ON THE SQUARE ROOTS OF TRIANGULAR NUMBERS

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1. BALANCING NUMBERS

We call an integer $n \in \mathbf{Z}^+$ a *balancing number* if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1}$$

for some $r \in \mathbf{Z}^+$. Here r is called the *balancer* corresponding to the balancing number n.

For example, 6, 35, and 204 are balancing numbers with balancers 2, 14, and 84, respectively.

It follows from (1) that, if n is a balancing number with balancer r, then

$$n^2=\frac{(n+r)(n+r+1)}{2} \tag{2}$$

and thus

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2}. \tag{3}$$

It is clear from (2) that n is a balancing number if and only if n^2 is a triangular number (cf. [2], p. 3). Also, it follows from (3) that n is a balancing number if and only if $8n^2+1$ is a perfect square.

2. FUNCTIONS GENERATING BALANCING NUMBERS

In this section we introduce some functions that generate balancing numbers. For any balancing number x, we consider the following functions:

$$F(x)=2x\sqrt{8x^2+1}, \tag{4}$$

$$G(x)=3x+\sqrt{8x^2+1}, \tag{5}$$

$$H(x)=17x+6\sqrt{8x^2+1}. \tag{6}$$

First, we prove that the above functions always generate balancing numbers.

Theorem 2.1: For any balancing number x, $F(x)$, $G(x)$, and $H(x)$ are also balancing numbers.

Proof: Since x is a balancing number, $8x^2+1$ is a perfect square, and

$$\frac{8x^2(8x^2+1)}{2}=4x^2(8x^2+1)$$

is a triangular number which is also a perfect square; therefore, its square root $2x\sqrt{8x^2+1}$ is a (an even) balancing number. Thus, for any given balancing number x, $F(x)$ is an even balancing number. Since $8x^2+1$ is a perfect square, it follows that

$$8(G(x))^2+1=(8x+3\sqrt{8x^2+1})^2$$

is also a perfect square; hence, $G(x)$ is a balancing number. Again, since $G(G(x))=H(x)$, it follows that $H(x)$ is also a balancing number. This completes the proof of Theorem 2.1.

It is important to note that, if x is any balancing number, then $F(x)$ is always even, whereas $G(x)$ is even when x is odd and $G(x)$ is odd when x is even. Thus, if x is any balancing number, then $G(F(x))$ is an odd balancing number. But

$$G(F(x)) = 6x\sqrt{8x^2+1}+16x^2+1.$$

The above discussion proves the following result.

Theorem 2.2: If x is any balancing number, then

$$K(x) = 6x\sqrt{8x^2+1}+16x^2+1 \tag{7}$$

is an odd balancing number.

3. FINDING THE NEXT BALANCING NUMBER

In the previous section, we showed that $F(x)$ generates only even balancing numbers, whereas $K(x)$ generates only odd balancing numbers. But $H(x)$ and $K(x)$ generate both even and odd balancing numbers. Since $H(6) = 204$ and there is a balancing number 35 between 6 and 204, it is clear that $H(x)$ does not generate the next balancing number for any given balancing number x. Now the question arises: "Does $G(x)$ generate the next balancing number for any given balancing number x?" The answer to this question is affirmative. More precisely, if x is any balancing number, then the next balancing number is $3x+\sqrt{8x^2+1}$ and, consequently, the previous one is $3x-\sqrt{8x^2+1}$.

Theorem 3.1: If x is any balancing number, then there is no balancing number y such that $x<y<3x+\sqrt{8x^2+1}$.

Proof: The function $G:[0,\infty)\to[1,\infty)$, defined by $G(x)=3x+\sqrt{8x^2+1}$, is strictly increasing since

$$G'(x) = 3+\frac{8x}{\sqrt{8x^2+1}} > 0.$$

Also, it is clear that G is bijective and $x<G(x)$ for all $x\geq 0$. Thus, G^{-1} exists and is also strictly increasing with $G^{-1}(x)<x$. Let $u=G^{-1}(x)$. Then $G(u)=x$ and $u=3x\pm\sqrt{8x^2+1}$. Since $u<x$, we have $u=3x-\sqrt{8x^2+1}$. Also, since $8(G^{-1}(x))^2+1=(8x-3\sqrt{8x^2+1})^2$ is a perfect square, it follows that $G^{-1}(x)$ is also a balancing number.

