Zeitschrift: L'Enseignement Mathématique

Band: 38 (1992)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TOEPLITZ SEQUENCES, PAPERFOLDING, TOWERS OF HANOI AND PROGRESSION-FREE SEQUENCES OF INTEGERS

Kapitel: 4. Towers of Hanoi and Toeplitz sequences

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DOI: https://doi.org/10.5169/seals-59494

If, instead of C one takes a "generalized" Morse sequence C', and if one defines

$$A'(n) = C'(n) + C'(n+1) \text{ modulo } 2 ,$$

then A' is also a Toeplitz sequence, as proved in [16].

4. Towers of Hanoi and Toeplitz sequences

The tower of Hanoi puzzle consists of three vertical pegs and of N circular disks of different diameters stacked in decreasing order on the first peg. At each step one may transfer the topmost disk from a peg to a different peg according to the rule: no disk is allowed to be on a smaller one. The game ends when all the disks are stacked on the second or third peg.

The sequence of moves for the classical (minimal) Hanoi tower algorithm can be generated in a very easy way as it is 2-automatic (see [4] and section 6), which essentially means that the k^{th} move can be predicted by a machine with bounded memory. More precisely number the pegs as I, II, III and define a (respectively b, c) to be the move which takes the topmost disk from peg I (respectively II, III) and puts it on peg II (respectively III, I). Let $\bar{a}, \bar{b}, \bar{c}$ be the respective opposite moves. Then the sequence of moves for N disks is the prefix of length $2^N - 1$ of an infinite sequence U which is 2-automatic. Moreover the following proposition is proved in [4]:

PROPOSITION. *The infinite sequence of moves U is equal to the Toeplitz transform of* $((a\bar{c}b\omega c\bar{b}a\omega b\bar{a}c\omega)^{\infty}, id)$.

Note that, keeping the notations of [4], the sequence U is indexed by $1, 2, \cdots$ and not by $0, 1, 2, \cdots$ as the sequences above.

5. Progression-free sequences and Toeplitz sequences

The question of finding a sequence of integers without arithmetic progressions of given length has been intensively studied (see its history in [14] and the included bibliography). In particular what is the "minimal" increasing sequence having this property?

One knows that, if k is a prime number, the minimal sequence of integers without any arithmetic progression of k terms is exactly the increasing sequence of the integers without the digit $k - 1$ in their base k expansion (cited