DISCOVERING THE SQUARE-TRIANGULAR NUMBERS

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Among the many mathematical gems which fascinated the ancient Greeks, the Polygonal Numbers were a favorite. They offered a variety of exciting problems of a wide range of difficulty and one can find numerous articles about them in the mathematical literature even up to the present time.

To the uninitiated, the polygonal numbers are those positive integers which can be represented as an array of points in a polygonal design. For example, the Triangular Numbers are the numbers $1, 3, 6, 10, \cdots$ associated with the arrays

$$\begin{array}{cccc} & & & x \\ & & x & x\,x \\ & x & x\,x & x\,x\,x \\ x, & x\,x, & x\,x\,x, & x\,x\,x\,x, \end{array} \quad \cdots .$$

The square numbers are just the perfect squares $1, 4, 9, 16, \cdots$ associated with the arrays:

$$\begin{array}{cccc} & & & x\,x\,x\,x \\ & & x\,x\,x & x\,x\,x\,x \\ & x\,x & x\,x\,x & x\,x\,x\,x \\ x, & x\,x, & x\,x\,x, & x\,x\,x\,x, \end{array} \quad \cdots .$$

Similar considerations lead to pentagonal numbers, hexagonal numbers and so on.

One of the nicer problems which occurs in this topic is to determine which of the triangular numbers are also square numbers, i. e., which of the numbers

$$1,\ 3,\ 6,\ 10,\ \cdots,\ \frac{n(n+1)}{2},\ \cdots$$

are perfect squares. There are several ways of approaching this problem [1]. I would like to direct your attention to a very elementary method using the discovery approach advocated so well by Polya [2].

A few very natural questions arise, such as, "Are there any square-triangular numbers?". This is easily answered since

$$1^2 = \frac{1\cdot 2}{2} = 1$$

is such a number. To show that more than this trivial case occurs, we find that

$$36 = 6^2 = \frac{8\cdot 9}{2}$$

is also a square triangular number. One would then naturally ask, "Are there infinitely many square-triangular numbers?". This is considerably more difficult to answer since a careful check reveals the next one to be

$$35^2 = \frac{49\cdot 50}{2} = 1225$$

and we see that they do not appear to be very dense. In seeking to answer the last question, one quite naturally asks, "Is there a formula which always yields such a number, or better yet, is there a formula which yields all such numbers?". This, in turn, leads us to ask, "Is there a pattern in these numbers which would help us guess a formula?".

To find a pattern from the three cases 1, 36, 1225, seems rather futile, so we apply a little (!) more arithmetic to find that the next two cases are

$$204^2 = \frac{288\cdot 289}{2} = 41{,}616$$

and

$$1189^2 = \frac{1681 \cdot 1682}{2} = 1,413,721 .$$

We now seek a pattern from the five cases: 1; 36; 1225; 41,616; 1,413,721. One is immediately discouragingly impressed by the relative scarcity of square-triangular numbers and the possibility of a nice easy-to-guess pattern seems quite remote; but, having gone this far, it does not hurt to at least pursue this course a little further. Let us introduce some notation to facilitate the work by calling S_n, T_n, and $(ST)_n$ the n^{th} square, triangular, and square-triangular numbers, respectively. Organizing our data to date, then, we have:

$$\begin{aligned}
1 &= (ST)_1 = S_1 = 1^2 = \frac{1\cdot 2}{2} = T_1 \\
36 &= (ST)_2 = S_2 = 6^2 = \frac{8\cdot 9}{2} = T_8 \\
1,225 &= (ST)_3 = S_{35} = 35^2 = \frac{49\cdot 50}{2} = T_{49} \\
41,616 &= (ST)_4 = S_{204} = 204^2 = \frac{288\cdot 289}{2} = T_{288} \\
1,413,721 &= (ST)_5 = S_{1189} = 1189^2 = \frac{1681\cdot 1682}{2} = T_{1681}
\end{aligned}$$

The adventurous reader is encouraged at this point to look for a pattern and formula on his own before reading any further.

