Lagrange inversion and Lambert W

D. J. Jeffrey, G. A. Kalugin and N. Murdoch
Department of Applied Mathematics,
The University of Western Ontario,
London, CANADA, N6A 5B7

***Abstract*—We show that Lagrange inversion can be used to obtain closed-form expressions for a number of series expansions of the Lambert W function. Equivalently, we obtain expressions for the nth derivative. Various integer sequences related to the series expansions now can be expressed in closed form.**

I. INTRODUCTION

The Lagrange inversion theorem [5] is a powerful way to study the inverse of a given function. If, moreover, the theorem can be combined with the generating function for an established family of quantities, usually integers but possibly more general algebraic quantities, then the expression for the inverse function gains an additional level of elegance. This is the case with respect to the Lambert W function.

The Lambert W function is an inverse function [2], [4], the inverse of $f : z \mapsto ze^z$. It is a multi-branched function (also called a multivalued function). The branches are indexed by an integer k, and written W_k. For $z \in \mathbb{C}$, the branches are defined by

$$W_k(z)e^{W_k(z)} = z \ , \tag{1}$$

$$W_k(z) \to \ln_k z \quad \text{for} \quad |z| \to \infty \ , \tag{2}$$

where $\ln_k z = \ln z + 2\pi i k$ is the kth branch of logarithm. The function $\ln z$ is the principal branch of logarithm, and is defined by the inequality $\forall z \in \mathbb{C},\ -\pi < \Im(\ln z) \le \pi$; it is the function implemented in Maple and Mathematica. We recall that the branch cut for $\ln z$ is $(-\infty, 0]$ by modern convention. Returning to W, we note that the branch cuts of W vary between branches. For the principal branch, W_0, the cut is $(-\infty, -1/e]$, while for other branches it is $(-\infty, 0]$. In addition, the 3 branches $k = -1, 0, -1$ share a singular point at $z_{\mathrm{s}} = -e^{-1}$.

Lambert W has found numerous applications in every branch of science. One well known application is to the number of rooted trees; the generating function is given by an expansion of W, namely,

$$W_0(x) = \sum_{n\ge1} (-n)^{n-1} \frac{x^n}{n!} \ . \tag{3}$$

The minus signs can be removed by defining the Tree function

$$T(x) = -W(-x) \ ,$$

and expanding T. A second well known application is to the solution of the problem of the iterated exponential (in the complex plane). Given

$$y = z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} \ ,$$

then in the region where the iteration converges, the value is

$$y = \frac{W(-\log z)}{-\log z} = \frac{T(\log z)}{\log z} \ .$$

Further examples of applications can be found in the paper [2].

A cognate function of W is the Wright ω function. It is defined by

$$\omega(z) = W_{\mathcal{K}(z)}\left(e^z\right) \ , \tag{4}$$

where

$$\mathcal{K}(z) = \left\lceil \frac{\Im z - \pi}{2\pi} \right\rceil$$

is the unwinding number [1]. Unlike Lambert W, Wright ω is not a branched function. The connection between ω and the branches of W is $W_k(z) = \omega(\ln_k z)$.

Several expansions for W have been studied in the past [3], and the coefficients appearing in these expansions have been catalogued in the Online Encyclopedia of Integer Sequences [10]. Recurrence relations have been derived for the coefficients, but not expressions in terms of known quantities. We supply such expressions here.

The Lagrange inversion theorem is usually stated as follows. For an analytic function $y = f(x)$ with $f(0) = 0$, $f'(0) \ne 0$, the inverse function $x = \check{f}(y)$ can be expressed as

$$\check{f}(y) = \sum_{n\ge1} \check{f}_n y^n \ , \tag{5}$$

where the coefficients are given by

$$\check{f}_n = \frac{1}{n!} \lim_{x\to0} \frac{d^{n-1}}{dx^{n-1}} \frac{x^n}{f(x)^n} \ . \tag{6}$$

We, however, shall use an alternative, equivalent, expression:

$$\check{f}_n = \frac{1}{n} \left[x^{n-1}\right] \left(\frac{f(x)}{x}\right)^{-n} \ , \tag{7}$$

where we have used the notation of Knuth that $[x^m]g(x)$ equals the coefficient of x^m in a Taylor series expansion of $g(x)$ in the variable x. The reason for preferring the second form is a practical one. Below we shall see that either expression leads to nested sums, and to extract expressions for $\check{f}_n$ it is easier to think in terms of expansions, rather than derivatives. Notice that all calculations require knowledge only of the function $f(x)$, not its inverse $\check{f}(x)$.

