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On Intervals $(kn, (k+1)n)$ Containing a Prime for All $n > 1$

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Abstract

We study values of k for which the interval $(kn, (k+1)n)$ contains a prime for every $n > 1$. We prove that the list of such integers k includes $1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 100,000,000$. Moreover, for every known k in this list, we give a good upper bound for the smallest $N_k(m)$, such that if $n \geq N_k(m)$, then the interval $(kn, (k+1)n)$ contains at least m primes.

1 Introduction and main results

In 1850, Chebyshev proved the famous Bertrand postulate (1845) that every interval $[n, 2n]$ contains a prime (for a very elegant version of his proof, see Redmond [10, Theorem 9.2]).

Other nice proofs were given by Ramanujan in 1919 [8] and Erdős in 1932 (reproduced in Erdős and Surányi [4, p. 171–173]). In 2006, El Bachraoui [1] proved that every interval $[2n, 3n]$ contains a prime, while Loo [6] proved the same statement for every interval $[3n, 4n]$. Moreover, Loo found a lower bound for the number of primes in the interval $[3n, 4n]$. In 1952, Nagura [7] proved that there is always a prime between n and $\frac{6}{5}n$ for $n \geq 25$. From his result, it follows that the interval $[5n, 6n]$ always contains a prime. In this paper we prove the following:

Theorem 1. *The list of integers k for which every interval $(kn, (k+1)n)$, $n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 100,000,000$.*

To prove Theorem 1, in Section 3 we introduce and study the so-called k-Chebyshev primes. We give them, and the generalized Ramanujan primes, the best estimates of the form p_{tn}, where p_n is the n-th prime. Note that the core of the proof of Theorem 1 is Proposition 9, which in turn depends on Proposition 8.

In passing, for every $k = 1, 2, 3, 5, 9, 14$, we give an algorithm for finding the smallest $N_k(m)$, such that for $n \geq N_k(m)$, the interval $(kn, (k+1)n)$ contains at least m primes.

Proof of Theorem 1 is completed in Section 7 by computer research of sequence A218831 in [13].

2 Case $k = 1$

Ramanujan [8] not only proved Bertrand's postulate, but also provided the smallest integers $\{R(m)\}$, such that if $x \geq R(m)$, then the interval $\left(\frac{x}{2}, x\right]$ contains at least m primes, or equivalently, $\pi(x) - \pi(x/2) \geq m$. It is easy to see that it is sufficient to consider *integer* x, and it is also evident that every term of $\{R(m)\}$ is prime. The numbers $\{R(m)\}$ are called *Ramanujan primes* [14]. It is sequence A104272 in [13]:

$$2, 11, 17, 29, 41, 47, 59, 67, 71, 97, \ldots \tag{1}$$

Since $\pi(x) - \pi(x/2)$ is not a monotonic function, to calculate the Ramanujan numbers one should have an effective upper bound for $R(m)$. Ramanujan [8] showed that

$$\pi(x) - \pi(x/2) > \frac{1}{\ln x}\left(\frac{x}{6} - 3\sqrt{x}\right),\ x > 300. \tag{2}$$

In particular, for $x \geq 324$, the left-hand side is positive and thus ≥ 1. Using direct descent, he found that $\pi(x) - \pi(x/2) \geq 1$ from $x \geq 2$. Thus $R(1) = 2$, which proves the Bertrand postulate. Further, e.g., for $x \geq 400$, the left-hand side of (2) is more than 1 and thus ≥ 2. Again, using direct descent, he found that $\pi(x) - \pi(x/2) \geq 2$ from $x \geq 11$. Thus $R(2) = 11$, etc. Sondow [14] found that $R(m) < 4m\ln(4m)$ and conjectured that

$$R(m) < p_{3m} \tag{3}$$

which was proved by Laishram [5]. Since, for $n \ge 2$, $p_n \le en \ln n$ (cf. [3, Section 4]), then (3) yields $R(m) \le 3em \ln(3m)$, $m \ge 1$. Let $x = 2n$. If $2n \ge R(m)$, then $\pi(2n) - \pi(n) \ge m$. Thus the interval $(n, 2n)$ contains at least m primes, if

$$n \ge \left\lceil \frac{R(m)+1}{2} \right\rceil = \begin{cases} 2, & \text{if } m = 1; \\ \frac{R(m)+1}{2}, & \text{if } m \ge 2. \end{cases}$$