Now we can complete the proof in two ways. The first is by the *method of induction*; the second is by the *method of infinite descent* used by Fermat ([2], p. 228).

By induction: We define $B_0=1$ (the reason is that $8\cdot 1^2+1=9$ is a perfect square) and $B_n = G(B_{n-1})$ for $n=1,2,\ldots$. Thus, $B_1=6$, $B_2=35$, and so on. Let H_i be the hypothesis that there is no balancing number between B_{i-1} and B_i. Clearly, H_1 is true. Assume H_i is true for $i=1, 2, \ldots, n$. We shall prove that H_{n+1} is true, i.e., there is no balancing number y such that $B_n<y<B_{n+1}$. Assume, to the contrary, that such a y exists. Then $G^{-1}(y)$ is a balancing number, and since G^{-1} is strictly increasing, it follows that $G^{-1}(B_n)<G^{-1}(y)<G^{-1}(B_{n+1})$, i.e., $B_{n-1}<G^{-1}(y)<B_n$, which is a contradiction to the assumption that H_n is true. So H_{n+1} is also true. Thus, if x is a balancing number, then $x=B_n$ for some n and there is no balancing number between x and $G(x)$.

By the method of infinite descent: Here assume H_n is false for some n. Then there exists a balancing number y such that $B_{n-1} < y < B_n$, and this implies that $B_{n-2} < G^{-1}(y) < B_{n-1}$. Finally, this would imply that there exists a balancing number B between B_0 and B_1, which is false. Thus, H_n is true for $n = 1, 2, \ldots$.

This completes the proof of Theorem 3.1.

Corollary 3.2: If x is any balancing number, then its previous balancing number is $3x - \sqrt{8x^2+1}$.

Proof: $G(3x - \sqrt{8x^2+1}) = x$.

4. ANOTHER FUNCTION GENERATING BALANCING NUMBERS

In this section we develop a function $f(x, y)$ of two variables generating balancing numbers such that all the functions $F(x)$, $G(x)$, $H(x)$, and $K(x)$ are obtained as particular cases of this function.

Let x be any balancing number. We try to find balancing numbers of the form

$$B = px + q\sqrt{8x^2+1},$$

where $p, q \in \mathbf{Z}^+$. In the previous section we have seen that most of the balancing numbers are of this form. Since B is a balancing number, $8B^2+1 = (8qx + p\sqrt{8x^2+1})^2 + 8q^2 - p^2 + 1$ must be a perfect square; this happens if $8q^2 - p^2 + 1 = 0$, i.e., $p = \sqrt{8q^2+1}$. Since $p \in \mathbf{Z}^+$, it follows that $8q^2+1$ must be a perfect square, and this is possible if q is a balancing number.

The above discussion proves the following theorem.

Theorem 4.1: If x and y are balancing numbers, then

$$f(x, y) = x\sqrt{8y^2+1} + y\sqrt{8x^2+1} \tag{8}$$

is also a balancing number.

Remark 4.2: ***(a)*** $f(x, x) = F(x)$; ***(b)*** $f(x, 1) = G(x)$; ***(c)*** $f(x, 6) = H(x)$; ***(d)*** $f(x, G(x)) = K(x)$.

5. RECURRENCE RELATIONS FOR BALANCING NUMBERS

We know that $B_1 = 6$, $B_2 = 35$, $B_3 = 204$, and so on. We have already assumed that $B_0 = 1$. In Section 3 we proved that, if B_n is the n^{th} balancing number, then

$$B_{n+1} = 3B_n + \sqrt{8B_n^2+1} \quad \text{and} \quad B_{n-1} = 3B_n - \sqrt{8B_n^2+1}.$$

It is clear that the balancing numbers obey the following recurrence relation:

$$B_{n+1} = 6B_n - B_{n-1}. \tag{9}$$

Using the recurrence relation (9), we can obtain some other interesting relations concerning balancing numbers.

Theorem 5.1:

(a) $B_{n+1} \cdot B_{n-1} = (B_n + 1)(B_n - 1)$.

(b) $B_n = B_k \cdot B_{n-k} - B_{k-1} \cdot B_{n-k-1}$ for any positive integer $k < n$.

(c) $B_{2n} = B_n^2 - B_{n-1}^2$.

(d) $B_{2n+1} = B_n(B_{n+1} - B_{n-1})$.