TABLE I
THE INITIAL VALUES OF THE COEFFICIENTS P_{nk} APPEARING IN (11).

$n \backslash k$	0	1	2	3	4
1	1				
2	2	1			
3	9	8	2		
4	64	79	36	6	
5	625	974	622	192	24

TABLE II
THE INITIAL VALUES OF THE COEFFICIENTS Q_{nk} APPEARING IN (12).

$n \backslash k$	0	1	2	3	4
1	1				
2	1	1			
3	2	4	3		
4	6	18	25	15	
5	24	96	190	210	105

II. DERIVATIVES OF W

Various expressions for the nth derivatives of Lambert W and Wright ω have been given in the literature. For W, at least 3 ways have been considered. These are

$$\frac{d^nW(z)}{dz^n} = \frac{\exp(-nW(z))}{(1+W(z))^n}\,\frac{P_n(W(z))}{(1+W(z))^{n-1}}\ , \tag{8}$$

$$= \frac{\exp(-nW(x))}{(1+W(x))^n}\, Q_n\left(\frac{1}{1+W(x)}\right)\ , \tag{9}$$

$$= \frac{\exp(-nW(x))}{(1+W(x))^n}\, R_n\left(\frac{W(x)}{1+W(x)}\right)\ . \tag{10}$$

In [2], [6] and [9, p 773, Answer to **50**], it has been shown that the functions P_n, Q_n, R_n are polynomials. Moreover, the coefficients of each of the polynomials can be used to define families of integer sequences. We write the polynomials in terms of sets of positive integers as

$$P_n(x) = (-1)^{n-1}\sum_{k=0}^{n-1} P_{nk}x^k\ , \tag{11}$$

$$Q_n(x) = (-1)^{n-1}\sum_{k=0}^{n-1} Q_{nk}x^k\ , \tag{12}$$

$$R_n(x) = (-1)^{n-1}\sum_{k=0}^{n-1} (-1)^k R_{nk}x^k\ . \tag{13}$$

TABLE III
THE INITIAL VALUES OF THE COEFFICIENTS R_{nk} APPEARING IN (13).

$n \backslash k$	0	1	2	3	4
1	1				
2	2	1			
3	9	10	3		
4	64	113	70	15	
5	625	1526	1450	630	105

Explicit values of the initial P_{nk} are given in table I; similarly values for Q_{nk} and R_{nk} are given in tables II and III. We note that these numerical values for the coefficients can also be found in the Online Encyclopedia of Integer sequences [10] as entries A042977, A054589, A048160 and A005264. The challenge it to find explicit expressions for the coefficients in the polynomials.

Theorem 1: The 3 sets of coefficients are related through

$$P_{nk} = \sum_{m=0}^{n-1-k} \binom{n-1-m}{k} Q_{nm}\ , \tag{14}$$

$$= \sum_{m=0}^{k} (-1)^m \binom{n-1-m}{k-m} R_{nm}\ , \tag{15}$$

$$Q_{nk} = \sum_{m=0}^{k} (-1)^{k-m} \binom{n-1-m}{k-m} P_{n(n-1-m)}\ , \tag{16}$$

$$R_{nk} = \sum_{m=0}^{k} (-1)^m \binom{n-1-m}{k-m} P_{nm}\ . \tag{17}$$

Proof. We write w for $W(z)$, and equate (8) and (9).

