ON THE REPRESENTATION OF THE INTEGERS AS A DIFFERENCE OF NONCONSECUTIVE TRIANGULAR NUMBERS

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1. INTRODUCTION

The problem of determining the set of integer solutions of a polynomial equation, over $\mathbf{Z}$, occurs frequently throughout much of the theory of numbers. Typically, the most common form of these problems involves quadratic functions in several variables, such as those dealing with the polygonal representation of the integers. The n^{th} polygonal number of order k is defined as the n^{th} partial sum of a sequence of integers in arithmetic progression, having a first term of one and a common difference of $k-2$, and so is given by $\frac{1}{2}[(k-2)n^2-(k-4)n]$. One of the earliest results in connection with representing the positive integers as sums of polygonal numbers was due to Gauss, who proved that every positive integer could be expressed as a sum of three triangular numbers. Despite these classical origins, many difficult and interesting problems dealing with polygonal representations of the integers are still unresolved at present (see [1]). In this paper we shall continue with the theme of polygonal representation but in a slightly different direction by examining the following problem involving the differences of triangular numbers denoted here by $T(x)=\frac{1}{2}x(x+1)$.

Problem: Given any $M \in \mathbf{Z}\setminus\{0\}$, for what values $x, y \in \mathbf{N}$ is it possible that $M=T(x)-T(y)$ such that $|x-y|>1$, and how many such representations can be found?

The fact that a number can be represented as a difference of triangular number is not at all surprising since, by definition, $M=T(M)-T(M-1)$; hence, the restriction $|x-y|>1$ in the problem statement. To establish the existence or otherwise of a representation for M, we will see that the problem can be reduced to solving the diophantine equation $X^2-Y^2=8M$ in odd integers. Although this equation is solvable for all $M \in \mathbf{Z}\setminus\{0\}$, there is a subset of $\mathbf{Z}\setminus\{0\}$, namely, $\{\pm 2^m : m \in \mathbf{N}\}$, for which the consecutive triangular number difference is the only possible representation. Apart from the set mentioned, all other $M \in \mathbf{Z}\setminus\{0\}$ will have a nonconsecutive triangular representation and, moreover, the exact number will be shown to equal $D-1$, where D is the number of odd divisors of M, which will require a combinatorial type argument to establish. We note that a somewhat similar problem to the one above was studied in [3] where, for a given $s \in \mathbf{N}$, it was asked for what $r \in \mathbf{N}\setminus\{0\}$ could $T(r+s)-T(s)$ be a triangular number. However, unlike our result, for every $r \in \mathbf{N}\setminus\{0\}$, there corresponded an infinite number $m \in \mathbf{N}\setminus\{0\}$ such that $T(m)$ was expressible as a difference of triangular numbers indicated. In addition to the above, we shall provide an alternate proof of a result of E. Lucas dated around 1873, namely, that all triangular numbers greater than one can never be a perfect cube. This result, as we shall see, will follow as a corollary of the main representation theorem.

2. MAIN RESULT

We begin in this section by introducing a preliminary definition and lemma which will be required later in developing a formula for the total number of nonconsecutive triangular number representations of the integers.

Definition 2.1: For a given $M \in \mathbf{N}\setminus\{0\}$, a factorization $M = ab$ with $a, b \in \mathbf{N}\setminus\{0\}$ is said to be nontrivial if $a \neq 1, M$. Two such factorizations, $a_1b_1 = a_2b_2 = M$, are distinct if $a_1 \neq a_2, b_2$.

The following result, which concerns counting the total number of distinct nontrivial factorizations $ab = M$, may be known, but it is included here for completeness. Note that in the subsequent definition for $d(M)$ we include both 1 and M when counting the total number of divisors of M.

Lemma 2.1: Let M be an integer greater than unity and $d(M)$ be the number of divisors of M. Then the total number $N(M)$ of nontrivial distinct factorizations of M is given by

$$N(M) = \begin{cases} \frac{d(M)-2}{2} & \text{for nonsquare } M, \\ \frac{d(M)-1}{2} & \text{for square } M. \end{cases}$$