Let $N_1(m)$ denote the smallest number such that if $n \ge N_1(m)$, then the interval $(n, 2n)$ contains at least m primes. It is clear that $N_1(1) = R(1) = 2$. If $m \ge 2$, formally the condition $x = 2n \ge 2N_1(m)$ is not stronger than the condition $x \ge R(m)$, since the latter holds for x even and odd. Therefore, for $m \ge 2$, we have $N_1(m) \le \frac{R(m)+1}{2}$. Let us show that, in fact, we have the equality

Proposition 2. *For $m \ge 2$,*

$$N_1(m) = \frac{R(m)+1}{2}. \tag{4}$$

Proof. Note that the interval $\left(\frac{R(m)-1}{2}, R(m)-1\right)$ cannot contain more than $m-1$ primes. Indeed, it is an interval of type $\left(\frac{x}{2}, x\right)$ for integer x and the following such interval is $\left(\frac{R(m)}{2}, R(m)\right)$. By definition, $R(m)$ is the *smallest* number such that if $x \ge R(m)$, then $\{(\frac{x}{2}, x)\}$ contains $\ge m$ primes. Therefore, the supposition that the interval $\left(\frac{R(m)-1}{2}, R(m)-1\right)$ contains $\ge m$ primes contradicts the minimality of $R(m)$. Since the following interval of type $(y,\ 2y)$ with integer $y \ge \frac{R(m)-1}{2}$ is $\left(\frac{R(m)+1}{2},\ R(m)+1\right)$, Eq. (4) then follows. □

So the sequence $\{N_1(m)\}$, by (1), is A084140 in [13]:

$$2, 6, 9, 15, 21, 24, 30, 34, 36, 49, \ldots \tag{5}$$

3 Generalized Ramanujan numbers

Our research is based on a generalization of Ramanujan's method. With this aim, we define generalized Ramanujan numbers (cf. [12, Section 10], and the earlier comment in A164952 in [13]).

Definition 3. Let $v > 1$ be a real number. A v-Ramanujan number, $R_v(m)$, is the smallest integer such that if $x \ge R_v(m)$, then $\pi(x) - \pi(x/v) \ge m$.

It is known [10] that all v-Ramanujan numbers are primes. In particular, $R_2(m) = R(m)$, $m = 1, 2, \ldots$ are the proper Ramanujan primes.

Definition 4. For a real number $v > 1$ the *v-Chebyshev number*, $C_v(m)$, is the smallest integer such that if $x \ge C_v(m)$, then $\vartheta(x) - \vartheta(x/v) \ge m \ln x$, where $\vartheta(x) = \sum_{p \le x} \ln p$ is the Chebyshev function.

Since $\frac{\vartheta(x)-\vartheta(x/v)}{\ln x}$ can increase by 1 only when x is prime, then all v-Chebyshev numbers are primes.

Proposition 5. *We have*

$$R_v(m) \le C_v(m). \tag{6}$$

Proof. Let $x \ge C_v(m)$. Then we have

$$m \le \frac{\vartheta(x)-\vartheta(x/v)}{\ln x} = \sum_{\frac{x}{v}<p\le x} \frac{\ln p}{\ln x} \le \sum_{\frac{x}{v}<p\le x} 1 = \pi(x)-\pi(x/v). \tag{7}$$

Thus, if $x \ge C_v(m)$, then *always* $\pi(x)-\pi(x/v) \ge m$. By Definition 3, this means that $R_v(m) \le C_v(m)$. □

Now we give upper bounds for $C_v(m)$ and $R_v(m)$.