For the unsuccessful guessers or those wishing to compare results, let us carry on by writing the numbers in various ways; in particular, we might look at them in prime factored form.

$$\begin{aligned}
1 &= (ST)_1 = S_1 = 1^2 = (1\cdot 1)^2 = \frac{1\cdot 2}{2} = \frac{2}{2} = T_1 \\
36 &= (ST)_2 = S_6 = 6^2 = (2\cdot 3)^2 = \frac{8\cdot 9}{2} = \frac{2^3\cdot 3^2}{2} = T_8 \\
1,225 &= (ST)_3 = S_{35} = 35^2 = (5\cdot 7)^2 = \frac{49\cdot 50}{2} = \frac{2\cdot 5^2\cdot 7^2}{2} = T_{49}
\end{aligned}$$

$$41{,}616 = (ST)_4 = S_{204} = 204^2 = (2^2 \cdot 3 \cdot 17)^2 = \frac{288 \cdot 289}{2} = \frac{2^5 \cdot 3^2 \cdot 17^2}{2} = T_{288}$$

$$1{,}413{,}721 = (ST)_5 = S_{1189} = 1189^2 = (29 \cdot 41)^2 = \frac{1681 \cdot 1682}{2} = \frac{2 \cdot 29^2 \cdot 41^2}{2} = T_{1681}$$

Is there a pattern now? We note that as far as patterns are concerned, the form of the S_n's is a little nicer than that of the T_n's, but essentially they are the same, so we shall concentrate on the S_n's.

Since we only have five cases at hand, and the sixth case is likely to be a bit far off, we must make the most of what we have. We might note that three of the cases are the square of exactly two factors whereas the trivial case $s_1 = 1^2$ could be written with any number of 1's and

$$S_{204} = (2^2 \cdot 3 \cdot 17)^2$$

could be reduced to the square of two factors if we dropped the requirement of prime factors. It might be worthwhile to write each S_n as the square of two factors. This allows no options except for S_{204} which could then be written in five non-trivial ways, namely,

$$S_{204} = (2 \cdot 102)^2 = (3 \cdot 68)^2 = (4 \cdot 51)^2 = (6 \cdot 34)^2 = (12 \cdot 17)^2 .$$

Do any of these fit into a pattern with the other four? If we looked only for the monotone increasing pattern of the factors we would choose $S_{204} = (12 \cdot 17)^2$. Now, looking at the data so arranged, we have:

$$\begin{aligned} S_1 &= (1 \cdot 1)^2 \\ S_6 &= (2 \cdot 3)^2 \\ S_{35} &= (5 \cdot 7)^2 \\ S_{204} &= (12 \cdot 17)^2 \\ S_{1189} &= (29 \cdot 41)^2 \end{aligned}$$

Look hard, now, for there is a very nice pattern here; and in fact, it is recursive of a Fibonacci type. Do you see that $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 =$

5, $2 + 5 = 7$, $5 + 7 = 12$, $5 + 12 = 17$, $12 + 17 = 29$, and $12 + 29 = 41$? Let us write this into our data as:

$$
\begin{aligned}
(ST)_1 &= (1\cdot 1)^2 \\
(ST)_2 &= (2\cdot 3)^2 &&= (1 + 1)^2\cdot(1 + 1 + 1)^2 &&= (1 + 1)^2\cdot(2\cdot 1 + 1)^2 \\
(ST)_3 &= (5\cdot 7)^2 &&= (2 + 3)^2\cdot(2 + 2 + 3)^2 &&= (2 + 3)^2\cdot(2\cdot 2 + 3)^2 \\
(ST)_4 &= (12\cdot 17)^2 &&= (5 + 7)^2\cdot(5 + 5 + 7)^2 &&= (5 + 7)^2\cdot(2\cdot 5 + 7)^2 \\
(ST)_5 &= (29\cdot 41)^2 &&= (12 + 17)^2\cdot(12 + 12 + 17)^2 &&= (12 + 17)^2\cdot(2\cdot 12 + 17)^2
\end{aligned}
$$

