RECREATIONAL MATHEMATICS

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A FIBONACCI CONSTANT

I would like to introduce a new constant, if it hasn't been done before. If you evaluate the continued fraction

$$1 + \cfrac{1}{2 + \cfrac{3}{5 + \cfrac{8}{13 + \cfrac{21}{\text{etc.} \\ \ddots}}}}$$

you obtain $1.3941865502 \cdots$. Readers of this Journal will note immediately that the terms of this continued fraction are successive Fibonacci terms. Perhaps someone will evaluate this constant to many more decimal places, give it a reasonably good name (or Greek letter), and discover some interesting properties of the number.

INSTANT DIVISION

If you wish to divide 717948 by 4 merely move the initial 7 to the other end, obtaining 179487. This is about as instant as you can be — or is it? Much larger numbers can be divided just as easily:

$$9,130,434,782,608,695,652,173$$

can be divided by 7 by transposing the initial 9 to the end, obtaining

$$1,304,347,826,086,956,521,739 \ .$$

An article by Charles W. Trigg [1] described three methods of finding the smallest integer N, such that when its initial digit, d, is transposed to the right end of the integer, the result is N/d. Trigg's article restricted d to a single digit and N as the smallest integer satisfying the condition. The idea of instant arithmetic is not new, having appeared in the Fibonacci Quarterly [2, 3] and elsewhere [4, 5].

I wondered if there were other integers, N, such that when any <u>one or more of the initial digits</u> were transposed intact to the right, the result would be N/k, where k is any integer. In other words, as an example, is there an integer which can be divided by 7 by moving its initial digits, 317, to the right? The answer is yes. Although not all integers possess the desired property, there are an infinite number of integers that do.

Trigg [1] shows that, for single-digit transposition

$$F = \frac{d^2}{10d - 1}$$

where d is the initial digit to be transposed to the right and F is the proper fraction which, when written as a decimal for one period, or cycle, represents the integer sought. If $d = 4$, for example, we have

$$F = 16/39 = .410256410256 \cdots .$$

Therefore, the smallest integer which can be divided by 4 by transposing the initial digit to the right is 410256.

Now, I will show how to find integers such that the transposition is not restricted to single digits, nor need N be divisible by the transposed digits. Following Trigg's format, let D represent the initial digit or digits to be transposed from left to right, k the divisor of N, the integer sought. Then

$$N = 0.D \cdots D \cdots D \cdots .$$

If D has n digits, we multiply by 10^n,

$$10^n N = D \cdots D \cdots D \cdots$$

and

$$N/k = 0.\cdots D\cdots D\cdots .$$

Therefore

$$10^n N - N/k = D,$$

or

$$N = \frac{Dk}{10^n k - 1} \tag{1}$$

This now allows us to find an N for any integer values of D and k. Here are several examples:

D	n	k	N = (with no decimal point)
6	1	2	12/19 = 631,578,947,368,421,052
9	1	7	63/69 = 9,130,434,782,608,695,652,173
73	2	37	2701/3699 = 730,197,350,635,306,839,686,401
7	1	317	2219/3169 = (see below)
317	3	7	2219/6999 = (see below)

2219/3169 = (with no decimal point) 700,220,889,870,621,647,207,320,921,-
426,317,450,299,779,110,129,378,352,792,679,078,573,682,-
549 (72 digits)

2219/6999 = (with no decimal point) 3,170,452,921,845,977,996,856,693,-
813,401,914,559,222,746,106,586,655,236,462,351,764,537,-
791,113,016,145,163,594,799,257,036,719,531,361,623,089,-
012,716,102,300,328,618,374,053,436,205,172,167,452,493,-
213,316,188,026,860,980,140,020,002,857,551,078,725,532,-
218,888,412,630,375,767,966,852,407,486,783,826,260,894,-
413,487,741,091,584,512,073,153,307,615,373,624,803,543,-
363,337,619,659,951,421,631,661,665,952,278,896,985,283,-
611,944,563,509,072,724,674,953,564,794,970,710,101,443,-
063,294,756,393,770,538,648,378,339,762,823,260,465,780,-
825,832,261,751,678,811,258,751,250,178,596,942,420,345,-
763,680,525,789,398,485,497,928,275,467,923,989,141,305,-
900,842,977,568,224,032,004,572,081,725,960,851,550,221,-
460,208,601,228,746,963,851,978,854,122,017,431,061,580,-
225,746,535,219 (583 digits)

Instant division in other bases can be done also. We have, for any base b

$$N = \frac{Dk}{b^n k - 1}$$

but, since b in base b is always 10, we have

$$N = \frac{Dk}{10^n k - 1}$$

So the same equation used before works in any base as long as d, n, k, $10^n k - 1$, and all calculations are in the given base. Some examples:

Base	D	n	k	N
Three	12	2	2	101/122 = 1202122110201001
Four	23	2	3	201/233 = 23032332220312131331101
Five	13	2	31	1003/3044 = 1310022231202000303

BIZLEY'S PROBLEM AND INSTANT MULTIPLICATION

In the solution to [5] the Editor notes that M. T. L. Bizley said a more difficult problem would be to determine all rational numbers q/p such that an integer can be found which will increase in the ratio p:q when the digit on the extreme left is moved to the extreme right. Trigg's work in [1] brought me to the general solution to the problem of instant division, and that general solution allowed me to solve Bizley's problem. A solution to Bizley's problem would automatically enable one to multiply instantly by transposing digits from left to right.

In Equation (1) above substitute q/p for k, obtaining

$$N = \frac{Dq}{10^n q - p} \tag{2}$$

A few solutions are given below.

D	p	q	N
1	3	2	1,176,470,588,235,294
2	3	2	2,352,941,176,470,588
3	7	11	3,203,883,495,145,631,067,961,165,048,543,689
4	3	2	4,705,882,352,941,176
4	2	3	428,571
11	7	3	11,262,798,634,812,286,689,419,795,221,843,003,412,969,-283,276,450,511,945,392,491,467,576,791,808,873,720,136,-518,771,331,058,020,477,815,699,658,703,071,672,354,948,-805,460,750,853,242,320,819 (146 digits)

In the examples above, moving D to the right multiplies N by p/q.

However, there are certain restrictions on the values of D, p, and q in Equation (2), otherwise the results obtained by using the equation are not solutions. For example, if we let $D = 6$, $p = 3$, and $q = 2$, we obtain

$$N = 7{,}058{,}823{,}529{,}411{,}764$$

which is not a solution for two reasons: the initial digit, 7, is not equal to D, nor is the integer produced by transposing the 7 to the right in the ratio 3:2 to the calculated N.

Tentatively, I have found that, for proper solutions Dq must be less than $(q/p)(10^n q - p)$. Perhaps readers can provide further insight, or provide definite criteria.

NOTE: In [3, problem 2] it is proven that there is no integer which is doubled when the initial digit is transposed to the right. However, I found several integers which almost meet the condition:

124999 ··· 999	and	125000 ··· 000
249999 ··· 999	and	250000 ··· 000
374999 ··· 999	and	375000 ··· 000

By including as many 9's or 0's as necessary, you can get as close to doubling as you desire. It is possible, however, to double by moving two or more digits to the right. Let $D = 10$, $p = 2$, and $q = 1$ to obtain

$N = 102{,}040{,}816{,}326{,}530{,}612{,}244{,}897{,}959{,}183{,}673{,}469{,}387{,}755.$