Proposition 6. *Let $x = x_v(m) \ge 2$ be any number for which*

$$\frac{x}{\ln x}\left(1-\frac{1300}{\ln^4 x}\right) \ge \frac{vm}{v-1}. \tag{8}$$

Then

$$R_v(m) \le C_v(m) \le x_v(m). \tag{9}$$

Proof. We use the following inequality of Dusart [3] (see his Theorem 5.2):

$$|\vartheta(x)-x| \le \frac{1300x}{\ln^4 x},\ x \ge 2.$$

Thus we have

$$\begin{aligned}\vartheta(x)-\vartheta(x/v) &\ge x\left(1-\frac{1}{v}-1300\left(\frac{1}{\ln^4 x}-\frac{1}{v\ln^4 \frac{x}{v}}\right)\right)\\ &\ge x\left(1-\frac{1}{v}\right)\left(1-\frac{1300}{\ln^4 x}\right).\end{aligned}$$

If now

$$x\left(1-\frac{1}{v}\right)\left(1-\frac{1300}{\ln^4 x}\right) \ge m\ln x,\ x \ge x_v(m),$$

then

$$\vartheta(x)-\vartheta(x/v) \ge m\ln x,\ x \ge x_v(m)$$

and, by Definition 4, $C_v(m) \le x_v(m)$. So, according to (6), we conclude that $R_v(m) \le x_v(m)$. □

Proposition 6 gives the terms of sequences $\{C_v(m)\},\ \{R_v(m)\}$ for every $v > 1,\ m \geq 1$. In particular, if $k = 1$ we find $\{C_2(m)\}$:

$$\begin{aligned}&11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 223,\\ &229, 233, 239, 241, 263, 269, 281, 307, 311, 347, 349, 367, 373, 401, 409,\\ &419, 431, 433, 443, \ldots\end{aligned} \tag{10}$$

This sequence requires a separate comment. We observe that up to $C_2(100) = 1489$ only two terms of this sequence ($C_2(17) = 223$ and $C_2(36) = 443$) are not Ramanujan numbers, and the sequence is missing only the following 6 Ramanujan numbers: 181,227,439,491,1283,1301 and no others up to the 104-th Ramanujan number 1489. The latter observation shows how much the ratio $\frac{\vartheta(x)}{\ln x}$ exactly approximates $\pi(x)$. Similar observations are also valid for the following sequences for $v = \frac{k+1}{k}$ (and undoubtedly require an additional special study):

for $k = 2,\ \{C_v(m)\}$,

$$13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \ldots; \tag{11}$$

for $k = 2,\ \{R_v(m)\}$,

$$2, 13, 37, 41, 67, 73, 97, 127, 137, 173, 179, 181, 211, 229, 239, \ldots; \tag{12}$$

for $k = 3,\ \{C_v(m)\}$,

$$29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \ldots; \tag{13}$$

for $k = 3,\ \{R_v(m)\}$,

$$11, 29, 59, 67, 101, 149, 157, 163, 191, 227, 269, 271, 307, 379, \ldots; \tag{14}$$

for $k = 5,\ \{C_v(m)\}$,

$$59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \ldots; \tag{15}$$

for $k = 5,\ \{R_v(m)\}$,

$$29, 59, 137, 139, 149, 223, 241, 347, 353, 383, 389, 563, 569, 593, \ldots; \tag{16}$$

for $k = 9,\ \{C_v(m)\}$,

$$223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots; \tag{17}$$

for $k = 9,\ \{R_v(m)\}$,

$$127, 223, 227, 269, 349, 359, 569, 587, 593, 739, 809, 857, 991, 1009, \ldots; \tag{18}$$

for $k = 14,\ \{C_v(m)\}$,

$$307, 347, 563, 569, 733, 821, 1427, 1429, 1433, 1439, 1447, 1481, \ldots; \tag{19}$$

for $k = 14,\ \{R_v(m)\}$,

$$127, 307, 347, 563, 569, 733, 1423, 1427, 1429, 1433, 1439, 1447, \ldots \tag{20}$$

Remark 7. In fact, Dusart [3, Theorem 5.2] gives several inequalities of the form

$$|\vartheta(x) - x| \le \frac{ax}{\ln^b x},\ x \ge x_0(a,b)$$

From a computing point of view, the values $a = 1300,\ b = 4$ from Dusart's theorem are not always the best. The analysis for $x \ge 25$ shows that the condition

$$x\left(1 - \frac{1}{v}\right)\left(1 - \frac{ax}{\ln^b x}\right) \ge m \ln x$$

is satisfied for the smallest $x_v(m) = x_v(m; a, b)$, using the following values of a and b from Dusart's theorem:

