k -regular Sequences

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Introduction

A sequence $(a(n))_{n\geq 0}$ over a finite alphabet Δ is said to be k -automatic if there exists a finite automaton with output

$$
M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)
$$

su
h that

$$
a(n)=\tau(\delta(q_0,(n)_k))
$$

for all $n \geq 0$.

Here

 \bullet Q is a finite nonempty set of states;

$$
\bullet \ \Sigma_k = \{0, 1, \ldots, k-1\};
$$

- \mathcal{L} is the transition function function function function function \mathcal{L}
- \bullet q_0 is the initial state;
- $\langle \cdot \cdot \rangle_{\hat{h}}$ is the called mean also is the presentation of $n,$
- $\bullet \ \tau : Q \to \Delta$ is the output mapping.

Example: The Thue-Morse Sequence

This sequence

 $(t(n))_{n>0} = 0110100110010110\cdots$

counts the number of 1 's (mod 2) in the base- 2 representation of n .

It is generated by the following finite automaton:

Automatic Sequences

- Automatic sequences were introduced by Cobham
- . Popularized and further studied by Mendès Fran
e, Allou
he, and others
- Extremely useful, with well-developed theory (e.g., theorem of Christol)
- However, they are somewhat restricted because of the restriction to a finite alphabet
- Want a generalization that preserves the flavor of automatic sequence, but over an infinite alphabet

The k -kernel

The <u>k-kernel</u> of a sequence $(a(n))_{n>0}$ is the set of subsequen
es

 \blacksquare ^e \cdots \cdots \cdots \cdots \cdots \cdots \cdots ^e g: the contract of the contrac

Theorem. (Eilenberg) A sequence $(a(n))_{n>0}$ is k -automatic if and only if the k -kernel is finite.

Example. Consider the Thue-Morse sequence $(t(n))_{n>0}$. Then clearly

$$
t(2^e n + r) \equiv t(n) + t(r) \pmod{2}
$$

so every sequence in the k -kernel is either $(t(n))_{n\geq0}$ or $(t(2n+1))_{n>0}$.

k -regular Sequences

To generalize automatic sequences, we use the k -kernel.

Instead of demanding that the k -kernel be finite, we instead ask that the set of sequen
es generated by the k -kernel be finitely generated.

Example 1. Consider the sequence $(s_2(n))_{n>0}$, where $s_2(n)$ is the sum of the bits in the base-2 representation of n . Then

$$
s_2(2^e n + r) = s_2(n) + s_2(r),
$$

so every sequence in the k -kernel is a $\mathbb Z$ -linear combination of the sequence $(s_2(n))_{n>0}$ and the constant sequence 1.

Properties of k-regular Sequences

Theorem. A sequence is k -regular and takes finitely many values if and only if it is k -automatic.

Theorem. If $(a(n))_{n>0}$ and $(b(n))_{n>0}$ are kregular sequences, then so are $(a(n)+b(n))_{n>0}$, $(a(n)b(n))_{n>0}$, and $(ca(n))_{n>0}$ for any c.

Theorem. Let $c, d \geq 0$ be integers. If $(a(n))_{n>0}$ is *k*-regular, then so is $(a(cn+d))_{n>0}$.

Theorem. The sequence $(a(n))_{n>0}$ is k-regular Iff it is κ -regular for any $e > 1$.

Examples of k **-regular Sequences**

Example 2. Families of Separating Subsets. Consider a set S containing n elements. If a family $F = \{A_1, A_2, \ldots, A_k\}$ of subsets of S has the property that for every pair (x, y) of distinct elements of S, we can find indices $1 \leq i, j \leq k$ such that

(i) $A_i \cap A_j = \emptyset$ and (ii) $x \in A_i$ and $y \in A_j$,

then we call F a separating family. Let $f(n)$ denote the minimum possible cardinality of F .