Before formalizing and trying to prove this guess, it would be well to test it as much as possible to see if it works at all. Our first test will be to see if $(29 + 41)^2(29 + 29 + 41)^2$ is a triangular number.

$$(29 + 41)^2(29 + 29 + 41)^2 = 70^2\cdot 99^2 = 4900\cdot 9801 = \frac{9800\cdot 9801}{2}$$

is triangular and our confidence in our guess is considerably strengthened. Our next test will be to see if this new square-triangular number is, in fact, the next one; i.e., is it $(ST)_6$? This involves checking to see if there are any squares between T_{1681} and T_{9800} which is hardly an inviting exercise in arithmetic. Therefore, let us use the sometimes wise advice that "If you can't prove it, generalize it."

In order to proceed on with a proof we introduce a bit more notation. Let a_n be defined by the recursive relation $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$. For $n = 1, 2, 3, 4, 5$, this gives us the sequence $1, 2, 5, 12, 29$ which we recognize as the first factors for $(ST)_1$, $(ST)_2$, $(ST)_2$, $(ST)_4$, $(ST)_5$, respectively. We also note that the second factors $1, 3, 7, 17, 41$ are $a_1 + a_0$, $a_2 + a_1$, $a_3 + a_2$, $a_4 + a_3$, $a_5 + a_4$, respectively.

Finally, before proceeding with our proof, we notice that in order to prove a positive integer m is a triangular number, it suffices to show that there exists a positive integer n such that

$$m = \frac{n(n + 1)}{2}$$

or equivalently that there exists positive integers a and b such that

$$m = \frac{ab}{2}$$

with $|a - b| = 1$.

We now attempt to prove the following conjecture which is a formalized generalization from our data.

<u>Conjecture A</u>: $(ST)_n = a_n^2(a_n + a_{n-1})^2$ for $n = 1, 2, 3, \cdots$

<u>Proof</u>: We will attempt the proof in two parts.

(1) The sequence of numbers $a_n^2(a_n + a_{n-1})^2$ for $n = 1, 2, 3, \cdots$ are square-triangular numbers.

(2) This sequence is in fact all of the square-triangular numbers.

Clearly, $a_n^2(a_n + a_{n-1})^2$ is a square number for all $n \geq 1$ so we concentrate on showing these numbers are also triangular for $n \geq 1$. Using mathematical induction, we first dispense with the case for $n = 1$ as $1^2(1 + 0)^2 = 1 = (2 \cdot 1)/2$ with $|2 - 1| = 1$.

Now assume $a_n^2(a_n + a_{n-1})^2$ is triangular with

$$a_n^2(a_n + a_{n-1})^2 = \frac{2a_n^2(a_n + a_{n-1})^2}{2}$$

and

$$|2a_n^2 - (a_n + a_{n-1})^2| = 1$$

for some $n \geq 1$. Then

$$a_{n+1}^2(a_{n+1} + a_n)^2 = \frac{2(2a_n + a_{n-1})^2(2a_n + a_{n-1} + a_n)^2}{2}$$

where

$$\begin{aligned}
&\left|2(2a_n + a_{n-1})^2 - (3a_n + a_{n-1})^2\right| \\
&\quad = \left|(8a_n^2 + 8a_na_{n-1} + 2a_{n-1}^2) - (9a_n^2 + 6a_na_{n-1} + a_{n-1}^2)\right| \\
&\quad = \left|-a_n^2 + 2a_na_{n-1} + a_{n-1}^2\right| \\
&\quad = \left|a_n^2 - 2a_na_{n-1} - a_{n-1}^2\right| = \left|2a_n^2 - (a_n + a_{n-1})^2\right| = 1 .
\end{aligned}$$

Therefore $a_{n+1}^2(a_{n+1} + a_n)^2$ is also triangular and (1) is proved.