Proof: Suppose $M = p_1^{m_1}p_2^{m_2}\cdots p_n^{m_n}$, then the total number of divisors of M is

$$d(M) = (1+m_1)(1+m_2)\cdots(1+m_n).$$

Clearly, if $d|M$, then $(M/d)|M$; thus, the required factorization $ab = M$ will be given by $(a,b) = (d, M/d)$ provided $d \neq 1, M$. Excluding $d = 1$ and $d = M$, we have $d(M)-2$ divisors d_i of M such that $1 < d_i < M$ for $i = 1, 2, \dots, (d(M)-2)$. Arrange these divisors in ascending order and consider the set of ordered pairs

$$I = \{(d_i, M/d_i) : i = 1, 2, \dots, (d(M)-2)\}.$$

If M is not a perfect square, then $2|d(M)$ and so there will be an even number of elements in I. Consider for each $i = 1, 2, \dots, (d(M)-2)/2$ the subset $I_i = \{(d_i, M/d_i), (M/d_i, d_i)\}$ and note that $I_i \cap I_j = \emptyset$ for $i \neq j$ together with

$$I = \bigcup_{i=1}^{(d(M)-2)/2} I_i.$$

As both ordered pairs in each particular I_i correspond to the same nontrivial factorization of M, which must be distinct from that in I_j for $i \neq j$, one can conclude that $N(M) = (d(M)-2)/2$. Suppose now that M is a square, then $d(M)$ will be odd, and so I contains an odd number of elements. Furthermore, there must exist a unique $j \in \{1, 2, \dots, (d(M)-2)\}$ such that $d_j = M/d_j$, from which it is clear that $I_j = \{(d_j, d_j)\}$. Considering now the set $I' = I \setminus I_j$ which contains only $d(M)-3$ elements, one again has

$$I' = \bigcup_{i=1, i\neq j}^{(d(M)-3)/2} I_i,$$

from which we can count $(d(M)-3)/2$ nontrivial distinct factorizations of M together with the one from I_j to obtain $N(M) = 1 + (d(M)-3)/2 = (d(M)-1)/2$. □

Using Lemma 2.1, we can now establish the required representation theorem.

Theorem 2.1: Let $M \in \mathbf{Z} \setminus \{0\}$, then the number of distinct representations of M as a difference of nonconsecutive triangular numbers is given by $N_\Delta(M) = D - 1$, where D is the number of odd divisors of M.

Proof: Without loss of generality, we may assume that M is a positive integer. Our aim here will be to determine whether there exists $x, y \in \mathbf{N} \setminus \{0\}$ such that $M = T(x) - T(y)$. By completing the square, observe that the previous equation can be recast in the form $8M = X^2 - Y^2$, where $X = 2x+1$ and $Y = 2y+1$. To analyze the solvability of this equation, suppose $ab = 8M$, where $a, b \in \mathbf{N} \setminus \{0\}$ and consider the following system of simultaneous linear equations:

$$\begin{aligned} X - Y &= a \\ X + Y &= b \end{aligned} \tag{1}$$

whose general solution is given by

$$(X, Y) = \left(\frac{a+b}{2}, \frac{b-a}{2}\right).$$

Now, for there to exist a representation of M as a difference of nonconsecutive triangular numbers, one must be able to find factorizations $ab = 8M$ for which the system (1) will yield a solution (X, Y) in odd integers.

Remark 2.1: We note that it is sufficient to consider only (1), since if for a chosen factorization $ab = 8M$ an odd solution pair (X, Y) is found, then the corresponding representation $M = T(x) - T(y)$ is also obtained if the right-hand side of (1) is interchanged. Indeed, one finds upon solving

$$\begin{aligned} X' - Y' &= b \\ X' + Y' &= a \end{aligned}$$

where $X' = 2x'+1$, $Y' = 2y'+1$ that

$$X' = \frac{a+b}{2} \quad \text{and} \quad Y' = \frac{a-b}{2} = -Y.$$

Thus, $x' = x$ while $y' = (-Y-1)/2 = -y-1$, so

$$T(y') = \frac{(-y-1)(-y)}{2} = T(y) \quad \text{and} \quad T(x') - T(y') = T(x) - T(y) = M.$$

We deal with the existence or otherwise of those factorizations $ab = 8M$ which give rise to an odd solution pair (X, Y) of (1). It is clear from the general solution of (1) that, for X to be an odd positive integer a, b must be at least chosen so that $a + b = 2(2s+1)$ for some $s \in \mathbf{N} \setminus \{0\}$. As ab is even, this can only be achieved if a and b are also both even. Furthermore, such a choice of a and b will also ensure that $Y = X - a$ is odd. With this reasoning in mind, it will be convenient to consider the following cases separately.

Case 1. $M = 2^n$, $n \in \mathbf{N} \setminus \{0\}$.