For example, the letters of the alphabet can be separated by only 9 subsets:

 $\{a, b, c, d, e, f, g, h, i\}$ $\{j, k, l, m, n, o, p, q, r\}$ $\{s, t, u, v, w, x, y, z\}$ $\{a, b, c, j, k, l, s, t, u\}$ $\{d, e, f, m, n, o, v, w, x\}$ $\{g, h, i, p, q, r, y, z\}$ $\{a, d, g, j, m, p, s, v, y\}$ $\{b, e, h, k, n, q, t, w, z\}$ ${c, f, i, l, o, r, u, x}$

Examples of k -regular Sequences

Cai Mao-Cheng showed that

$$
f(n) = \min_{0 \le i \le 2} f_i(n),
$$

where

$$
f_i(n) = 2i + 3\lceil \log_3 n/2^i \rceil.
$$

The first few terms of this sequence are given in the following table:

A priori, it is not clear that f is 3-regular, since the minimum of two k -regular sequences is not necessarily k -regular. However, in this case it is possible to prove the following characterization:

Examples of k **-regular Sequences**

Theorem. Let it is a set of the substitution of the substitution of the substitution of the substitution of the ^j < j+1 , i.e., i.e \blacksquare (n) \blacksquare (n) \blacksquare ⁸ \blacksquare >: \sim 1; if \sim 1; if <u>je za obrazu za obrazu i predsjednik kapeta</u> < n ⁴ ³ j1 ; \sim 3j \sim j1 < n ² ³ <u>je za obraz o</u> . . \sim 3, if \sim ^j < n ³ j+1

From this, it now easily follows that $f(n)$ is 3-regular.

Example 3. Mallows showed there there is a unique monotone sequence $(a(n))_{n>0}$ of nonnegative integers such that $a(a(n)) = 2n$ for $n \neq 1$. Here are the first few terms of this sequence:

It can be shown that $a(z+1) = 3 \cdot 2$ +j for \blacksquare . The \blacksquare is the \blacksquare is the \blacksquare , and a 2 $\,$ \blacksquare . \blacksquare . \blacksquare . \blacksquare \sim $-$ ⁺ 2j for \sim 2 \sim 2

We have

$$
a(4n) = 2a(2n)
$$

\n
$$
a(4n + 1) = a(2n) + a(2n + 1)
$$

\n
$$
a(4n + 3) = -2a(n) + a(2n + 1) + a(4n + 2)
$$

\n
$$
a(8n + 2) = 2a(2n) + a(4n + 2)
$$

\n
$$
a(8n + 6) = -4a(n) + a(2n + 1) + a(4n + 2)
$$

Hence this sequence is also 2 -regular.

Example 4. A greedy partition of the natural numbers into sets avoiding arithmetic progressions.

Suppose we consider the integers $0, 1, 2, \ldots$ in turn, and place each new integer i into the set of lowest index S_k $(k \geq 0)$ so that S_k never contains three integers in arithmetic progression. For example, we put 0 and 1 in S_0 , but placing 2 in S_0 would create an arithmetic progression of size 3 (namely, $\{0,1,2\}$), so we put 2 in S_1 , etc.

 $N = \frac{1}{2}$ as follows: $\frac{1}{2}$ as $\frac{1}{2}$ as $\frac{1}{2}$ as follows: $a_k = n$ if k is placed into set S_n . Here are the first few terms of this sequence:

Gerver, Propp, and Simpson showed that $a_{3k+r} =$ $|(3a_k + r)/2|$ for $k \ge 0, 0 \le r < 3$. It follows $\sum_{k=1}^{\infty} h/k > 0$

Example 5. Merge sort.

Consider sorting a list of n numbers as follows:

- sort the first half of the list recursively;
- sort the second half of the list recursively;
- merge the two sorted lists together.

The total number of omparisons needed is given by $T(1) = 0$ and

$$
T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rfloor) + n - 1
$$

for $n \geq 2$.

It is now not hard to see that $T(n)$ is 2-regular, and in fact

$$
T(n) = n\lceil \log_2 n \rceil - 2^{\lceil \log_2 n \rceil} + 1
$$

for $n \geq 1$.

Inferring k-regular Sequences

Given a sequence $(s_n)_{n>0}$, how can we determine if it is k -regular?