Now let

$$m_1^2 = \frac{k_1(k_1 + 1)}{2}$$

be an arbitrary square-triangular number. There are two cases which can be considered, namely k_1 even or k_1 odd. It is immaterial which we consider first, as we will be alternating back and forth from one to the other in a descending sequence of square-triangular numbers which will terminate finally at $(ST)_1 = 1$. To be definite, let k_1 be odd which implies

$$\frac{k_1 + 1}{2}$$

is an integer and

$$\left(k_1, \frac{k_1 + 1}{2}\right) = 1$$

(we are using the common notation of letting (a, b) denote the greatest common divisor of a and b). Therefore

$$m_1^2 = k_1 \frac{(k_1 + 1)}{2}$$

a square implies that both k_1 and $(k_1 + 1)/2$ are squares. Let

$$\frac{k_1 + 1}{2} = b_1^2$$

and $k_1 = c_1^2$ with $b_1, c_1 \geq 1$. Now $k_1 \geq 1$ implies $b_1 \leq c_1$ and, in fact, $b_1 = c_1$, if and only if

$$k_1 = \frac{k_1 + 1}{2}$$

if and only if $k_1 = 1$ if and only if $m_1^2 = 1$, in which case

$$m_1^2 = a_1^2(a_1 + a_0)^2 = (ST)_1 ,$$

and we are done. Consider then $b_1 < c_1$ and define $b_2 = c_1 - b_1$, $c_2 = 2b_1 - c_1$, and $m_2^2 = b_2^2 c_2^2$.

Since

$$2b_1^2 - c_1^2 = k_1 + 1 - k_1 = 1,$$

we factor and get

$$(\sqrt{2}\,b_1 - c_1)(\sqrt{2}\,b_1 + c_1) = 1 ,$$

where $b_1, c_1 \geq 1$ implies $\sqrt{2}\,b_1 + c_1 > 0$, which implies that $\sqrt{2}b_1 - c_1 > 0$, so $\sqrt{2}b_1 > c_1$. Now $3/2 > \sqrt{2}$ so it also follows that $3b_1/2 > c_1$ which is equivalent to $3b_1 > 2c_1$ and thus $2b_1 - c_1 > c_1 - b_1$. Also, $c_1 > b_1$ implies both that $c_1 - b_1 > 0$ and that $b_1 > 2b_1 - c_1$, which then gives us the inequality $c_1 > b_1 > 2b_1 - c_1 > c_1 - b_1 > 0$, or equivalently, $c_1 > b_1 > c_2 > b_2 > 0$.

Furthermore,

$$m_2^2 = b_2^2 c_2^2 = \frac{2b_2^2 c_2^2}{2} ,$$

where

$$|2b_2^2 - c_2^2| = |2(c_1 - b_1)^2 - (2b_1 - c_1)^2| = |c_1^2 - 2b_1^2| = |-1| = 1 ,$$

so m_2^2 is also a square-triangular number and is necessarily smaller than m_1^2 from the last inequality. Now let

$$m_2^2 = \frac{2b_2^2c_2^2}{2} = \frac{k_2(k_2+1)}{2} ,$$

with $2b^2 = k_2$ (since $2b_2^2 - c_2^2 = -1$ gives $c_2^2 = 2b_2^2 + 1$) and we get k_2 even as predicted earlier. It might be observed that in this case $m_2^2 \neq 1$ which is equivalent to our fact that $c_2 > b_2$.

