A Generalization of the Genocchi Numbers with Applications to Enumeration of Finite Automata Technical Report No. 2001–449

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Abstract

We consider a natural generalization of the well-studied Genocchi numbers. This generalization proves useful in enumerating the class of deterministic finite automata (DFA) which accept a finite language. We also link our generalization to the method of Gandhi polynomials for generating Genocchi numbers.

1 Introduction and Motivation

The study of Genocchi numbers and their combinatorial interpretations has received much attention (see [4, 5, 7, 9, 11, 16]). In this paper, we give another combinatorial interpretation of the Genocchi numbers, as well as suggest a generalization of the Genocchi numbers.

The Genocchi numbers may be defined in terms of the generating function

$$\frac{2t}{e^t + 1} = 1 + \sum_{n \geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

They may also be defined in the following way (cf. [11, 16]). Let the Gandhi polynomials be defined by :

$$\begin{aligned} A(n+1, k) &= k^2 A(n, k+1) - (k-1)^2 A(n, k) \\ A(1, k) &= k^2 - (k-1)^2 \end{aligned}$$

Then $|G_{2n}| = A(n-1, 1)$. The first few values of $|G_{2n}|$ are 1,1,3,17,155.

Our motivation comes from automata theory. We are interested in the number of finite languages recognized by deterministic finite automata (DFAs) with n states. It is easy to see that if a DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts a finite language, then there exists an ordering of the elements of Q, say $Q = \{0, 1, 2, \ldots, n\}$ with $q_0 = 0$ and such that $\delta(i, a) > i$ for all $i \in Q$ and $a \in \Sigma$. Thus, we are interested in the number of labeled directed graphs with labeled edges on n vertices in which all edges (u, v) satisfy $u < v$.

There have been previous generalizations of the Genocchi and Euler numbers (e.g. Dumont [6], Dumont and Randrianarivony [8], Horadam [13], Thakare [17] and Karande and Thakare [14]) but apparently none deal with the extension we propose here. To our knowledge, this generalization has not been suggested.

2 Definitions and Background

We first recall some definitions from automata theory and formal languages. For any terms not covered here, the reader may consult Hopcroft and Ullman [12]. Let Σ denote a finite alphabet. Then Σ^* is the set of all finite strings over Σ. The empty string is denoted by ϵ. A language L over Σ is a subset of Σ^*. A deterministic finite automaton (DFA) is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states, Σ is a finite alphabet of symbols, $q_0 \in Q$ is the initial state and $F \subset Q$ is the set of final states. The transition function δ is a function $\delta : Q \times \Sigma \to Q$. It is extended to $Q \times \Sigma^* \to Q$ in the following manner. For any $w \in \Sigma^*$ and $a \in \Sigma$, $\delta(wa, q) = \delta(\delta(w, q), a)$ for all states $q \in Q$ (we set $\delta(q, \epsilon) = q$ for all $q \in Q$).

A string $w \in \Sigma^*$ is accepted by M if $\delta(q_0, w) \in F$. The language accepted by a DFA M is the set of all strings accepted by M, denoted by $L(M)$:

$$L(M) = \{w \in \Sigma^* \ : \ \delta(q_0, w) \in F\}$$

We say that a DFA M accepts a language $L \subseteq \Sigma^*$ if $L = L(M)$.

We now proceed with our generalization of the Genocchi numbers. An alternate definition for the Genocchi numbers given by Dumont [4] is as follows: Define $B_{n,k}$ by

$$B_{n,k} = \sum \binom{2}{k_1}\binom{4-k_1}{k_2-k_1}\binom{6-k_2}{k_3-k_2}\cdots\binom{2(n-1)-k_{n-2}}{k_{n-1}-k_{n-2}}$$

where the sum is taken over all $k_1, \ldots, k_{n-1}$ such that $1 \le k_1 < k_2 < \cdots < k_{n-2} < k_{n-1} \le 2n - k$ and $k_j \le 2j$ for all j. Then $G_{2n} = B_{n,2}$, or, in particular

$$G_{2n} = \sum \binom{2}{k_1}\binom{4-k_1}{k_2-k_1}\binom{6-k_2}{k_3-k_2}\cdots\binom{2(n-2)-k_{n-3}}{k_{n-2}-k_{n-3}}$$