$$\frac{P_n(w)}{(1+w)^{n-1}} = Q_n\left(\frac{1}{1+w}\right)\ .$$

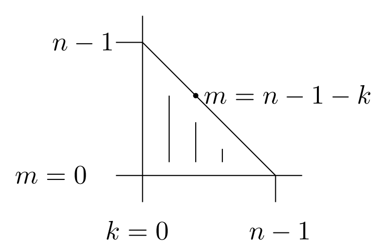
Clearing the fraction and substituting for P_n and Q_n gives

$$\sum_{k=0}^{n-1} P_{nk}w^k = (1+w)^{n-1}\sum_{k=0}^{n-1}\frac{Q_{nk}}{(1+w)^k}\ . \tag{18}$$

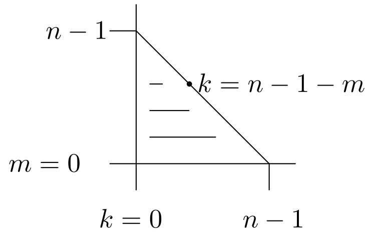
Combining the $(1+w)$ terms, we use the binomial theorem

$$\sum_{k=0}^{n-1} Q_{nk}(1+w)^{n-1-k} = \sum_{k=0}^{n-1} Q_{nk}\sum_{m=0}^{n-1-k}\binom{n-1-k}{m}w^m\ .$$

We now have to rearrange the sums. We can understand this with a diagram. The values taken by k run along a horizontal axis, and by m along a vertical.



The triangle shows the ranges of the indexes being summed, and the vertical lines show that we sum first over m, the inner sum, and after over k. The sum over m stops at $m = n-1-k$ as shown. We now change the order of summing so that the sum over k is done first.



This diagram corresponds to the sum

$$\sum_{m=0}^{n-1} \sum_{k=0}^{n-1-m} Q_{nk} \binom{n-1-k}{m} w^m \ .$$

To obtain the theorem statement, the k and m, which are dummy variables, must be swapped.

$$\sum_{k=0}^{n-1} \sum_{m=0}^{n-1-k} Q_{nm} \binom{n-1-m}{k} w^k \ .$$

It is now straightforward to equate coefficients of w^k in (18) and obtain the theorem statement (14). The other statements are proved in the same manner. ☐

We introduce the next theorem with a general discussion. Given an analytic function $y=f(x)$ and its inverse $x=\check{f}(y)$, it is well known that $f(\check{f}(y)) = y$ and $\check{f}(f(x)) = x$. For Lagrange inversion, both functions are known through series expansions, and we consider a consequence of this. Substituting into (5), we see

$$x = \sum_{k\geq 1} \check{f}_k [f(x)]^k \ .$$

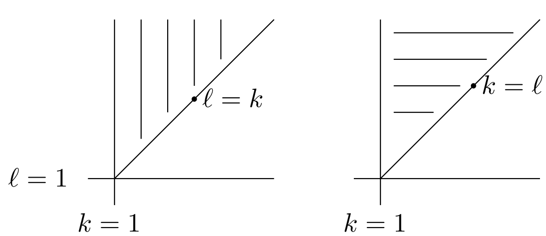
Denoting the expansion of $f(x)^k$ by

$$f(x)^k = \sum_{\ell \geq k} f_\ell^{(k)} x^\ell \ ,$$

where, since $f(0)=0$ by assumption, we start the sum at $\ell = k$, we obtain the identity

$$x = \sum_{k\geq 1} \check{f}_k \sum_{\ell\geq k} f_\ell^{(k)} x^\ell \ . \tag{19}$$

To invert the order of summation, we again use a diagram.