In this instance, consider $8M = 2^{n+3} = ab$, where $(a, b) = (2^i, 2^{n+3-i})$ for $i = 0, 1, \ldots, n+3$ with $a + b = 2(2^{i-1} + 2^{n+2-i}) = 2(2s+1)$ only when $i = 1, n+2$. However, since both factorizations are equivalent, we need only investigate the solution of (1) when $(a, b) = (2, 2^{n+2})$. Thus, one finds that $(X, Y) = (1+2^{n+1}, 2^{n+1}-1)$ and so $(x, y) = (2^n, 2^n - 1)$. Hence, there exists only the trivial representation $M = T(M) - T(M-1)$.

Case 2. $M \neq 2^n$.

Clearly, $M = 2^m(2n+1)$ for an $n \in \mathbf{N} \setminus \{0\}$ and $m \in \mathbf{N}$. However, as there are more available factorizations of $8M$, due to the presence of the term $2n+1$, it will be necessary to consider the following subcases based on the possible factorizations $cd = 2n+1$.

Subcase 1. $(c, d) = (1, 2n+1)$.

Here consider $8M = 2^{m+3}(2n+1) = ab$, where $(a, b) = (2^i, 2^{m+3-i}(2n+1))$ for $i = 0, 1, \ldots, m+3$ with $a+b = 2(2^{i-1} + 2^{m+2-i}(2n+1)) = 2(2s+1)$ only when $i = 1, m+2$. Solving (1) with $(a, b) = (2, 2^{m+2}(2n+1))$, one finds that $(x, y) = (2^m(2n+1), 2^m(2n+1)-1)$, which corresponds to a consecutive triangular number difference of M, while for $(a, b) = (2^{m+2}, 2(2n+1))$ we have $(x, y) = (2^m + n, n - 2^m)$ and so

$$M = T(2^m + n) - T(y'), \tag{2}$$

where $y' = 2^m - n - 1$ if $y < 0$ and $y' = y$ otherwise. In either situation, one has $|x - y'| > 1$, giving a nonconsecutive triangular number representation of M.

Subcase 2. $(c, d), c \neq 1, 2n+1$.

Here consider $8M = 2^{m+3}cd = ab$, where $(a, b) = (2^i c, 2^{m+3-i}d)$ for $i = 0, 1, \ldots, m+2$ with $a+b = 2(2^{i-1}c + 2^{m+2-i}d) = 2(2s+1)$ when $i = 1, m+2$. Solving (1) with $(a, b) = (2c, 2^{m+2}d)$, one has $(X, Y) = (c + 2^{m+1}d, 2^{m+1}d - c)$, from which it is immediate that

$$(x, y) = \left(\frac{c-1}{2} + 2^m d, 2^m d - \frac{c+1}{2}\right)$$

and so

$$M = T\left(\frac{c-1}{2} + 2^m d\right) - T(y'), \tag{3}$$

where $y' = \frac{c+1}{2} - 2^m d - 1$ if $y < 0$ and $y' = y$ otherwise. Alternatively, when $(a, b) = (2^{m+2}c, 2d)$, one has $(X, Y) = (2^{m+1}c + d, d - 2^{m+1}c)$, from which we obtain

$$(x, y) = \left(2^m c + \frac{d-1}{2}, \frac{d-1}{2} - 2^m c\right)$$

and again

$$M = T\left(2^m c + \frac{d-1}{2}\right) - T(y'), \tag{4}$$

where $y' = 2^m c - \frac{d-1}{2} - 1$ if $y < 0$ and $y' = y$ otherwise. In either of the representations in (3) and (4), it is again easily seen that $|x - y'| > 1$. Consequently, for every distinct factorization $cd = 2n+1$ with $c \neq 1$, $(2n+1)$, we can expect at most two representations of $M = 2^m(2n+1)$ as a difference of two nonconsecutive triangular numbers.

