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ON UNITARY PERFECT NUMBERS

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1. Introduction.

This paper extends (in Theorem 2 below) some results on unitary perfect numbers given in a recent paper by M.V. Subbarao and L J. Warren [1]. Let $\sigma^*(N)$ denote the sum of the unitary divisors of N, that is, divisors d of N for which $(d, N/d) = 1$. It is easily seen that if N has the prime decomposition

$$N = p_1^{a_1} p_2^{a_2} \ldots\ldots p_r^{a_r}, \text{ then}$$

$$\sigma^*(N) = \left(1 + p_1^{a_1}\right)\left(1 + p_2^{a_2}\right)\ldots\left(1 + p_r^{a_r}\right) \text{ for } n > 1.$$

A number N is said to be unitary perfect if $\sigma^*(N) = 2N$. The only five unitary perfect numbers known at present are 6, 60, 90, 87360 and $146361946186458562560000 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19\ 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$ the last being due to C.R. Wall [3]

Now some further notation must be introduced. We write $N = 2^m n$ where $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ with $p_1 < p_2 < \ldots < p_r$, m

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a positive integer, the p_i's are odd primes and the a_i's are positive integers. Also, we write $n = n_1 n_2 n_3$ where every prime factor of n_1 is of the form 1 (mod 4), every prime factor of n_2 is of the form 3 (mod 4) and has even exponent, and every factor of n_3 is of the form 3 (mod. 4) but has odd exponent. Let a, b and c denote the number of distinct primes in n_1, n_2, and n_3. Now $K(a, b, c)$ is the class of all odd numbers $n = n_1 n_2 n_3$ associated with a, b, and c ; and

1.1 $$B(a, b, c) = \max \{\sigma^*(x)/x\},$$

where $x \varepsilon K(a, b, c)$. At this point, we would like to point out that result (2.1) of the paper [1] cited earlier needs correction. It was stated in that paper, "if $n = n_1 n_2 n_3$ and $n' = n_1' \; n_2' \; n_3'$ are both members of $K(a, b, c)$, then

1.2 $$\sigma^*(n)/n \geqslant \sigma^*(n')/n'$$

whenever $n'_1 \geqslant n_1, n'_2 \geqslant n_2$, and $n'_3 \geqslant n_3$." A counter example is $n = 25 \cdot 41$ and $n' = 5 \cdot 41^2$. The correct statement should be as follows :

if

$$n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \text{ and } n' = q_1^{b_1} q_2^{b_2} \ldots q_r^{b_r},$$

then the inequality (1.2) holds whenever

$$q_1^{b_1} \geqslant p_1^{a_1}, q_2^{b_2} \geqslant p_2^{a_2}, \ldots, q_2^{b_r} \geqslant p_r^{a_r}.$$

The main tool used in the paper [1] was the following :

Lemma. If $N = 2^m n$ is unitary perfect, then

1.3 $$p_r/(2^m+1) \text{ if } a_1 = a_2 = \ldots = a_{r-1} = 1 \;;$$

1.4 $$a + b + c \leqslant m + 1 \text{ and equality holds when } c = 0;$$

1.5 $B(a, b, c) \geqslant 2^{m+1}/(2^m+1)$ for at least one set of values of a, b, c satisfying (1.4).

Using this lemma the following theorem was proved in [1].

Theorem 1. Let $N = 2^m p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ be unitary perfect.

(1) If $r = 1$, then $N = 6$.

(2) If $m = 1$, then $N = 6$ or 90.

(3) If $m = 2$, then $N = 60$.

(4) If $r = 2$, then $N = 60$ or 90.

(5) It is not possible for $m = 3, 4, 5$, or 7.

(6) It is not possible for $r = 3$ or 5.

(7) If $m = 6$, then $N = 87{,}360$.

(8) If $r = 4$, then $N = 87{,}360$.

2. An extension of theorem 1.

Our main object is to extend the results of this theorem. We prove :

Theorem 2. Let $N = 2^m p_1^{a_1} p_2^{a_2} \ldots p_r^{a}$ be unitary perfect.

(1) It is not possible for $m = 8, 9, 10$.

(2) If is not possible for $r = 6$.

Proof. In proving this we proceed as follows : first, we list all possible classes $K(a, b, c)$ using lemma 1; then for each class the lowest possible primes in the classes a, b, and c are selected; next this number N is modified by eliminating certain factors by observing the exponents of 2 and 3 occurring in $\sigma^*(N)$. For all of these, $\sigma^*(n)/n$ is calculated using a desk calculator and found to be less than $2^{m+1}/(2^m+1)$. These extensive calculations are too long to be shown here but are in the possession of the authors. We briefly illustrate below the nature of calculations involved, by considering tbe case $m = 8$.

The list of all possible classes $K(a, b, c)$ is as follows :

	a	b	c
(1)	1	8	0
(2)	2	7	0
(3)	3	6	0
(4)	4	5	0
(5)	5	4	0
(6)	6	3	0
(7)	7	2	0
(8)	8	1	0
(9)	9	0	0
(10)	1	6	1
(11)	2	5	1
(12)	3	4	1
(13)	4	3	1
(14)	5	2	1
(15)	6	1	1
(16)	7	0	1
(17)	1	4	2
(18)	2	3	2
(19)	3	2	2
(20)	4	1	2
(21)	5	0	2
(22)	6	0	1
(23)	1	2	3
(24)	2	1	3
(25)	3	0	3

We now check each of these cases. We offer case (18) as an example : $K(2, 3, 2)$.

We select the lowest primes $N = 2^{8} \cdot 257 \cdot 5 \cdot 3 \cdot 7 \cdot 11^{2} \cdot 19^{2} \cdot 23^{2}$ and see if it satisfies (1·5). If it does, we now modify N to get the correct number of exponents of 2 and 3 which yields

$$N = 2^{8} \cdot (257)^{2} \cdot 5 \cdot 3 \cdot 19 \cdot 7^{2} \cdot 11^{2} \cdot 23^{2},$$

$$\sigma^{*}(N) = (2^{8}+1)\,(257^{2}+1)\,(6)\,(4)\,(20)\,(50)\,(122)\,(530),$$

9 and 1 are the powers of 2 and 3 respectively in both $2N$ and $\sigma^{*}(N)$. We now calculate $\sigma^{*}(n)/n$ and find that it is less than $2^{9}/2^{8}+1$. Therefore in view of the lemma stated earlier, there are no unitary perfect numbers in K (2, 3, 2).

3. Remarks.

Using theorem 2, it can be shown that after 87360, there is no unitary perfect number with less than twenty digits. It is very likely that the next unitary perfect number after 87,360 is Wall's number which has 24 digits. The sixth unitary perfect (which we believe to exist) must indeed be much bigger than Wall's number !

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