The left diagram shows the sum in (19) with the sums over ℓ first extending to infinity. The right diagram shows the sum first over k. Thus we see

$$x = \sum_{\ell\geq 1} \sum_{k=1}^{\ell} \check{f}_k f_\ell^{(k)} x^\ell \ .$$

Equating coefficients of powers of x, we obtain

$$\sum_{k=1}^{\ell} \check{f}_k f_\ell^{(k)} = \delta_{1\ell} \ , \tag{20}$$

where $\delta_{1\ell}$ is the usual Kronecker delta. As an example, consider $y=f(x)=xe^x$ and $x=\check{f}(y)=W(y)$. Then from (3), $\check{f}_k = (-k)^{k-1}/k!$ and

$$f(x)^k = x^k e^{kx} = x^k \sum_{m\geq 0} \frac{(kx)^m}{m!} = \sum_{\ell\geq k} \frac{k^{\ell-k} x^\ell}{(\ell-k)!} \ .$$

This gives the identity

$$\sum_{k=1}^{\ell} \frac{(-1)^{k-1} k^{\ell-1}}{k!(l-k)!} = \delta_{1\ell} \ .$$

This approach is used in the next theorem, which is a variation on that in [8].

Theorem 2: The P_{nk} defined in (11) can be expressed as

$$P_{nk} = \sum_{m=0}^{k} \binom{2n-1}{k-m} \sum_{q=0}^{m} \frac{(-1)^q (q+n)^{m+n-1}}{q!(m-q)!} \ . \tag{21}$$

Proof. We differentiate (3) to obtain $d^nW(x)/dx^n$ and then use the above type of substitution to convert the result to an expression in $W(x)$. For $n\geq 1$, we have

$$\frac{d^nW(x)}{dx^n} = \sum_{k\geq n} \frac{(-k)^{k-1} x^{k-n}}{(k-n)!} = \sum_{m\geq 0} \frac{(-m-n)^{m+n-1} x^m}{m!} .$$

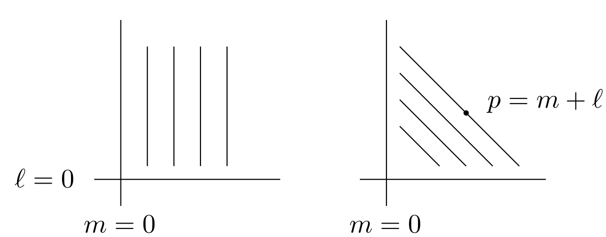
We abbreviate $W(x)$ to w to save space. From (8) we have

$$\begin{aligned} P_n(w) &= (1+w)^{2n-1} e^{nw} \sum_{m\geq 0} \frac{(-m-n)^{m+n-1} (we^w)^m}{m!} \ , \\ &= (1+w)^{2n-1} \sum_{m\geq 0} \frac{(-m-n)^{m+n-1}}{m!} w^m e^{(m+n)w} \ . \end{aligned}$$

We concentrate on the sum, and expand the exponential.

$$\sum_{m\geq 0} \frac{(-m-n)^{m+n-1}}{m!} w^m \sum_{\ell\geq 0} \frac{((m+n)w)^\ell}{\ell!} =$$
$$\sum_{m\geq 0} \sum_{\ell\geq 0} \frac{(-1)^{m+n-1} (m+n)^{m+n-1+\ell}}{m!} \frac{w^{m+\ell}}{\ell!} \ .$$

To rewrite the summation, we again use a diagram.



We set $p=m+\ell$ and replace $\ell = p-m$ to get

$$\sum_{p\geq 0} \sum_{m=0}^{p} \frac{(-1)^{m+n-1} (m+n)^{p+n-1}}{m!} \frac{w^p}{(p-m)!} \ .$$

At this stage, we still have an infinite series. After the final multiplication, the series becomes finite. We return to P_n and expand the remaining term:

$$P_n(w) = \sum_{q=0}^{2n-1} \binom{2n-1}{q} w^q \times \sum_{p\geq 0}\sum_{m=0}^{p} \frac{(-1)^{m+n-1}(m+n)^{p+n-1}}{m!}\frac{w^p}{(p-m)!} \ .$$

To re-order the summations, we can be lazy and note that the upper limit of the q summation can be extended to infinity, because the binomial factor will be zero for all the additional cases. Then we can reuse the last procedure. With $r = p + q$,

$$P_n(w) = \sum_{r\geq 0}\sum_{p=0}^{r} \binom{2n-1}{r-p} \times \sum_{m=0}^{p} \frac{(-1)^{m+n-1}(m+n)^{p+n-1}}{m!}\frac{w^r}{(p-m)!} \ . \quad (22)$$