Consider the change of variables $i_1 = k_1$ and $i_\ell = k_\ell - k_{\ell-1}$ for $2 \le \ell \le n-3$. Then the following is an equivalent formula for G_{2n}:

$$\sum_{i_1=1}^{2}\binom{2}{i_1}\sum_{i_2=1}^{4-i_1}\binom{4-i_1}{i_2}\sum_{i_3=1}^{6-(i_1+i_2)}\binom{6-(i_1+i_2)}{i_3}\cdots\sum_{i_{n-2}=1}^{2(n-2)-\sum_{\ell=1}^{n-3} i_\ell}\binom{2(n-2)-\sum_{\ell=1}^{n-3} i_\ell}{i_{n-2}} \tag{2.1}$$

Given this representation, we may make the main definition of the paper, by replacing the appearance of multiples of 2 by multiples of an arbitrary integer $k \ge 2$:

Definition 2.1 *Define the following sequence of integers $G_{2n}^{(k)}$ for all $k \geq 2$ and $n \geq 1$:*

$$G_{2n}^{(k)} = \sum_{i_1=1}^{k} \binom{k}{i_1} \cdots \sum_{i_j=1}^{kj-\sum_{\ell=1}^{j-1} i_\ell} \binom{kj - \sum_{\ell=1}^{j-1} i_\ell}{i_j} \cdots \sum_{i_{n-2}=1}^{k(n-2)-\sum_{\ell=1}^{n-3} i_\ell} \binom{k(n-2) - \sum_{\ell=1}^{n-3} i_\ell}{i_{n-2}}$$

Call the sequence $\{G_{2n}^{(k)}\}_{n\geq 1}$ the k-th generalized Genocchi numbers.

n	1	2	3	4	5	6	7
$G_{2n}^{(2)}$	1	1	3	17	155	2073	38227
$G_{2n}^{(3)}$	1	1	7	145	6631	566641	81184327
$G_{2n}^{(4)}$	1	1	15	1025	209135	100482849	97657699279
$G_{2n}^{(5)}$	1	1	31	6721	5850271	15060446401	94396946822431

Figure 2.1: Small values of $G_{2n}^{(k)}$.

Note that Horadam defines the *generalized Genocchi numbers of order k* [13] which do not appear to be related to our definition in any meaningful way. Figure 2.1 shows the first few values of $G_{2n}^{(k)}$ for small values of k. The method of generating Genocchi numbers by Gandhi polynomials [11, 16] may also be generalized to generate the sequences $G_{2n}^{(k)}$.

Lemma 2.2 *Given the following Gandhi polynomials*

$$\begin{aligned} A_k(n+1,r) &= r^k A_k(n,r+1) - (r-1)^k A_k(n,r) \\ A_k(1,r) &= r^k - (r-1)^k \end{aligned}$$

then $G_{2n}^{(k)} = A_k(n-1,1)$.

Proof. Define

$$P_k(s,r) = \sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \cdots \sum_{i_s=1}^{sk-\sum_{\ell=1}^{s-1} i_\ell} \binom{sk - \sum_{\ell=1}^{s-1} i_\ell}{i_s} (r-1)^{sk-\sum_{\ell=1}^{s} i_\ell}$$

For all $r \geq 1$ and $s \geq 1$. Then note that by Definition 2.1,

$$G_{2n}^{(k)} = P_k(n-1,1). \tag{2.2}$$

Also note that by the binomial theorem,

$$\begin{aligned} P_k(1,r) &= \sum_{i_1=1}^{k} \binom{k}{i_1} (r-1)^{k-i_1} \\ &= r^k - (r-1)^k \\ &= A_k(1,r) \end{aligned} \tag{2.3}$$

Now we consider the general case of $P_k(s,r)$. For convenience, denote $J_r = rk - \sum_{\ell=1}^{r-1} i_\ell$. Then

$$P_k(s,r) = \sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \cdots \sum_{i_s=1}^{J_s} \binom{J_s}{i_s} (r-1)^{J_s - i_s}. \tag{2.4}$$