$a = 3.965,\ b = 2$ for x in range $(25,\ 7 \cdot 10^7]$;
$a = 1300,\ b = 4$ for x in range $(7 \cdot 10^7,\ 10^9]$;
$a = 0.001,\ b = 1$ for x in range $(10^9,\ 8 \cdot 10^9]$;
$a = 0.78,\ b = 3$ for x in range $(8 \cdot 10^9,\ 7 \cdot 10^{33}]$;
$a = 1300,\ b = 4$ for $x > 7 \cdot 10^{33}$,

which minimizes the amount of calculations for v-Chebyshev primes.

4 Bounds of type (3)

Proposition 8. *We have*

$$C_2(m-1) \le p_{3m},\ m \ge 2; \tag{21}$$

$$R_{\frac{3}{2}}(m) \le p_{4m},\ m \ge 1;\ C_{\frac{3}{2}}(m-1) \le p_{4m},\ m \ge 2; \tag{22}$$

$$R_{\frac{4}{3}}(m) \le p_{6m},\ m \ge 1;\ C_{\frac{4}{3}}(m-1) \le p_{6m},\ m \ge 2; \tag{23}$$

$$R_{\frac{6}{5}}(m) \le p_{11m},\ m \ge 1;\ C_{\frac{6}{5}}(m-1) \le p_{11m},\ m \ge 2; \tag{24}$$

$$R_{\frac{10}{9}}(m) \le p_{31m},\ m \ge 1;\ C_{\frac{10}{9}}(m-1) \le p_{31m},\ m \ge 2; \tag{25}$$

$$R_{\frac{15}{14}}(m) \le p_{32m},\ m \ge 1;\ C_{\frac{15}{14}}(m-1) \le p_{32m},\ m \ge 2. \tag{26}$$

Proof. Firstly, let us find some values of $m_0 = m_0(k)$, such that, at least, for $m \ge m_0$ all formulas (21)–(26) hold. According to (8) and (9), it is sufficient to show that, for $m \ge m_0$, we can take p_{tm}, where $t = 3, 4, 6, 11, 31, 32$ for formulas (21)-(26) respectively, in the capacity of $x_v(m)$. As we noted in Remark 7, in order to find possibly smaller values of m_0, we use the bound

$$\frac{x}{\ln x}\left(1 - \frac{3.965}{\ln^2 x}\right) \ge \frac{vm}{v-1} \tag{27}$$

instead of (8). In order to get $x = p_{mt}$ satisfying this inequality, note that [11]

$$p_n \geq n \ln n.$$

Therefore, it is sufficient to consider p_{mt} satisfying the inequality

$$\ln p_{tm} \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right).$$

On the other hand, for $n \geq 2$, (see [3, (4.2)])

$$\ln p_n \leq \ln n + \ln \ln n + 1.$$

Thus, it is sufficient to choose m so large that the following inequality holds

$$\ln(tm) + \ln \ln(tm) + 1 \leq \left(1 - \frac{1}{v}\right) t \ln(tm) \left(1 - \frac{3.965}{\ln^2(tm \ln(tm))}\right),$$

or, since $1 - \frac{1}{v} = \frac{1}{k+1}$, that

$$\frac{\ln(tm) + \ln \ln(tm) + 1}{\ln(tm)(1 - \frac{3.965}{\ln^2(tm \ln(tm))})} \leq \frac{t}{k+1}. \tag{28}$$

For example, let $k = 1,\ t = 3$. We can choose $m_0 = 350$. Then the left-hand side of (28) equals $1.4976 \cdots < 1.5$. This means that at least for $m \geq 350$, the estimate (21) is valid. Using a computer for $m \leq 350$, we obtain (21) for $m \geq 2$. Other bounds are proved in the same way. □

5 Bounds and formulas for $N_k(m)$

Proposition 9.

