P_2D_2}$$

note: $e<1$

Notes