Further, note that $J_r = k - i_{r-1} + J_{r-1}$. We may consider the inner most sum of (2.4), again using the binomial theorem:

$$\begin{aligned}
& \sum_{i_s=1}^{J_s} \binom{J_s}{i_s} (r-1)^{J_s-i_s} \\
= & \sum_{i_s=0}^{J_s} \binom{J_s}{i_s} (r-1)^{J_s-i_s} - (r-1)^{J_s} \\
= & \ r^{J_s} - (r-1)^{J_s} \\
= & \ r^k \left(r^{J_{s-1}-i_{s-1}}\right) - (r-1)^k \left((r-1)^{J_{s-1}-i_{s-1}}\right)
\end{aligned}$$

Thus substituting this into our expression for $P_k(s,r)$ gives

$$\begin{aligned}
P_k(s,r) &= \sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \cdots \sum_{i_s=1}^{J_s} \binom{J_{s-1}}{i_s} (r-1)^{J_s-i_s} \\
&= \sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \cdots \sum_{i_{s-1}=1}^{J_{s-1}} \binom{J_{s-1}}{i_{s-1}} \cdot \left(r^k \left(r^{J_{s-1}-i_{s-1}}\right)\right) \\
&- \sum_{i_1=1}^{k} \binom{k}{i_1} \sum_{i_2=1}^{2k-i_1} \binom{2k-i_1}{i_2} \cdots \sum_{i_{s-1}=1}^{J_{s-1}} \binom{J_{s-1}}{i_{s-1}} \cdot \left((r-1)^k \left((r-1)^{J_{s-1}-i_{s-1}}\right)\right) \\
&= r^k P_k(s-1, r+1) - (r-1)^k P_k(s-1, r) \qquad (2.5)
\end{aligned}$$

From (2.3) and (2.5), we can conclude that $A_k(n,r) = P_k(n,r)$. Thus, the result follows from (2.2). ■

2.1 Alternate Expressions for $G_{2n}^{(k)}$

In this section, we consider exponential generating functions for $G_{2n}^{(k)}$. Recall that

$$G(t) = \frac{2t}{e^t + 1} = \sum_{n \geq 0} G_n^{(2)} \frac{t^n}{n!} \tag{2.6}$$

For this, we will consider a generalization of the Stirling numbers given by Comtet [2].

The equivalence of the generating function (2.6) and the expression for Genbocchi numbers in terms of Gandhi polynomials was proved by Riordan and Stein [16]. In particular, let A_k denote the Gandhi polynomials of Lemma 2.2. Then Riordan and Stein have shown the following holds for all $1 \leq j \leq n$:

$$A_2(n,1) = \sum_{\ell=0}^{j-1} (-1)^\ell x_1 x_2 \ldots x_{j-\ell} h_\ell(x_1, \cdots, x_{j-\ell}) A_2(n-j, j+1-\ell)$$

where $x_i = i^2$ and $h_i(x_1, \ldots, x_i)$ is the homogeneous product sum symmetric function (see [15]). We may immediately conclude that

$$A_k(n,1) = \sum_{\ell=0}^{j-1} (-1)^\ell x_{1,k} x_{2,k} \ldots x_{j-\ell,k} h_\ell(x_{1,k}, \cdots, x_{j-\ell,k}) A_k(n-j, j+1-\ell) \tag{2.7}$$

where $x_{i,j} = i^j$. Consider the following definition:

Definition 2.3 *Define the following numbers $T_k(n,i)$ for all $k \geq 2$, $n \geq 1$ and all i as follows:*

$$T_k(1,1) = 1$$

$$T_k(n,i) = 0 \quad \forall i \notin [1,n]$$

$$T_k(n,i) = i^k T_k(n-1,i) + T_k(n-1,i-1) \quad \forall i \in [1,n]$$

Call the numbers $T_k(n,i)$ the k-th central factorial numbers.