We now address the problem of finding the exact number $N_\Delta(M)$ of representations for an M in Case 2. Primarily, this will entail determining whether any duplication occurs between the various representations given in (2), (3), and (4). Recall that two factorizations $a_i b_i = a_j b_j = 8M$

are said to be distinct if $a_i \neq a_j, b_j$ for $i \neq j$. First, it will be necessary to show that any two distinct factorizations of $8M$ considered in Case 2 will always produce two different triangular representations for M. To this end, we need to demonstrate that if in $\mathbf{Z}\setminus\{0\}$ $a_ib_i = a_jb_j$, with $a_i \neq a_j$, b_j for $i \neq j$, then one has $a_i + b_i \neq a_j + b_j$. Suppose to the contrary that $a_i + b_i = a_j + b_j$, then there must exist an $r \in \mathbf{Z}\setminus\{0\}$ such that $a_j = a_i + r$ and $b_i = b_j + r$. Substituting these equations into the equality $a_ib_i = a_jb_j$, one finds $a_i(b_j + r) = (a_i + r)b_j$. Hence, r must be a nonzero integer solution of

$$r(a_i - b_j) = 0. \tag{5}$$

However, this is impossible because $r = 0$ is the only possible solution of (5) since $a_i - b_j \neq 0$; a contradiction. Consequently, if for two distinct factorizations $a_ib_i = a_jb_j = 8M$, one solves (1) to produce corresponding odd solution pairs (X_i, Y_i) and (X_j, Y_j), then we must have

$$X_i = (a_i + b_i)/2 \neq (a_j + b_j)/2 = X_j$$

and so $x_i \neq x_j$. Moreover, as $x_i, x_j \geq 0$, we immediately see that $T(x_i) \neq T(x_j)$, hence

$$T(y_i) = T(x_i) - M \neq T(x_j) - M = T(y_j).$$

Thus, in order to calculate $N_\Delta(M)$ for $M = 2^m(2n+1)$, one must determine the total number of distinct factorizations $ab = 8M$ examined in Case 2. Recall that in Subcase 2 the only triangular representation of M was found by the factorization $(a, b) = (2^{m+2}, 2(2n+1))$. Clearly, this cannot be repeated by the factorizations $(a, b) = (2c, 2^{m+2}d)$ or $(a, b) = (2^{m+2}c, 2d)$ in Subcase 2 since $c \neq 1,\ 2n+1$. Now, if $2n+1$ is not a perfect square, that is, $c \neq d$, then $2c \neq 2^{m+2}c, 2d$ and so, by the above, each factorization $cd = 2n+1$ with $c \neq 1, 2n+1$ will produce two unique representations of M as a difference of two nonconsecutive triangular numbers. Consequently, in this instance, by combining both subcases we see that $N_\Delta(M)$ must be one more than twice the total number of nontrivial distinct factorizations $cd = 2n+1$. Thus, if one denotes by D the total number of divisors of $2n+1$, then by Lemma 2.1,

$$N_\Delta(M) = 1 + 2\left(\frac{D-2}{2}\right) = D - 1.$$

However, if $2n+1$ is a perfect square, then $(2c, 2^{m+2}d)$ and $(2^{m+2}c, 2d)$ will be equivalent factorizations when $c = d$. So by Lemma 2.1 only $\frac{D-1}{2} - 1$ of the factorizations in Subcase 2 will produce two distinct triangular representations of M. Hence, counting the remaining factorization $(2c, 2^{m+2}c)$ together with the one in Subcase 1, we find that

$$N_\Delta(M) = 2\left(\frac{D-1}{2} - 1\right) + 2 = D - 1.$$

To conclude, we note that the formula $N_\Delta(M) = D - 1$ also holds for all integers 2^m, where $m = 0, 1, \ldots$, since by Case 1, $N_\Delta(2^m) = 0$ while, clearly, $D - 1 = 0$ because 1 is the only odd divisor of 2^m. □

Example 2.1: For a given integer M whose prime factorization is known, one can use equations (2), (3), and (4) to determine all of the $D-1$ representations of M as a difference of triangular numbers. To illustrate this, we shall calculate the representations in the case of a square and nonsquare number. Beginning with, say $M = 2^2 \cdot 5 \cdot 7^2$, we have $N_\Delta(980) = 5$. So, if $cd = 5 \cdot 7^2$ and $m = 2$, then apart from $(c, d) = (1, 5 \cdot 7^2)$ each of the factorizations $(c, d) \in \{(5, 7^2), (5 \cdot 7, 7)\}$ will produce two distinct representations via (3) and (4). The remaining representation can be calculated using (2), with $n = (5 \cdot 7^2 - 1)/2$. Consequently, one obtains that

$$\begin{aligned} 980 &= T(126) - T(118) = T(198) - T(193) \\ &= T(45) - T(10) = T(143) - T(136) = T(44) - T(4). \end{aligned}$$