$$N_k(1) = 2,\ k = 2, 3, 5, 9, 14. \tag{29}$$

For $m \geq 2,\ k \geq 1$,

$$N_k(m) \leq \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil; \tag{30}$$

besides, if $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$, *then*

$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k}{k+1} \tag{31}$$

and, if $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$, *then*

$$N_k(m) = \left\lceil \frac{R_{\frac{k+1}{k}}(m)}{k+1} \right\rceil = \frac{R_{\frac{k+1}{k}}(m) + k - 1}{k+1}. \tag{32}$$

Proof. If $m \geq 2$, formally, the condition $x = (k+1)n \geq (k+1)N_k(m)$ is not stronger than the condition $x \geq R_{\frac{k+1}{k}}(m)$, since the first one is valid only for x multiple of $k+1$. Therefore, for $m \geq 2$, (30) holds. It allows calculation of terms in the sequence $\{N_k(m)\}$ for $k > 1,\ m \geq 2$. Since $N_k(1) \leq N_k(2)$, then, having $N_k(2)$, we can also prove (29) using direct calculation. Now let $R_{\frac{k+1}{k}}(m) \equiv 1 \pmod{k+1}$. Note, that for $y = (R_{\frac{k+1}{k}}(m) - 1)/(k+1)$ the interval

$$(ky,\ (k+1)y) = \left(\frac{k}{k+1}\left(R_{\frac{k+1}{k}}(m) - 1\right),\ R_{\frac{k+1}{k}}(m) - 1\right) \tag{33}$$

cannot contain more than $m-1$ primes. Indeed, it is an interval of type $\left(\frac{k}{k+1}x, x\right)$ for integer x, and the following such interval is

$$\left(\frac{k}{k+1}\left(R_{\frac{k+1}{k}}(m)\right), R_{\frac{k+1}{k}}(m)\right).$$

By definition, $R_{\frac{k+1}{k}}(m)$ is the *smallest* number such that if $x \geq R_{\frac{k+1}{k}}(m)$, then $\{(\frac{k}{k+1}x, x)\}$ contains $\geq m$ primes. Therefore, the supposition that the interval (33) contains $\geq m$ primes contradicts the minimality of $R_{\frac{k+1}{k}}(m)$. Since the following interval of type $(ky,\ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 1)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k),\ R_{\frac{k+1}{k}}(m) + k\right),$$

then (31) follows.

Finally, let $R_{\frac{k+1}{k}}(m) \equiv 2 \pmod{k+1}$. Again, for $y = (R_{\frac{k+1}{k}}(m) - 2)/(k+1)$ the interval

$$(ky,\ (k+1)y) = \left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2),\ R_{\frac{k+1}{k}}(m) - 2\right) \tag{34}$$

cannot contain more than $m-1$ primes. Indeed, comparing interval (34) with interval (33), we see that they contain the same integers except for $R_{\frac{k+1}{k}}(m) - 2$, which is multiple of $k+1$. Therefore, they contain the same number of primes, and this number does not exceed $m-1$. Again, since the following interval of type $(ky,\ (k+1)y)$ with integer $y \geq \frac{k}{k+1}(R_{\frac{k+1}{k}}(m) - 2)$ is

$$\left(\frac{k}{k+1}(R_{\frac{k+1}{k}}(m) + k - 1),\ R_{\frac{k+1}{k}}(m) + k - 1\right),$$

then (32) follows. □

As a corollary from (29), (31) and (32), we obtain the following formula in case $k = 2$.

Proposition 10.

$$N_2(m) = \begin{cases} 2, & \text{if } m = 1; \\ \left\lceil \frac{R_{\frac{3}{2}}(m)}{3} \right\rceil, & \text{if } m \geq 2. \end{cases} \tag{35}$$

Note that, if $k \geq 3$ and $R_{\frac{k+1}{k}}(m) \equiv j \pmod{k+1},\ 3 \leq j \leq k$, then, generally speaking, (30) is not an equality. Evidently, $N_k(m) \geq N_k(m-1)$, and it is interesting that the equality is attainable (see sequences (37)–(40) below).

