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# A Note on Composition and Recursion

Oboifeng Dira
Department of Mathematics, University of Botswana, Gaborone, Botswana P/bag 00704
Email: dira@mopipi.ub.bw

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**Abstract.** Two results and some applications are presented concerning the COMPOSE operator of N Sloane applied to arithmetic progressions and to doubly recursive sequences.

**Keywords:** Recurrent integer sequence; Arithmetic progression; Composition of power series.

## 1. The Operator Compose and Power Series

In the OEIS N Sloane has introduced the operator COMPOSE for two sequences that performs the operation of composition of the two associated formal power series, see for example the references [1, 2].

In this note I investigate the self composition of an arithmetic progression with initial term 1 and step length $d$.

Also I study the self composition of a doubly recursive sequence $x_{n+1} = ax_n + bx_{n-1}$ with initial terms $x_0 = 0, x_1 = 1$.

It is proved that in both cases the resulting self composition is a sequence with a four fold recursion $x_{n+1} = c_0x_n + c_1x_{n-1} + c_2x_{n-2} + c_3x_{n-3}$ where the coefficients $c_i,\ i = 0, 1, 2, 3$ are explicit functions of the original step length $d$ (or of $a, b$ respectively).

The technique used is the composition of generating series and their associated formal power series.

Applications are the proof of an empirical observation of Harding in A229587 and the introduction of several new sequences not yet in the OEIS.

## 2. Arithmetic Progressions

The generating series for two arithmetic progression with initial term 1 in offset position one and with step lengths $d, e$ respectively are given as

$$\begin{aligned}
& g(x) = \frac{x+(e-1)x^2}{(1-x)^2}\ , f(x) = \frac{x+(d-1)x^2}{(1-x)^2}, \\
& g(f(x)) \\
= & \frac{f(x)+(e-1)f(x)^2}{(1-f(x))^2} \\
= & \frac{(1-x)^2(x+(d-1)x^2)+(e-1)x^2(1+(d-1)x)^2}{((1-x)^2-x-(d-1)x^2)^2} \\
= & \frac{x+(d+e-4)x^2+(2de-4d-2e+5)x^3+(d-1)(de-d-e+2)x^4}{(1-3x-(d-2)x^2)^2}.
\end{aligned}$$

Note that in the denominator the number $e$ does not enter.

In the particular case self composing $g(x)$ given by

$$g(x) = \frac{x+(d-1)x^2}{(1-x)^2}\ .$$

I get

$$\begin{aligned}
g(g(x)) &= \frac{g(x)+(d-1)g(x)^2}{(1-g(x))^2} \\
&= \frac{(1-x)^2(x+(d-1)x^2)+(d-1)x^2(1+(d-1)x)^2}{((1-x)^2-x-(d-1)x^2)^2} \\
&= \frac{x+(2d-4)x^2+(2d^2-6d+5)x^3+(d-1)(d^2-2d+2)x^4}{(1-3x-(d-2)x^2)^2}\ .
\end{aligned}$$

In the special case when $d = e = 1$ then we obtain the sequence A030267 and we can reconfirm its generating series as well as the characteristic polynomial $(1-3x+x^2)^2$ which leads to the recursive formula

$$a_{n+1} = 6a_n - 11a_{n-1} + 6a_{n-2} - a_{n-3}.$$

In the special case when $d = e = 2$ we obtain (except for the first term $x$) the as yet unknown generating series

$$\frac{x+x^3+2x^4}{(1-3x)^2}$$

of A229587 [3] and we can confirm the empirical relation $a_{n+1} = 6a_n - 9a_{n-1}$ for $n \geq 5$ given by Mr R H Harding.

For the cases $d = e \geq 3$ the resulting sequences seem to be new to OEIS, and we attach a short table of the first few cases.

| $d$ | $g(x)$ | $g(g(x))$ | OEIS |
|---|---|---|---|
| 1 | 1,2,3,4,5,6,7... | 1,4,14,46,145,444,1331,3926,11434 | A030267 |
| 2 | 1,3,5,78,9,11,... | 1,6, 28, 116, 444, 1620, 5724, 19764 | A229587 |
| 3 | 1,4,7,10,13,... | 1,8,46,224,973,3986,15715,60326,227062 | D.N.E. |
| 4 | 1,5,9,13,17,21,.. | 1,10,68,376,1792,8016,34352,143024 | D.N.E. |
| 5 | 1,6,11,16,21,26,.. | 1,12,94,578,2961,14232,65259,290358, | D.N.E. |
| 6 | 1,7,13,19,25,31,.. | 1,14,124,836,4540,23204,112636,530276, | D.N.E. |
| 7 | 1,8,15,22,29,36,.. | 1,16,158,1156,6589,35550,181259,896534 | D.N.E. |