Note that this definition is consistent with the central factorial numbers when $k = 2$ (see [5, Eq. (3), p. 307]). Now the relation

$$h_\ell(x_{1,k}, \ldots, x_{j-\ell,k}) = h_\ell(x_{1,k}, \ldots, x_{j-\ell-1,k}) + x_{j-\ell,k} h_{\ell-1}(x_{1,k}, \ldots, x_{j-\ell,k})$$

implies that if we take (2.7) with $j = n$ then

$$A_k(n,1) = \sum_{\ell=1}^{n} (-1)^{\ell+1} (\ell!)^k T_k(n,\ell)$$

By Lemma 2.2, this gives us a represtentation for the generalized Genocchi numbers in terms of the generalized central factorial numbers which is of independent interest (compare with [16, Eq. (2), p. 382]):

$$G_{2n+2}^{(k)} = \sum_{\ell=1}^{n} (-1)^{\ell+1} (\ell!)^k T_k(n,\ell) \tag{2.8}$$

Figure 2.2 gives the value of $T_3(n,i)$ for $1 \leq i \leq n \leq 7$.

$n \setminus i$	1	2	3	4	6	7	
1	1						
2	1	1					
3	1	9	1				
4	1	73	36	1			
5	1	585	1045	100	1		
6	1	4681	28800	7445	255	1	
7	1	37449	782281	505280	35570	441	1

Figure 2.2: Small values of $T_3(n,i)$

2.1.1 Generalized Stirling Numbers

In our pursuit of a generating function for G_{2n}^k, we may now turn to the generalization of Stirling numbers given by Comtet [2]. This generalization of the Stirling numbers includes the k-th central factorial numbers $T_k(n,i)$.

Let $\xi = (\xi_0, \xi_1, \ldots)$ be an infinite sequence. The *generalized Stirling numbers of the second kind* $S_\xi(n,k)$, are given implicitly by

$$x^n = \sum_{k=0}^{n} S_\xi(n,k)(x-\xi_0)(x-\xi_1)\cdots(x-\xi_{n-1}).$$

The following identity is given by Comtet [2]:

$$S_\xi(n,k) = S_\xi(n-1,k-1) + \xi_k S_\xi(n-1,k-1).$$

Comtet also gives the following generating function:

$$\sum_{n\geq k} S_\xi(n,k)\frac{t^n}{n!} = \sum_{j=0}^{k} \frac{e^{t\xi_j}}{(\xi_j)_k}$$

where $(\xi_j)_k$ is given by

$$(\xi_j)_k = \prod_{l=0, l\neq j}^{k} (\xi_j - \xi_l).$$

This equation was also known to Bach [1, Eq. (13) p. 215].

Thus, for the generalized central factorial numbers $T_k(n,i)$, we may choose $\xi^{(k)} = (\xi_0^{(k)}, \xi_1^{(k)}, \ldots)$ such that $\xi_j^{(k)} = j^k$. Then we get that

$$\sum_{n\geq i} T_k(n,i)\frac{t^n}{n!} = \sum_{j=0}^{i} \frac{\exp(tj^k)}{\prod_{m=0,m\neq j}^{i}(j^k - m^k)}$$

This yields the following expression using (2.8):

$$\sum_{n=0}^{\infty} G_{2n+2}^{(k)}\frac{t^n}{n!} = \sum_{\ell=0}^{\infty}(-1)^{\ell+1}(\ell!)^k \sum_{j=0}^{\ell} \frac{\exp(tj^k)}{\prod_{m=0,m\neq j}^{l}(j^k - m^k)}$$

Unfortunately, we have not been able to extend this to a generating function for G_{2n}^k in the manner of Riordan and Stein [16].

3 Combinatorial Interpretations

In this section, we go through several combinatorial interpretations of the generalized central factorial numbers and the generalized Genocchi numbers.

3.1 Quasi-Permutations

Recall the following definition (cf. [5, p. 306]):

Definition 3.1 *A set $P \subset \{1,\ldots,n\} \times \{1,\ldots,n\}$ is a quasi-permutation of $\{1,\ldots,n\}$ if there exists a permutation p of $\{1,\ldots,n\}$ such that P is a subset of the following set*

$$\{(i,p(i)) \;:\; i \in \{1,\ldots,n\}, p(i) > i\}$$

Let $|P|$ denote the cardinality of P as a set. For any subset $P \subset \{1,\ldots,n\} \times \{1,\ldots,n\}$, let $Y(P) = \{i \;:\; \exists i'$ such that $(i',i) \in P\}$, the projection of P on the second component. Similarly, $X(P) = \{i \;:\; \exists i'$ such that $(i,i') \in P\}$.