Applying the definition of P_{nk} completes the proof. □

Corollary 1: It should be noted that in (22), the sum over r does not have an upper limit. It is known that $\deg P_n = n-1$, and hence we have for $n > 1$ that,

$$\sum_{r\geq n}\sum_{p=0}^{r}\sum_{m=0}^{p} \binom{2n-1}{r-p} \frac{(-1)^{m+n}(m+n)^{p+n-1}w^r}{m!(p-m)!} = 0 \ .$$

Using Lagrange inversion, we obtain a new expression for R_{nk} in terms of Stirling partition numbers.

Definition 1: Stirling r-partition numbers, also called r-associated Stirling numbers of the second kind, are defined by the generating function

$$\left(\exp(z) - \sum_{m=0}^{r-1} \frac{z^m}{m!}\right)^k = k! \sum_{n\geq rk} \left\{ {n \atop k} \right\}_{\geq r} \frac{z^n}{n!} \ . \quad (23)$$

An equivalent definition is the number of partitions of a set of n objects into k subsets with at least r members.

Theorem 3: For the expansion defined in (10) and (13), we have the following explicit form in terms of Stirling 2-partition numbers.

$$R_{nk} = \sum_{p=0}^{n-1-k} n^p \binom{n+k-1}{p} \left\{ {n-1-p+k \atop k} \right\}_{\geq 2}$$

Proof. Since the Taylor series for W around $z = a$ is

$$W(z) = \sum_{n=0} \left.\frac{d^n W(z)}{dz^n}\right|_{z=a} \frac{(z-a)^n}{n!} \ ,$$

a series expansion for W gives the derivatives.

In order to apply Lagrange inversion at $z = a$, we write $z = a+x$ and $W(a) = A$. Then we write $W(z) = w+A$. So

$$We^W = (w+A)e^{w+A} = x + a = x + Ae^A \ .$$

Hence

$$(w+A)e^w - A = e^{-A}x \ .$$

From the above expressions, it is clear that the natural variable to work in is $y = e^{-A}x/(1 + A)$. Therefore we multiply through by $B = 1/(1 + A)$. We note also that $1 - B = AB$.

$$y = Bwe^w + AB(e^w - 1) = w + O(w^2) \ ,$$

so the inverse is

$$w = \sum_{n\geq 1} w_n y^n \ ,$$

with

$$\begin{aligned} w_n &= \frac{1}{n}[w^{n-1}] \left(Be^w + AB\frac{e^w-1}{w}\right)^{-n} \\ &= \frac{1}{n}[w^{n-1}] \left(e^w + AB\frac{e^w - 1 - we^w}{w}\right)^{-n} \end{aligned} \quad (24)$$

We write $C = AB = W(a)/(1 + W(a)$, and note that C is the argument of R_n in (10).

$$\begin{aligned} w_n &= \frac{1}{n}[w^{n-1}]e^{-nw}\left(1 - C\,\frac{e^{-w}-1+w}{w}\right)^{-n} \\ &=[w^{n-1}]\frac{e^{-nw}}{n}\sum_{k\geq 0}\binom{-n}{k}C^k\left(\frac{e^{-w}-1+w}{-w}\right)^k \end{aligned}$$

This is where we combine Lagrange inversion with the idea of generating functions. The expression above has been arranged so that we have the generating function for Stirling partition numbers. We now expand everything:

$$\begin{aligned} w_n =& \frac{1}{n}[w^{n-1}]\sum_{p\geq 0}\frac{(-nw)^p}{p!}\times \\ & \sum_{k\geq 0}\binom{-n}{k}C^k\frac{k!}{(-w)^k}\sum_{q\geq 2k}\left\{ {q \atop k} \right\}_{\geq 2}\frac{(-w)^q}{q!} \\ =& \frac{1}{n}[w^{n-1}]\sum_{p\geq 0}\frac{(-nw)^p}{p!}\times \\ & \sum_{k\geq 0}\binom{-n}{k}C^k k!\sum_{q\geq k}\left\{ {q+k \atop k} \right\}_{\geq 2}\frac{(-w)^q}{(q+k)!} \end{aligned}$$