In the case $d \neq e$ there are many other examples which do appear in the OEIS. In the table below we have used the following notations:

We use D.N.E. to denote sequences that have not yet been published in the OEIS. We use the operator $\triangleleft$ to denote left shift, i.e. $(1, 3, 5, 7, 9, ...)^{\triangleleft} = (3, 5, 7, 9, ...)$. We use the juxtaposition follows: if have Axyz $= (3, 5, 8, 10, ..)$ then we write 1,Axyz $= (1, 3, 5, 8, 10, ...)$.

For example in the case $d = 1, e = 2$ essentially the example A054444, $1, 5, 20, 71, 235, 744, 2285, 6865, 20284$ is obtained, except for one shift which then has the generating function

$$\frac{x - x^2 + x^3}{(1 - 3x + x^2)^2}.$$

In the case $d \neq e$ there are many other examples which do appear in the OEIS.

| $d$ | $f(x)$ | $e$ | $g(x)$ | $g(f(x))$ | OEIS |
|---|---|---|---|---|---|
| 0 | 1,1,1,1,1,.. | 1 | 1,2,3,4,5,... | 1,3,8,20,48,112,... | A001792 |
| 0 | 1,1,1,1,1,... | 2 | 1,3,5,7,9,... | 1,4,12,32,80,192, ... | A001787 |
| 0 | 1,1,1,1,1,... | 3 | 1,4,7,10,.. | 1,5,16,44,112,272,... | A053220 |
| 0 | 1,1,1,1,1,... | 4 | 1,5,9,13,.. | 1,6,20,56,144,352,.... | A014480 |
| 0 | 1,1,1,1,1,.. | -1 | 1,0,-1,-2,.. | 1,1,0,-4,-16,-48,-128,.. | 1,A159964 |
| 0 | 1,1,1,1,1,.. | -2 | 1,-1,-3,-5,.. | 1,0,-4,-16,-48,-128,.. | A159964 |
| 0 | 1,1,1,1,1,... | -3 | 1,-2,-5,-8,.. | 1,-1,-8,-28,-80,-208,... | A130129$^{\sigma}$ |
| 1 | 1,2,3,4,5.. | 0 | 1,1,1,1,... | 1,3,8,21,55,144,... | A001906 |
| 1 | 1,2,3,4,5,.. | 2 | 1,3,5,7,9,... | 1,5,20,71,235,744,... | A054444 |
| 2 | 1,3,5,7,9,.. | 0 | 1,1,1,1,... | 1,4,12,36,108,... | A003946 |
| 3 | 1,4,7,10,13,.. | 0 | 1,1,1,1,... | 1,5,16,53,175,578,... | A108300 |
| 4 | 1,5,9,13,17,... | 0 | 1,1,1,1,... | 1,6,20,72,256,912,... | A189604 |
| -1 | 1,0,-1,-2,-3,... | 0 | 1,1,1,1,... | 1,1,0,-3,-9,-18,-27,-27,0.. | A057681$^{\triangleleft}$ |

For the special case $d = 2$ if $a_n$ denotes the sequence associated to the composition $g \circ f$ then an explicit formula for $a_n$ where $n \geq 3$ has been found (and included for $e = 2$ in A229587) as follows:

$$a_n = 4 \cdot 3^{n-4} \cdot [4e \cdot n - 3(2e - 3)] \ .$$

## 3. Doubly Recursive Sequence With Offset 0, 1

Any sequence $x_{n+1} = ax_n + bx_{n-1}$ and $x_0 = 0, x_1 = 1$ has the generating series

$$g(x) = \frac{x}{1 - ax - bx^2} \ .$$

By self composing $g(x)$, we have

$$\begin{aligned} g(g(x)) &= \frac{g}{1 - ag - bg^2} \\ &= \frac{x(1 - ax - bx^2)}{(1 - ax - bx^2)^2 - ax(1 - ax - bx^2) - bx^2} \\ &= \frac{x - ax^2 - bx^3}{1 - 3ax + (2a^2 - 3b)x^2 + 3abx^3 + b^2x^4} \ . \end{aligned}$$

For example for the Fibonacci numbers A000045 we obtain the composition sequence

$$1, 2, 6, 17, 50, 147, 434, 1282, 3789, 11200, ...$$

which does not seem to be recorded in OEIS.