We can generalize a theorem of Dumont [5, Thm. 1, p. 309] (which is itself inspired by a theorem of Foata and Schützenberger [10, Prop. 2.8., p. 38]) concerning combinatorial interpretations of the central factorial numbers as follows:

Theorem 3.2 *The quantity $T_k(n,i)$ is equal to the number of k-tuples $(Q_1,Q_2,\ldots,Q_k)$ of quasi-permutations of $\{1,\ldots,n\}$ such that*

- $|Q_j| = n - i$ *for all j with $1 \le j \le k$*
- *for all $1 \le j,j' \le k$, $Y(Q_j) = Y(Q_{j'})$.*

Proof. The proof is a simple generalization of that of Dumont [5, Thm. 1, p. 309]. ∎

Note that a simple calculation will show that the result of Dumont concerning tuples of permutations [5, Thm. 2, p. 310] does not generalize obviously to k-th generalized Genocchi numbers.

3.2 Finite language DFAs over 2 letters

We start by defining a set of directed graphs which will be of interest:

Definition 3.3 *Let $\mathcal{G}_{n,k}$ define the set of digraphs satisfying the following conditions: For all $G = (V,E) \in \mathcal{G}_{n,k}$,*

- *There are n vertices, labeled $\{1,\ldots,n\}$.*
- *The edges of E are labeled with an integer from the set $\{1,2,\ldots,k\}$. Thus an edge of E is given by an element of $V \times \{1,2,\ldots,k\} \times V$.*
- *All the edges of E are directed and satisfy the following: if $e = (u,a,v) \in E$ and $u \neq n$ then e is directed from u to v and $u < v$. If $u = n$ then necessarily $v = n$.*
- *G is initially connected, that is, for each vertex v, there exists a directed path from 1 to v.*
- *The graph G is complete: For each vertex v and each integer i ($1 \le i \le k$), there exists an edge with source v and label i.*

Given (2.1), we can prove the following:

Theorem 3.4 *For all $n \ge 1$, $|\mathcal{G}_{n,2}| = G_{2n}$.*

Proof. The sum given in (2.1) represents the number of ways of connecting each of the vertices $2, \ldots, n$ with a lower numbered vertex. We can see this as follows. Consider vertex 2. In order for vertex 2 to be connected to vertex 1, at least one of the 2 edges leaving vertex 1 must enter vertex 2. We let i_1 of them enter 2, and account for all possible combinations.

Now for vertex 3, at least 1 of the $4 - i_1$ edges leaving vertex 1 and 2 which have yet to be assigned must enter vertex 3, let i_2 of them enter vertex 3.

We continue this process for the first $n-1$ vertices. The result is the sum (2.1). The vertex n is initially connected since by definition all edges leaving vertex $n-1$ must enter vertex n. ■

We can also give a direct proof of Theorem 3.4. Recall [7] that a *surjective step function (SSF) of size* $2n$ is a function $f : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}$ such that

- f is increasing function, ie $f(i) \geq i$.
- the image of f is exactly $\{2, 4, 6, \ldots, 2n\}$.

Theorem 3.5 *(Dumont, [5, 7]): The number of surjective step functions of size* $2n$ *is* $G_{2(n+1)}$.

We show a bijection between all SSFs of size $2(n-1)$ and $\mathcal{G}_{n,2}$.