If, on the other hand, $M = (2 \cdot 5 \cdot 7)^2$, then; $N_\Delta(4900) = 8$. So again, for $(c, d) = (1, (2 \cdot 5 \cdot 7)^2)$ and $m = 2$, there corresponds one representation calculated via (2) with $n = ((2 \cdot 5 \cdot 7)^2 - 1)/2$. Apart from $(c, d) = (5 \cdot 7, 5 \cdot 7)$, all of the factorizations $(c, d) \in \{(5, 5 \cdot 7^2), (5^2, 7^2), (5^2 \cdot 7, 7)\}$ will each produce two distinct representations via (3) and (4). However, for $(c, d) = (5 \cdot 7, 5 \cdot 7)$, the representations given in (3) and (4) are identical as $c = d$. Thus, we obtain

$$\begin{aligned} 4900 &= T(616) - T(608) = T(982) - T(977) = T(142) - T(102) = T(208) - T(183) \\ &= T(124) - T(75) = T(115) - T(59) = T(703) - T(696) = T(157) - T(122). \end{aligned}$$

We now use Theorem 2.1 to deduce that all triangular numbers greater than unity cannot be a perfect cube. To achieve this end, the following two technical lemmas will be required, the first of which gives a necessary and sufficient condition for a positive integer to be a triangular number.

Lemma 2.2: An integer M greater than unity is triangular if and only if out of the $D-1$ distinct representations of $M = T(x) - T(y')$, with $|x - y'| > 1$, there exists one in which $y' = 0$.

Proof: Clearly, if $M = T(x) - T(0)$ for some $x \in \mathbb{N}$, then M is triangular. Conversely, assume M is a triangular number. To show there exists a representation of the above form, with $y' = 0$, it will be sufficient to find a factorization $ab = 8M$ such that the system of equations in (1) has a solution (X, Y) with $Y = 1$. From the general solution

$$(X, Y) = \left(\frac{a+b}{2}, \frac{b-a}{2}\right),$$

this is equivalent to finding positive integers a, b which simultaneously satisfy $b - a = 2$ and $ab = 8M$. Solving for b in terms of a from the first equation and substituting the result into the second, one finds upon simplifying that $0 = a^2 + 2a - 8M$. Hence, $a = -1 + \sqrt{1 + 8M}$; however, this must be a positive integer since $1 + 8M$ is a perfect square greater than unity. Consequently, $b = 2 + a$ is also a positive integer. □

Lemma 2.3: If c is an odd cube greater than unity, then neither $(c+1)/2$ nor $(c-1)/2$ can be perfect cubes.

Proof: To demonstrate the result, it is equivalent to show that the diophantine equations $X^3 - 2Y^3 = 1$ and $X^3 - 2Y^3 = -1$ have no solutions (X, Y) with $X > 1$. By Theorem 5 of [2], we have that $x^3 + dy^3 = 1$ $(d > 1)$ has at most one integer solution (x, y) with $xy \neq 0$. Now, since $(x, y) = (-1, 1)$ is such a solution of $x^3 + 2y^3 = 1$, it can be the only one with $xy \neq 0$. Making the substitution $X = x$, $Y = -y$, we deduce that $(X, Y) = (-1, -1)$ is the only integer solution $X^3 - 2Y^3 = 1$ while, if we take $X = -x$, $Y = y$, then $(X, Y) = (1, 1)$ can be the only integer solution of $X^3 - 2Y^3 = -1$. Hence, in either case, no other integer solutions (X, Y) exists where $X > 1$. □

Combining the previous two lemmas, we can now prove the desired result, which is stated here in terms of the solvability of a diophantine equation.

Corollary 2.1: The only solutions of the diophantine equation $x(x+1)=2y^3$ are given by $(x, y)=(1,1), (-2,1), (-1,0), (0,0)$.