Example 11. Let $k = 3,\ m = 2$. Then $v = \frac{4}{3}$ and, by (14), $R_{\frac{4}{3}}(2) = 29 \equiv 1 \pmod 4$. Therefore, by (31), $N_3(2) = \frac{29+3}{4} = 8$. Indeed, interval $(3 \cdot 7,\ 4 \cdot 7)$ already contains only prime 23.

Example 12. Let $k = 3,\ m = 3$. Then, by (14), $R_{\frac{4}{3}}(3) = 59 \equiv 3 \pmod 4$. Here $N_3(3) = 11$ which is essentially less than $\left\lceil R_{\frac{4}{3}}(3)/4 \right\rceil = 15$. Indeed, each interval

$$(3 \cdot 15,\ 4 \cdot 15),\ (3 \cdot 14,\ 4 \cdot 14),\ (3 \cdot 13,\ 4 \cdot 13),\ (3 \cdot 12,\ 4 \cdot 12),\ (3 \cdot 11,\ 4 \cdot 11)$$

contains more than 2 primes and only interval $(3 \cdot 10,\ 4 \cdot 10)$ contains only 2 primes.

In any case, Proposition 9 allows us to calculate terms of sequence $\{N_k(m)\}$ for every considered value of k. So, we obtain the following few terms of $\{N_k(m)\}$:

for $k = 2$,

$$2, 5, 13, 14, 23, 25, 33, 43, 46, 58, 60, 61, 71, 77, 80, 88, 103, 104, \ldots; \tag{36}$$

for $k = 3$,

$$2, 8, 11, 17, 26, 38, 40, 41, 48, 57, 68, 68, 70, 87, 96, 100, 108, 109, \ldots; \tag{37}$$

for $k = 5$,

$$2, 7, 17, 24, 25, 38, 41, 58, 59, 64, 65, 73, 95, 97, 103, 106, 107, 108, \ldots; \tag{38}$$

for $k = 9$,

$$2, 14, 23, 23, 34, 36, 57, 58, 60, 60, 77, 86, 100, 100, 102, 123, 149, \ldots; \tag{39}$$

for $k = 14$,

$$2, 11, 24, 37, 38, 39, 50, 96, 96, 96, 96, 97, 97, 125, 125, 132, 178, 178, \ldots \tag{40}$$

Remark 13. If, as in [1, 6], instead of intervals $(kn,\ (k+1)n)$, we consider intervals $[kn,\ (k+1)n]$, then sequences (5) and (36)–(38) would begin with 1.

6 Method of small intervals

If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $(x,\ (1+\frac{1}{\Delta})x]$ contains a prime, then we can calculate a bounded number of the first terms of sequences (5) and (36)–(40). Indeed, put $x_1 = kn$, such that $n \geq \frac{x_0}{k}$. Then $(k+1)n = \frac{k+1}{k}x_1$ and, if $1 + \frac{1}{\Delta} < \frac{k+1}{k}$, i.e., $\Delta > k$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \subset (kn, (k+1)n).$$

Thus, if $n \geq \frac{x_0}{k}$, then the interval $(kn, (k+1)n)$ contains a prime, and using method of finite descent, we can find $N_k(1)$. Further, put $x_2 = (1+\frac{1}{\Delta})x_1$. Then interval $(x_2,\ (1+\frac{1}{\Delta})x_2]$ also contains a prime. Thus the union

$$\left(x_1,\ (1+\frac{1}{\Delta})x_1\right] \cup \left(x_2,\ (1+\frac{1}{\Delta})x_2\right] = \left(x_1, (1+\frac{1}{\Delta})^2 x_1\right]$$

contains at least two primes. This means that if $(1+\frac{1}{\Delta})^2 x_1 < (k+1)n$ or $(1+\frac{1}{\Delta})^2 < 1+\frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^2 x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least two primes; again, using method of finite descent, we can find $N_k(2)$ etc. If $(1+\frac{1}{\Delta})^m < 1+\frac{1}{k}$, then