In general the resulting composed sequence $y_n$ then will have a four fold recursion $y_{n+1} = 3ay_n + (3b - 2a^2)y_{n-1} - 3aby_{n-2} - b^2y_{n-3}$, at least for $n \geq 3$.

For the cases $a, b \geq 1$ the resulting composition sequences seem to be all new to OEIS, and we attach a short table of the first few cases.

| $a$ | $b$ | $g(x)$ | OEIS | $g(g(x))$ | OEIS |
|---|---|---|---|---|---|
| 1 | 2 | 1,1,3,5,11,21,43,.. | A001045 | 1,2,8,26,94,320,.. | D.N.E. |
| 2 | 1 | 1,1,3,7,17,41,.. | A001333 | 1,2,8,30,118,470,.. | D.N.E. |
| 2 | 2 | 1,1,4,10,28,76,.. | A026150 | 1,2,10,41,184,824,.. | D.N.E. |
| 2 | 3 | 1,1,5,13,41,121,.. | A046717 | 1,2,12,52,260,.. | D.N.E. |
| 3 | 1 | 1,1,4,13,43,142,.. | A003688 | 1,2,10,47,232,.. | D.N.E. |
| 3 | -1 | 1,1,2,5,13,34,89,.. | A001519 | 1,2,6,21,78,299,.. | D.N.E. |
| 3 | 3 | 1,1,6,21,81,306,.. | A108306 | 1,2,14,73,426,.. | D.N.E. |
| 4 | 1 | 1,1,5,21,89,377,.. | A015448 | 1,2,12,68,404,.. | D.N.E. |
| 4 | 2 | 1,1,6,26,116,516,.. | A164549 | 1,2,14,83,526,.. | D.N.E. |
| 4 | 3 | 1,1,7,31,145,673,.. | A086901 | 1,2,16,98,658,.. | D.N.E. |
| 4 | 4 | 1,8,36,176,848,.. | A164545 | 1,2,18,113,800,.. | D.N.E. |

## 4. Conclusion

In the previous sections the following result was shown.

**Theorem 4.1.**

(a) *The self composition of an arithmetic progression starting with offset* 1 *and step length* $d$ *is a fourfold recursion sequence* $a_n$ *that satisfies*

$$a_{n+1} = 6a_n + (2d-13)a_{n-1} - (6d-12)a_{n-2} - (d-2)a_{n-3}$$

*and has generating function*

$$\frac{x + (2d-4)x^2 + (2d^2-6d+5)x^3 + (d-1)(d^2-2d+2)x^4}{(1-3x-(d-2)x^2)^2} .$$

*In case* $d = 2$ *this reduces to the two fold recursion* $a_{n+1} = 6a_n - 9a_{n-1}$ *and the explicit formula* $a_n = 4 \cdot 3^{n-4}(8n-3)$ *and essentially (except for the first term) reproduces the sequence A229587.*

(b) *The self composition of a doubly recursive sequence with offset* $0, 1$ *and recursion* $x_{n+1} = ax_n + bx_{n-1}$ *is a fourfold recursion sequence* $y_n$ *that satisfies*

$$y_{n+1} = 3ay_n + (3b-2a^2)y_{n-1} - 3aby_{n-2} - b^2y_{n-3} ,$$

*and it has generating function*

$$\frac{x - ax^2 - bx^3}{1 - 3ax + (2a^2-3b)x^2 + 3abx^3 + b^2x^4} .$$

## References


[1] T. Bier, Some inversion formulas for functions and power series, *Southeast Asian Bull. Math.* 4 (1997)329-336

[2] S. Plouffe, A method for obtaining the algebraic generating function from a series, FRSAC, *Formal Power Series and Algebraic Combinatorics*, Florence, 1993 and also in arXiv site: http :// arxiv .org/abs/0912.0072 vl.

[3] OEIS (The Encyclopeadia of Integer Sequences), Founded by N.J. Sloane, entry A 229587, entry by Ron. H. Hardin.