- construct a matrix in which the rows are elements of the k -kernel, and attempt to do row reduction
- \bullet as elements further out in the k-kernel are examined, the number of columns of the matrix that are known in all entries decreases
- if rows that are previously linearly independent suddenly become dependent with the elimination of terms further out in the sequen
e, then no relation can be accurately deduced; stop and retry after omputing more terms
- \bullet if the subsequence $(s(\kappa'\ n+c))_{n>0}$ is not linearly dependent on the previous sequences, try adding the subsequences $(s(\kappa^j(\kappa n +a)+$ $(c))_{n>0}$ for $0 \leq a < k$

Inferring k-regular Sequences

· when no more linearly independent sequences an be found, you have found hypotheti
al relations for the sequence

Inferring k **-regular Sequences**

• (N. Strauss, 1988) Define

$$
r(n) = \sum_{0 \le i < n} \binom{2i}{i},
$$

- \bullet let $\nu_3(n)$ be the exponent of the highest power of 3 that divides n .
- The first few terms of $\nu_3(r(n))$ are:

 $0, 1, 2, 0, 2, 3, 1, 2, 4, 0, 1, 2, 0, 3, 4, 2, 3, 5, 1, 2, \ldots$

- A 3-regular sequence recognizer easily produces the following conjectured relations (where $f(n) = \nu_3(r(n+1))$):
- $f(3n+2) = f(n) + 2;$
- $f(9n) = f(9n + 3) = f(3n);$
- $f(9n+1) = f(9n+4) = f(9n+7) = f(3n) +$ $\overline{1}$.

 With a little more work, one arrives at the conjecture

$$
\nu_3(r(n)) = \nu_3(n^2 \binom{2n}{n}).
$$

- proved by Allou
he and JOS.
- A beautiful proof of this identity using 3-adic analysis was also given by Don Zagier.
- Zagier showed that if we set

$$
F(n) = \frac{\sum_{0 \le k \le n-1} {2k \choose k}}{n^2 {2n \choose n}},
$$

then $F(n)$ extends to a 3-adic analytic function from \mathbb{Z}_3 to $-1 + 3\mathbb{Z}_3$, and has the expansion:

$$
F(-n) = -\frac{(2n-1)!}{(n!)^2} \sum_{0 \le k \le n-1} \frac{(k!)^2}{(k-1)!}.
$$

A "Mechanically-Produced" Conjecture

Let

$$
a(n) = \sum_{0 \le k \le n} \binom{n}{k} \binom{n+k}{k}.
$$

Let $b(n) = \nu_3(a(n))$. Then computer experiments suggest:

 $b(n) =$

(b(bn=3
) ⁺ (bn=3 mod 2); if n 0; ² (mod 3); \blacksquare) + \blacksquare

This has been verified for $0 \le n \le 10,000$.

Open Problems on k-regular Sequences

 $1.1.1.1$ Prove or disprove or disprove or dispersion of $1.1.1$ $\sum_{i=1}^{\infty}$ $\frac{1}{i}$ $\frac{1}{i}$ $\frac{1}{i}$ $\frac{1}{i}$ $\frac{1}{i}$ a 2-regular sequence.

Comment. Suppose a(n) ⁼ ^b 2 + \circ 2 \cdot \cdot \cdot 2-regular. Define $b(n) := a(n + 1) - a(n)$ for $n \geq 1$. Then $(b(n))_{n>0}$ would be 2-automatic, and is over the alphabet $\{0, 1\}$. The 1's in b are in positions $c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 11,$ $c_5 = 22$, $c_6 = 45$, $c_7 = 90$, etc. Then $c_{i+1} - 2c_i$ is the i 'th bit in the binary expansion of $\sqrt{2}$.

2. Suppose S and T are k-regular sequences and $T(n) \neq 0$ for all n. Prove or disprove: if $S(n)/T (n)$ is always an integer, then $S(n)/T (n)$ is k -regular.

Comment. This is an analogue of van der Poorten's Hadamard quotient theorem.

Open Problems on k-regular Sequences

3. Prove or disprove: the 5-term analogue of the Gerver-Propp-Simpson sequence is not 5regular.

Comment. Computer experiments show that if it is, the \mathbb{Z} -module generated by the 5-kernel must have large rank.

4. Prove or disprove: if a sequence $(a(n))_{n>0}$ is simultaneously k - and *l*-regular, where k and l are multiplicatively independent, then $(a(n))_{n>0}$ satisfies a linear recurrence.

Theorem. (Allouche, 1999) If $(a(n))_{n>0}$ is simultaneously k - and *l*-regular, then it is kl regular.

Open Problems on k-regular Sequences

5. Prove or disprove: if q is a polynomial taking integer values and p is a prime, then $(\nu_p(q(n)))_{n>0}$ is either ultimately periodic or not p -regular.

Comment. If we understood, for example, the sequence $\nu_5(n^-+1)$, then we would understand the 5-adic expansion of $\sqrt{-1}$.

For Further Reading

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