Let $f : \{1, 2, \ldots, 2n-2\} \to \{2, 4, 6, \ldots, 2n-2\}$ be a surjective step function of size $2(n-1)$. Then define the graph $G_f = (V_f, E_f)$ as follows: $V_f = \{1, \ldots, n\}$ and

$$\begin{aligned} E_f &= \{(n, a, n) \ : \ a \in \{0,1\}\} \\ &\cup \ \{(i, 0, \frac{f(2i)}{2} + 1) \ : \ 1 \leq i < n\} \\ &\cup \ \{(i, 1, \frac{f(2i-1)}{2} + 1) \ : \ 1 \leq i < n\} \end{aligned}$$

Lemma 3.6 *Let f be a SSF of size $2(n-1)$ and G_f the resulting graph. If $(u, v) \in E_f$ with $u \neq n$ then $u < v$ and further G_f is initially connected.*

Proof. Let i be any integer such that $i \neq n$. Then the two edges with source i are $(i, 0, \frac{f(2i)}{2} + 1)$ and $(i, 1, \frac{f(2i-1)}{2} + 1)$. Then

$$\frac{f(2i)}{2} + 1 \geq \frac{2i}{2} + 1 = i + 1 > i$$

$$\frac{f(2i-1)}{2} + 1 \geq \frac{2i-1}{2} + 1 = i + \frac{1}{2} > i$$

by the fact that f is increasing.

Now we show G_f is initially connected. We proceed by induction on i. Assume that all vertices with label less than i are connected to vertex 1. Now consider vertex $i \leq n$. Since f is surjective on $\{2, 4, 6, \ldots, 2(n-1)\}$, there exists some j such that $f(j) = 2(i-1)$. Consider the parity of j.

case 1: j **is even**: Let j' be an integer such that $2j' = j$. Then $j' \geq 1$ and $f(2j') = 2i - 2$ implies $i = \frac{f(2j')}{2} + 1$ and thus $(j', 0, i) \in E_f$. By above, $j' < i$, and thus by induction i is connected to vertex 1.
case 2: j **is odd**: Let j' be an integer such that $2j' - 1 = j$. Then $j' \geq 1$ and $f(2j' - 1) = 2(i-1)$, so again $i = \frac{f(2j'-1)}{2} + 1$ and thus $(j', 0, 1) \in E_f$. Once again $j' < i$ and thus by induction i is connected to vertex 1. ■

Lemma 3.7 *Let $G = (V, E) \in \mathcal{G}_{n,2}$. Then there exists a SSF f of size $2(n-2)$ such that $G = G_f$.*

Proof. Obvious. ■

Thus, we have a direct bijection demonstrating Theorem 3.4. We now return to our motivation, DFAs which recognize finite languages. Adding final states in all possible ways, we have the following corollary:

Corollary 3.8 *The number of finite languages over a two letter alphabet accepted by a DFA with n states is at most $2^{n-1}G_{2n}^{(2)}$.*

Unfortunately, this bound is asymptotically worse than that given by Domaratzki *et al.* [3]. This is due to the fact that many of the languages recognized by distinct DFAs will be the same, due to relabeling of states. However, the upper bound $2^{n-1}G_{2n}^{(2)}$ is better than the bound given by Domaratzki *et al.* for the values $n \leq 50$.

3.3 Finite language DFAs over k letters

The argument of Theorem 3.4 can be easily extended to graphs over a k letter alphabet. In fact, if we repeat the same argument to get the following result:

Theorem 3.9 $|\mathcal{G}_{n,k}| = G_{2n}^{(k)}$.

Corollary 3.10 *The number of finite languages over a k letter alphabet accepted by a DFA with n states is at most $2^{n-1}G_{2n}^{(k)}$.*

3.4 k-th Surjective step functions

We may adapt the combinatorial interpretation of Dumont [5] to generalized Genocchi numbers:

Definition 3.11 *A k-th surjective step function (k-SSF) of size kn is a increasing surjective function $f : \{1, 2, \cdots, kn\} \to \{k, 2k, 3k, \ldots, kn\}$*

Lemma 3.12 *There are $G_{2(n+1)}^{(k)}$ k-th surjective step functions of size kn.*

4 Conclusions and Further Work

In this paper, we have considered a new generalization of the Genocchi numbers. This generalization has proved useful in our attempts to enumerate the number of finite languages recognized by DFAs with n states. However, this work is only a slight progress towards this goal, and any effective enumeraion must be an unlabelled enumeration of DFAs, which appears to be very difficult, even in the case of recognition of finite languages.

Our introduction of these new classes of Genocchi numbers is also not complete, as we have not given an expontential generating function for the sequences $G_{2n}^{(k)}$.

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