Now continue in the same manner by defining $b_3 = c_2 - b_2$, $c_3 = 2b_2 - c_2$, and $m_3^2 = b_3^2c_3^2$. In this case, $b_3 \leq c_3$ since if $b_3 > c_3$, then by substitution $c_2 - b_2 > 2b_2 - c_2$ which implies $c_2 > 3b_2/2$. Recalling that

$$2b_2^2 - c_2^2 = c_1^2 - 2b_1^2 = -1 ,$$

which is equivalent to $c_2^2 - 2b_2^2 = 1$, we get by using $c_2 > 3b_2/2$ that

$$(3b_2/2)^2 - 2b_2^2 < 1 .$$

This implies

$$\frac{b_2^2}{4} < 1 ,$$

which implies b_2 is a positive integer with square less than 4, or that $b_2 = 1$. This, however, yields $c_2^2 - 2\cdot1^2 = 1$ or $c_2^2 = 3$ in which case $c_2 = \sqrt{3}$ must be a positive integer, which is false. Thus the hypothesis that $b_3 > c_3$ is false and $b_3 \leq c_3$ as claimed. We might note that $b_3 = c_3$ is equivalent to $c_2 = 3b_2/2$ which implies

$$1 = c_2^2 - 2b_2^2 = (3b_2/2)^2 - 2b_2^2 = \frac{b^2}{4} ,$$

which implies that $b_2 = 2$ and also that

$$c_2 = \frac{3}{2} \cdot 2 = 3 .$$

This, in turn, gives us $b_3 = c_3 = 1$ as well as $b_1 = 5$, $c_1 = 7$, so

$$m_1^2 = 5^2 \cdot 7^2 = a_3^2(a_3 + a_2)^2 = (ST)_3$$

and we are done.

In general, with $b_3 \leq c_3$,

$$m_3^2 = b_3^2 c_3^2 = \frac{2b_3^2 c_3^2}{2} ,$$

with

$$\left|2b_3^2 - c_3^2\right| = \left|2(c_2 - b_2)^2 - (2b_2 - c_2)^2\right| = \left|c_2^2 - 2b_2^2\right| = 1$$

so m_3^2 is again a square-triangular number. Since

$$c_2 > b_2 > 2b_2 - c_2 \geq c_2 - b_2 > 0 ,$$

or equivalently

$$c_2 > b_2 > c_3 \geq b_3 > 0 ,$$

m_3^2 is again smaller than m_2^2. If we let

$$m_3^2 = \frac{2b_3^2 c_3^2}{2} = \frac{(k_3 + 1)k_3}{2},$$

with $k_3 = c_3^2$ (since from above $2b_3^2 - c_3^2 = 1$ gives $2b_3^2 = c_3^2 + 1$) we have $2b_3^2 = k_3 + 1$. Thus k_3 is now odd as in the first case and one can proceed in exactly the same manner generating new and smaller square-triangular numbers until we finally arrive at $m_n^2 = b_n^2 c_n^2 = 1$ with $b_n = c_n = 1$.

This gives us

$$1 = b_n = c_{n-1} - b_{n-1}$$

and

$$1 = b_n = c_{n-1} - b_{n-1}$$

which gives $b_{n-1} = 2$ and $c_{n-1} = 3$ when solved. It follows that

$$2 = b_{n-2} = c_{n-2} - b_{n-2}$$

and

$$3 = c_{n-1} = 2b_{n-2} - c_{n-2} ,$$

which yields $b_{n-2} = 5$ and $c_{n-2} = 7, \cdots$. In general, for $j \geq 2$, $b_{j+1} = c_j - b_j$,

$$c_{j+1} = 2b_j - c_j ,$$

and

$$b_j = c_{j-1} - b_{j-1}, \qquad c_j = 2b_{j-1} - c_{j-1} .$$

Therefore,

$$\begin{aligned} 2b_j + b_{j+1} &= 2(c_{j-1} - b_{j-1}) + (c_j - b_j) = 2c_{j-1} - 2b_{j-1} + (2b_{j-1} - c_{j-1}) \\ &\quad - (c_{j-1} - b_{j-1}) = b_{j-1} \end{aligned}$$

and

$$\begin{aligned} b_j + b_{j-1} &= b_j + (2b_j + b_{j+1}) = 3b_j + b_{j+1} = 3c_{j-1} - 3b_{j-1} \\ &\quad + (2b_{j-1} - c_{j+1}) - (c_{j-1} - b_{j-1}) = c_{j-1} . \end{aligned}$$

We have just done the computation for an induction proof that $b_j = a_{n-j+1}$ and $c_j = a_{n-j+1} + a_{n-j}$ for $j = 1, 2, \cdots, n$. In particular, for $j = 1$, it follows that

$$m_1^2 = b_1^2 c_1^2 = a_n^2 (a_n + a_{n-1})^2$$

and m_1^2 is in our sequence as claimed, and (2) is proved.