$$\left(x_1,\ (1+\frac{1}{\Delta})^m x_1\right] \subset (kn, (k+1)n)$$

and the interval $(kn, (k+1)n)$ contains at least m primes and we can find $N_k(m)$. In this way, we can find $N_k(m)$ for $m < \frac{\ln(1+\frac{1}{k})}{\ln(1+\frac{1}{\Delta})}$. In 2002, Ramaré and Saouter [9] proved that the interval $(x(1-28314000^{-1}),\ x)$ always contains a prime if $x > 10726905041$, or, equivalently, the interval $(x,\ (1+28313999^{-1})x)$ contains a prime if $x > 10726905419$. This means that, e.g., we can find $N_{14}(m)$ for $m \leq 1954471$. Unfortunately, this method cannot give the exact bounds and formulas for $N_k(m)$ as (30)–(32).

We can also consider a more general application of this method. Consider a fixed infinite set P of primes which we call P-primes. Furthermore, consider the following generalization of v-Ramanujan numbers.

Definition 14. For $v > 1$, a (v, P)-Ramanujan number, $R_v^{(P)}(m)$, is the smallest integer such that if $x \geq R_v^{(P)}(m)$, then $\pi_P(x) - \pi_P(x/v) \geq m$, where $\pi_P(x)$ is the number of P-primes not exceeding x.

Note that every (v, P)-Ramanujan number is P-prime. If we know a theorem of the type: for $x \geq x_0(\Delta)$, the interval $\left(x,\ (1+\frac{1}{\Delta})x\right]$ contains a P-prime, then using the above described algorithm, we can calculate a bounded number of the first (v, P) -Ramanujan numbers. For example, let P be the set of primes $p \equiv 1 \pmod 3$. From the result of Cullinan and Hajir [2] it follows, in particular, that for $x \geq 106706$, the interval $(x,\ 1.048x)$ contains a P-prime. Using the same algorithm, we can calculate the first 14 $(2, P)$-Ramanujan numbers. They are

$$7, 31, 43, 67, 97, 103, 151, 163, 181, 223, 229, 271, 331, 337. \tag{41}$$

Analogously, if P is the set of primes $p \equiv 2 \pmod 3$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$11, 23, 47, 59, 83, 107, 131, 167, 227, 233, 239, 251, 263, 281, \dots; \tag{42}$$

if P is the set of primes $p \equiv 1 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$13, 37, 41, 89, 97, 109, 149, 229, 233, 241, 257, 277, 281, 317, \dots; \tag{43}$$

and, if P is the set of primes $p \equiv 3 \pmod 4$, then the sequence of $(2, P)$-Ramanujan numbers begins

$$7, 23, 47, 67, 71, 103, 127, 167, 179, 191, 223, 227, 263, 307, \dots; \tag{44}$$

Let $N_k^{(P)}(m)$ denote the smallest number such that for $n \geq N_k^{(P)}(m)$, the interval $(kn, (k+1)n)$ contains at least m P-primes. It is easy to see that formulas (30)–(32) hold for $N_k^{(P)}(m)$ and $R_{\frac{k+1}{k}}^{(P)}(m)$. In particular, in cases $k = 1, 2$ we have the formulas

$$N_1^{(P)}(m) = \frac{R_2^{(P)}(m) + 1}{2}, \quad N_2^{(P)}(m) = \left\lceil \frac{R_{\frac{3}{2}}^{(P)}(m)}{3} \right\rceil. \tag{45}$$

Therefore, the following sequences for $N_1^{(P)}(m)$, correspond to sequences (41)–(44) respectively:

$$4, 16, 22, 34, 49, 52, 76, 82, 91, 112, 115, 136, 166, 169, \dots; \tag{46}$$

$$6, 12, 24, 30, 42, 54, 66, 84, 114, 117, 120, 126, 132, 141, \dots; \tag{47}$$

$$7, 19, 21, 45, 49, 55, 75, 115, 117, 121, 129, 139, 141, 159, \dots; \tag{48}$$

$$4, 12, 24, 34, 36, 52, 64, 84, 90, 96, 112, 114, 132, 154, \dots \tag{49}$$

7 The proof of Theorem 1

For $k \geq 1$, let $a(k)$ denote the least integer $n > 1$ for which the interval $(kn,\ (k+1)n)$ contains no prime; if no such n exists, we put $a(k) = 0$. Taking into account (29), note that