Proof: Note that, as $x(x+1)\geq 0$ for all $x\in\mathbf{Z}$, one may assume, without loss of generality, that x and y are positive integers. We shall first establish that no integer solution (x, y) exists for $x>1$. To this end, let M be a triangular number greater than unity and *assume* it is a perfect cube. In order to derive the necessary contradiction, we will show that all of the $D-1$ representations of $M=T(x)-T(y')$ have $y'\neq 0$, which is in violation of Lemma 2.2. Now, since $M=2^m(2n+1)$ for some $n\in\mathbf{N}\setminus\{0\}$ and $m\in\mathbf{N}$, 2^m and $2n+1$ must be perfect cubes because $(2^m, 2n+1)=1$. Considering the representation given in (2), suppose $y'=0$, then either $n=2^m$ or $n=2^m-1$. Taking $n=2^m=Y^3>1$, we then have $2n+1=2Y^3+1=X^3$ for some $X\in\mathbf{N}\setminus\{0\}$, which is impossible by the argument used to establish Lemma 2.3. Similarly, if $n=2^m-1$, then $2n+1=2Y^3-1=X^3$ for some $X\in\mathbf{N}\setminus\{0\}$, which again is impossible. Hence, for the representation in (2), $y'\neq 0$. Writing now $M=2^m cd$, where $c\neq 1$, $2n+1$ and setting $y'=0$ in the representation given in (3), we must have either $2^m d=\frac{c+1}{2}$ or $2^m d=\frac{c-1}{2}$. Multiplying both sides of these equations by c, one deduces $M=T(c)$ or $M=T(c-1)$. Now, since $(c, \frac{c+1}{2})=1$ and $(c, \frac{c-1}{2})=1$, we conclude that either c and $\frac{c+1}{2}$ or c and $\frac{c-1}{2}$ are a pair of perfect cubes; a contradiction by Lemma 2.3. Thus, for the representation in (3), $y'\neq 0$. By setting $y'=0$ in the remaining representation given in (4), one can similarly arrive at the contradictory conclusion that either d and $\frac{d+1}{2}$ or d and $\frac{d-1}{2}$ are a pair of perfect cubes. Thus, for the representation in (4), $y'\neq 0$. Consequently, via Lemma 2.2, M is not a triangular number; a contradiction. Therefore, M cannot be a perfect cube and so $x(x+1)=2y^3$ has no integer solutions (x, y) with $x>1$. The solutions indicated can now be found upon inspecting the solvability for the remaining integers $x\in[-2, 1]$. □

Remark 2.2: The above argument could be applied in exactly the same manner to investigate the solvability of the diophantine equation $x(x+1)=2y^n$ for $n\geq 4$, provided one could ascertain for each such n the solvability of $X^n-2Y^n=\pm 1$ in integers (X, Y).

In conclusion, we consider some further consequences of Theorem 2.1. The first of these gives a necessary and sufficient condition for a positive integer to be an odd prime and follows directly from the fact that a number $p\in\mathbf{N}$ is an odd prime if and only if $D=2$.

Corollary 2.2: An integer $p\in\mathbf{N}\setminus\{0\}$ is prime if and only if $N_\Delta(p)=1$.

In connection with the representation of primes as a difference of the polygonal numbers of order $k=6$, namely, the hexagonal numbers, we have the following.

Corollary 2.3: Let $p\in\mathbf{N}\setminus\{0\}$ be a prime number. If $p\equiv 1 \pmod 4$, then there exists exactly one representation of p as a difference of hexagonal numbers, while no such representation exists if $p\equiv 3 \pmod 4$.

Proof: By definition, the n^{th} hexagonal number is equal to $T(2n-1)$. Thus, the problem of representing an integer as a hexagonal number difference is equivalent to finding a triangular number difference $T(m_1)-T(m_2)$, where both m_1 and m_2 are odd integers. For a prime p, the only possible triangular representations are those of the form given in Case 2 Subcase 1; that is, if $p = 2^0(2n+1)$ for some $n \in \mathbf{N}\setminus\{0\}$, then $p = T(1+n) - T(n-1)$. Now, if $p \equiv 1 \pmod 4$, then clearly $2|n$, and so both $1+n$ and $n-1$ are odd integers. However, for $p \equiv 3 \pmod 4$, one must have $n = 2s+1$ for some $s \in \mathbf{N}\setminus\{0\}$, and so $1+n$ and $n-1$ are even integers. □

Clearly, in comparison with the triangular case, a larger subset of $\mathbf{Z}\setminus\{0\}$ fails to have a representation as a difference of hexagonal numbers. Consequently, in view of this, one might consider the following conjecture.

Conjecture 2.1: Denote the n^{th} polygonal number of order k by $P_k(n)$ and consider the set $A_k = \{M \in \mathbf{Z}\setminus\{0\} : M = P_k(n_1) - P_k(n_2) \text{ for some } n_1, n_2 \in \mathbf{N}\}$. Does the set inclusion $A_{k+1} \subseteq A_k$ hold for all $k = 3, 4, \ldots$, and, if so, is $\bigcap_{k=3}^{\infty} A_k \neq \emptyset$?

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