Since the sequence is monotonically increasing, we have that

$$(ST)_n = a_n^2 \cdot (a_n + a_{n-1})^2$$

as claimed, and our Conjecture A is true.

Thus the empirical data of five cases led us to guess a very nice recursion formula which turned out to be valid. So even though the square-triangular numbers are very sparse, not only in relation to the positive integers, but also in relation to either the square or triangular numbers themselves, there are still infinitely many of them and they behave very well. In fact, they behave beautifully.

There are many other very nice relationships in these numbers which are left for the reader to derive and/or prove. A few of these are listed here to whet the appetite.

(i) $(ST)_1 = a_1^2(a_1 + a_0)^2$

$(ST)_2 = a_2^2(a_2 + a_1)^2 = (2a_1 + a_0)^2(3a_1 + a_0)^2$

$\vdots$

$(ST)_n = a_n^2(a_n + a_{n-1})^2 = (2a_{n-1} + a_{n-2})^2(3a_{n-1} + a_{n-2})^2$ for $n \geq 2$.

(ii) $(ST)_n$ is odd if and only if n is odd if and only if a_n is odd.

(iii) $a_n = \dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$ and $(ST)_n = \left(\dfrac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{4\sqrt{2}}\right)^2$.

(iv) $2a_n^2 - (a_n + a_{n-1})^2 = (-1)^{n-1}$ for $n \geq 1$.

(v) If
$$(ST)_n = a_n^2(a_n + a_{n-1})^2 = S_{u_n} = u_n^2 = T_{v_n} = \frac{v_n(v_n+1)}{2},$$
then $v_{n+1} = u_{n+1} + v_n + u_n$ for $n \geq 1$. This may be proven with or without (vi) below.

(vi) The sequences of u_n's and v_n's are generated by the recursive formulae:

$u_0 = 0$, $u_1 = 1$, and $u_n = 6u_{n-1} - u_{n-2}$ for $n \geq 2$,

$v_0 = 0$, $v_1 = 1$, and $v_n = 6v_{n-1} - v_{n-2} + 2$ for $n \geq 2$.

(vii)
$$u_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}$$
$$v_n = \frac{(4+3\sqrt{2})(3+2\sqrt{2})^{n-1} - (4-3\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}} - \frac{1}{2}$$

(viii) The square-triangular numbers are precisely the numbers x^2y^2 such that $x^2 - 2y^2 = 1$ or $x^2 - 2y^2 = -1$ with x and y positive integers. These types of Diophantine equations are commonly known as Pell's Equations.

Having seen these very nice results, the mathematician naturally asks, "What about the triangular-pentagonal numbers, square-pentagonal numbers, and so on?". This is not at present completely answered, but many in-roads have been made by some outstanding mathematicians. In particular, W. Sierpinski devoted some time to this problem [3], but perhaps the nicest result so far obtained is one derived by Diane (Smith) Lucas as an undergraduate at Washington State University. In a paper (not yet published) she obtained the very beautiful result that for $3 \leq m < n$, there exist infinitely many numbers which are both n-gonal and m-gonal if and only if

(i) $m = 3$ and $n = 6$

or

(ii) $(m - 2)(n - 2)$ is not a perfect square.

With the machinery she developed, it is quite easy to derive for example, that the n^{th} pentagonal-triangular number

$$(P_{5,3})_n = \frac{(2 - \sqrt{3})(97 + 56\sqrt{3})^n + (2 + \sqrt{3})(97 - 56\sqrt{3})^n - 4}{48}$$

which is a result obtained by Sierpinski.

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