$a(k)=0$ for $k=1,2,3,5,9,14$. Consider sequence $\{a(k)\}$. Its first few terms are (A218831 in [13])

$$0,0,0,2,0,4,2,3,0,2,3,2,2,0,6,2,2,3,2,6,3,2,4,2,2,7,2,2,4,3,\ldots \tag{50}$$

Calculations of $a(k)$ for $k\in[1,15]$, except for $k=1,2,3,5,9,14$, give positive values of $a(k)$. Computer calculations of $a(k)$ in the range $\{16,\ldots,10^8\}$ show that all values of $a(k)$ in this range are positive and belong to the interval $[2,16]$. This completes the proof. □

In conclusion, we present a distribution of numbers of values $a(k)=2,3,\ldots,16$ within intervals $\{[1,10^7(i)]\}, i=1,\ldots,10$. All these numerical results are obtained using the following *Mathematica* program:

```
start=2;cutOff=100;
a218831=Table[
        NestWhile[#+1&,start,
                Union[PrimeQ[Range[# k+1,# (k+1)-1]]]!={False}&,
                1,cutOff],
        {k,1,100000000}]/.{cutOff+start->0};
```

We have for $a(k)=2$ the numbers

$$\begin{gathered}8729394, 17566347, 26437886, 35330619, 44238546,\\ 53158353, 62087802, 71025543, 79969616, 88921064.\end{gathered}$$

In general, here we have a simple explicit formula: the number of $a(k)=2, k\le K$ is $K+1-\pi(2K+1)$. Further, let

$$A_t(K)=|\{k\le K: a(k)=t\}|.$$

In cases $t\ge 3$ we have no explicit formulas. But, taking into account the distribution of primes into residue classes, a rough argument suggests that $A_t(K)\asymp c_tK(\ln K)^{2-t}$. For example, for $a(k)=3$ within the considered intervals we have the numbers

$$\begin{gathered}1061880, 2050703, 3014798, 3963752, 4901317,\\ 5830488, 6752801, 7668802, 8580597, 9486975,\end{gathered}$$

and one can hope that $c_3\approx 1.7\cdots$. In other cases we have

$$t=4:\ 173835, 321315, 461745, 597249, 729660, 859605, 987238, 1113288, 1237558, 1360344;$$

$$t=5:\ 25108, 45086, 63177, 80407, 97199, 113213, 128850, 144474, 159648, 174577;$$

$$t=6:\ 7312, 12542, 17150, 21536, 25714, 29734, 33616, 37243, 40952, 44503;$$

$$t=7:\ 1753, 2918, 3841, 4749, 5590, 6373, 7201, 7950, 8691, 9378;$$

$$t=8:\ 449, 703, 918, 1109, 1309, 1507, 1670, 1810, 1977, 2141;$$

$$t=9: \ 149, 216, 278, 342, 400, 440, 508, 558, 606, 647;$$
$$t=10: \ 73, 109, 138, 164, 186, 203, 222, 232, 249, 262;$$
$$t=11: \ 18, 25, 29, 31, 35, 36, 42, 46, 48, 49;$$
$$t=12: \ 13, 15, 17, 19, 21, 25, 26, 29, 30, 31;$$
$$t=13: \ 2, 3, 3, 3, 3, 3, 3, 4, 7, 7;$$
$$t=14: \ 4, 5, 6, 6, 6, 6, 7, 7, 7, 8;$$
$$t=15: \ 0, 2, 3, 3, 3, 3, 3, 3, 3, 3;$$
$$t=16: \ 4, 5, 5, 5, 5, 5, 5, 5, 5, 5.$$

For those t when the difference

$$\frac{A_t(10^8)}{10^8}(8\ln 10)^{t-2} - \frac{A_t(10^7)}{10^7}(7\ln 10)^{t-2}$$

remains less than 0.5, we can get an impression about the change of c_t depending on t. So, $c_2 = 1$ and approximately $c_3 = 1.7,\ c_4 = 4.6,\ c_5 = 11,\ c_6 = 49$.

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