At this point, we reorder the summations.

$$\begin{aligned} w_n =& \frac{1}{n}[w^{n-1}]\sum_{k\geq 0}\sum_{q\geq k}\sum_{p\geq 0}(-1)^{p+q}\frac{(n)^p}{p!}\times \\ & \binom{-n}{k}C^k k!\left\{ {q+k \atop k} \right\}_{\geq 2}\frac{(w)^{p+q}}{(q+k)!} \end{aligned}$$

Now we put $s = p + q$, ie, $q = s - p$, with $s \geq k$.

$$\begin{aligned} w_n =& \frac{1}{n}\sum_{k=0}^{n-1}\sum_{p=0}^{n-k-1}\frac{n^p}{p!}\frac{(-1)^{n-1}k!}{(n-1-p+k)!} \\ & \times\binom{-n}{k}C^k\left\{ {n-1-p+k \atop k} \right\}_{\geq 2} \end{aligned}$$

We finally re-express the binomial factor in more conventional form:

$$w_n = \sum_{k=0}^{n-1} \sum_{p=0}^{n-k-1} \frac{n^p}{n!} (-1)^{n+k-1} \times \binom{n+k-1}{p} C^k \left\{ n-1-p+k \atop k \right\}_{\geq 2} .$$

By comparison with (10), we see we have the expression for R_{nk} in the theorem. □

III. Wright omega

The derivatives of ω were given in [2] as

$$\frac{d^n\omega}{dz^n} = \frac{p_n(\omega)}{(1+\omega)^{2n-1}} , \qquad n \neq 0 , \tag{25}$$

$$p_n(\omega) = \sum_{k=0}^{n-1} (-1)^k \left\langle\!\!\left\langle n-1 \atop k \right\rangle\!\!\right\rangle \omega^{k+1} , \tag{26}$$

where $\langle\!\langle \dot{\cdot} \rangle\!\rangle$ denotes a second-order Eulerian number [7, §6.2]. The degree of p_n appears to be n, but for $n \geq 2$, it is $n-1$. In order to describe patterns succinctly, it is better to separate the $n=1$ case. Thus we can write

$$\frac{d\omega}{dz} = \frac{\omega}{1+\omega} , \tag{27}$$

$$\frac{d^n\omega}{dz^n} = \frac{p_n(\omega)}{(1+\omega)^{2n-1}} , \qquad n \geq 2 , \tag{28}$$

$$p_n(\omega) = \sum_{k=1}^{n-1} (-1)^{k-1} \left\langle\!\!\left\langle n-1 \atop k-1 \right\rangle\!\!\right\rangle \omega^k . \tag{29}$$

We give a different expression based on the pattern of (9).

$$\frac{d^n\omega}{dz^n} = \frac{\omega}{(1+\omega)^n} \, q_n\left(\frac{1}{1+\omega}\right) , \tag{30}$$

where q_n is a polynomial of degree $n-1$.

Definition 2: The Stirling r-cycle numbers, also called r-associated Stirling numbers of the first kind, are defined by the generating function

$$\left(\ln\left(\frac{1}{1-z}\right) - \sum_{m=1}^{r-1} \frac{z^m}{m} \right)^k = k! \sum_{n \geq rk} \left[n \atop k \right]_{\geq r} \frac{z^n}{n!} . \tag{31}$$

An equivalent definition is the number of permutations of n objects into m cycles, each cycle having a cardinality $\geq r$.

