THE NUMBER OF k-DIGIT FIBONACCI NUMBERS

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Define $a(k)$ to be the number of k-digit Fibonacci numbers. For $n>5$, we have $1.6F_{n-1} < F_n < 1.7F_{n-1}$. Thus, if F_n is the least k-digit Fibonacci number, we have $F_{n+5} > 1.6^5 F_n > 10.48\cdot 10^{k-1}$. On the other hand, $F_{n+3} < 1.7^4 F_{n-1} < 1.7^4\cdot 10^{k-1} < 8.36\cdot 10^{k-1}$. Therefore, F_{n+5} always has at least $k+1$ digits, but F_{n+3} always has k digits. Hence, for $k>1$, we always have $a(k)=4$ or $a(k)=5$. Define $A(x)$ to be the number of $k\le x$ such that $a(k)=5$. Guthmann [1] proved the following theorem.

Theorem 1: For $x\to\infty$, we have

$$A(x)=\alpha x+O(1),$$

where

$$\alpha=\log 10/\log((1+\sqrt{5})/2)-4=0.78497\ldots.$$

His proof uses Baker's bound on linear forms in logarithms. Here we will give a very short proof of this statement and generalize it to residue classes. Since, except for $k=1$, we have $a(k)=4$ or 5, we get

$$\#\{n|F_n<10^x\}=\sum_{k\le x}a(k)=4(x-A(x))+5A(x)+O(1).$$

On the other hand, we have $F_n\sim\frac{1}{\sqrt{5}}\varphi^n$; thus, the left-hand side is $x\frac{\log 10}{\log\varphi}+O(1)$ and solving for $A(x)$ gives the theorem.

Now define $A(x,q,l)$ to be the number of $k\le x$, $k\equiv l \pmod q$, such that $a(k)=5$. Using this notation, we claim the following theorem.

Theorem 2: For any fixed q, we have

$$A(x,q,l)\sim\frac{\alpha}{q}x,$$

where α is defined as above.

We first note that $F_{n+4}/F_n\to\varphi^4$. If F_n is the least Fibonacci number with k digits, then $a(k)=5$ if and only if $F_{n+4}<10^k$. Now let $\varepsilon>0$ be fixed. Then we consider three cases:

1. $10^{k-1}<F_n<\left(\frac{10}{\varphi^4}-\varepsilon\right)10^{k-1}$. If n is sufficiently large, this implies $F_{n+4}<10^k$, thus $a(k)=5$.
2. $\left(\frac{10}{\varphi^4}-\varepsilon\right)10^{k-1}<F_n<\left(\frac{10}{\varphi^4}+\varepsilon\right)10^{k-1}$. In this case, we might have $a(k)=4$ or $a(k)=5$.
3. $F_n>\left(\frac{10}{\varphi^4}+\varepsilon\right)10^{k-1}$. In this case we have, for n sufficiently large, $F_{n+4}>10^k$; thus, only F_n, $\ldots, F_{n+3}$ have k digits, which implies $a(k)=4$. We also note that, in this case, we have $F_n<(\varphi+\varepsilon)10^{k-1}$, since otherwise F_{n-1} would also have k digits.

If we consider only case 1, we get a lower bound for $A(x,q,l)$. Thus we have, for $x>x_0(\varepsilon)$, the estimate

$$A(x,q,l) \geq \#\left\{k \leq x, k \equiv l \pmod q \mid \exists n: 10^{k-1} < F_n < \left(\frac{10}{\varphi^4} - \varepsilon\right)10^{k-1}\right\}.$$

We set $k = k'q + l$, and taking logarithms we get

$$A(x,q,l) \geq \#\left\{k' \leq \frac{x-l}{q} \mid \exists n: (k'q + l - 1)\log 10 + \varepsilon < n \log\varphi < (k'q + l)\log 10 - 4\log\varphi - \varepsilon\right\},$$

which is equivalent to

$$A(x,q,l) \geq$$

$$\#\left\{k' \leq \frac{x-l}{q} \mid \exists n: n\log\varphi - l\log 10 + 4\log\varphi + \varepsilon < k'q\log 10 < n\log\varphi - l\log 10 + \log 10 - \varepsilon\right\}.$$

Since $\frac{q\log 10}{\log\varphi}$ is irrational, the fractional part of $\frac{k'q\log 10}{\log\varphi}$ is uniformly distributed (mod 1). k' is counted if and only if the fractional part of $\frac{k'q\log 10}{\log\varphi}$ is contained in some interval of length

$$\frac{\log 10 - 4\log\varphi - 2\varepsilon}{\log\varphi} \geq \alpha - 5\varepsilon.$$

Hence, for $y > y_0$, the number of $k' < y$ with $a(k'q + l) = 5$ is $\geq (\alpha - 6\varepsilon)y$. If $k' < \frac{x-l}{q}$, then $k \leq x$; thus, we get the lower bound $A(x,q,l) \geq (\alpha - 6\varepsilon)\frac{x}{q}$. In the same way we get the upper bound $A(x,q,l) \leq (\alpha - 6\varepsilon)\frac{x}{q}$, if $\varepsilon \to \infty$, we obtain the statement of Theorem 2.

ACKNOWLEDGMENT

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REFERENCE

1. A. Guthmann. "Wieviele k-stellige Fibonaccizahlen gibt es?" *Arch. Math.* **59.4** (1992):334-340.

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