Theorem 4: The polynomial q_n is given for $n \geq 1$ by

$$q_n(x) = \sum_{k=0}^{n-1} (-1)^{n+k-1} \left[n+k-1 \atop k \right]_{\geq 2} x^k . \tag{32}$$

Proof. It is well known that ω obeys $\omega(z) + \ln\omega(z) = z$. Setting $z = a + y$, $\omega(a) = A$, $\omega(z) = w + A$, we obtain

$$y = w + \ln(1 + \omega/A) .$$

We set $v = w/A$ and invert $y = Av + \ln(1+v)$. Thus

$$v = \sum_{n\geq 1} v_n y^n , \tag{33}$$

$$v_n = (1/n) \left[v^{n-1} \right] (A + \ln(1+v)/v)^{-n} . \tag{34}$$

TABLE IV
The initial values of the coefficients of $q_n(x)$ appearing in (32).

n \ k	0	1	2	3	4
1	1				
2	-2	3			
3	6	-20	15		
4	-24	130	-210	105	
5	120	-924	2380	-2520	945

To connect this with a generating function, we set $C = 1 + A$.

$$\begin{aligned} \left(A + \frac{\ln(1+v)}{v}\right)^{-n} &= C^{-n} \left(1 + \frac{\ln(1+v) - v}{Cv}\right)^{-n} \\ &= C^{-n} \sum_{k\geq 0} \binom{-n}{k} \left(\frac{\ln(1+v)-v}{Cv}\right)^k . \end{aligned}$$

Thus

$$\begin{aligned} v_n &= \frac{1}{n} \left[v^{n-1} \right] \sum_{k\geq 0} C^{-n-k} \binom{-n}{k} k! \sum_{p\geq 2k} \left[p \atop k \right]_{\geq 2} \frac{v^{p-k}}{p!} , \\ &= \frac{1}{n} \sum_{k\geq 0} C^{-n-k} \binom{-n}{k} k! \left[n+k-1 \atop k \right]_{\geq 2} \frac{1}{(n+k-1)!} . \end{aligned}$$

Simplifying the expression proves the theorem. □

Corollary 2: For $n \geq 2$, We can write more compactly:

$$\frac{d^n\omega}{dz^n} = \frac{\omega}{(1+\omega)^{n+1}} \sum_{k=0}^{n-2} \left[n+k \atop k+1 \right]_{\geq 2} \frac{(-1)^{n+k}}{(1+\omega)^k} , n \geq 2 . \tag{35}$$

Table IV gives the first coefficients explicitly. They appear in [10], both in A259456 and A111999.

References

[1] R.J. Bradford, R.M. Corless, J.H. Davenport, D.J. Jeffrey, S.M. Watt. Reasoning about the elementary functions of complex analysis. *Annals Maths Artificial Intelligence*, 36:303–318, 2002.

[2] Robert M. Corless, Gaston H. Gonnet, D. E. G. Hare, David J. Jeffrey, and Donald E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[3] Robert M. Corless, David J. Jeffrey and Donald E. Knuth. A Sequence of Series for the Lambert W Function. *Proceedings of the ACM ISSAC, Maui*, 195–203,1997.

[4] Robert M. Corless and David J. Jeffrey. The Lambert W function. In N. J. Higham, editor, *The Princeton Companion to Applied Mathematics*. Princeton University Press, 2015.

[5] C. Carathéodory. *Theory of functions of a complex variable, 2nd. ed.* Chelsea, New York, 1958.

[6] D. Dumont, and A. Ramamonjisoa. *Grammaire de Ramanujan et Arbres de Cayley*. The Electronic J. Combinatorics, 3, R17, 1996.

[7] R.L. Graham, D.E. Knuth and O. Patashnik. *Concrete Mathematics, 2nd Edition*. Addison—Wesley, 1994.

[8] German A. Kalugin and David J. Jeffrey. Unimodal sequences show Lambert W is Bernstein. Comptes Rendus/Mathematical Reports, Vol 33(2) June 2011. arXiv:1011.5940v1

[9] D.E. Knuth. *The Art of Computer Programming*. Vol. 4A, Combinatorial Algorithms, Part 1, Addison—Wesley, 2011.

[10] N.J.A. Sloane. `https://oeis.org/` , The On-Line Encyclopedia of Integer Sequences.