Proof: From (9), it follows that

$$\frac{B_{n+1} + B_{n-1}}{B_n} = 6. \tag{10}$$

Replacing n by $n-1$ in (10), we get

$$\frac{B_{n-1} + B_{n-2}}{B_{n-1}} = 6. \tag{11}$$

From (10) and (11), we obtain $B_n^2 - B_{n-1} \cdot B_{n+1} = B_{n-1}^2 - B_{n-2} \cdot B_n$. Now, iterating recursively, we see that $B_n^2 - B_{n-1} \cdot B_{n+1} = B_1^2 - B_0 \cdot B_2 = 36 - 1 \cdot 35 = 1$. Thus, $B_n^2 - 1 = B_{n+1} \cdot B_{n-1}$, from which (a) follows.

The proof of (b) is based on induction. Clearly, (b) is true for $n > 1$ and $k = 1$. Assume that (b) is true for $k = r$, i.e., $B_n = B_r \cdot B_{n-r} - B_{r-1} \cdot B_{n-r-1}$. Thus,

$$\begin{aligned} B_{r+1} \cdot B_{n-r-1} - B_r \cdot B_{n-r-2} &= (6B_r - B_{r-1})B_{n-r-1} - B_r \cdot B_{n-r-2} \\ &= 6B_r \cdot B_{n-r-1} - B_{r-1} \cdot B_{n-r-1} - B_r \cdot B_{n-r-2} \\ &= B_r(6B_{n-r-1} - B_{n-r-2}) - B_{r-1} \cdot B_{n-r-1} \\ &= B_r \cdot B_{n-r} - B_{r-1} \cdot B_{n-r-1} = B_n, \end{aligned}$$

showing that (b) is true for $k = r+1$. This completes the proof of (b).

The proof of (c) follows by replacing n by $2n$ and k by n in (b). Similarly, the proof of (d) follows by replacing n by $2n+1$ and k by n in (b). This completes the proof of Theorem 5.1.

6. GENERATING FUNCTION FOR BALANCING NUMBERS

In Section 5 we obtained some recurrence relations for the sequence of balancing numbers. In this section our aim is to find a nonrecursive form for B_n, $n = 0, 1, 2, \ldots$, using the generating function for the sequence B_n.

Recall that the generating function for a sequence $\{x_n\}$ of real numbers is defined by

$$g(s) = \sum_{n=0}^{\infty} x_n s^n.$$

Thus,

$$x_n = \frac{1}{n!} \frac{d^n}{ds^n} g(s) \bigg|_{s=0} \quad \text{(see [5], p. 29).}$$

Theorem 6.1: The generating function of the sequence B_n of balancing numbers is $g(s) = \frac{1}{1-6s+s^2}$ and, consequently,

$$\begin{aligned} B_n &= 6^n - \binom{n-1}{1} 6^{n-2} + \binom{n-2}{2} 6^{n-4} - \cdots + (-1)^{[\frac{n}{2}]} \binom{n - [\frac{n}{2}]}{[\frac{n}{2}]} 6^{n-[\frac{n}{2}]} \\ &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} 6^{n-2k}, \end{aligned} \tag{12}$$

where [ ] denotes the greatest integer function.

Proof: From (9) for $n=1,2,\dots$, we have $B_{n+1}-6B_n+B_{n-1}=0$. Multiplying each term by s^n and taking summation over $n=1$ to $n=\infty$, we obtain

$$\frac{1}{s}\sum_{n=1}^{\infty}B_{n+1}s^{n+1}-6\sum_{n=1}^{\infty}B_ns^n+s\sum_{n=1}^{\infty}B_{n-1}s^{n-1}=0$$

which, in terms of $g(s)$, yields

$$\frac{1}{s}(g(s)-1-6s)-6(g(s)-1)+sg(s)=0.$$

Thus,

$$\begin{aligned}g(s)&=\frac{1}{1-6s+s^2}=(1-(6s-s^2))^{-1}\\&=1+(6s-s^2)+(6s-s^2)^2+(6s-s^2)^3+\cdots.\end{aligned}\tag{13}$$

When n is even, the terms containing s^n in (13) are $(6s-s^2)^{n/2}, (6s-s^2)^{(n/2)+1},\dots,(6s-s^2)^n$, and in this case the coefficient of s^n in $g(s)$ is

$$6^n-\binom{n-1}{1}6^{n-2}+\binom{n-2}{2}6^{n-4}-\cdots+(-1)^{n/2}.\tag{14}$$

When n is odd, the terms containing s^n in (13) are $(6s-s^2)^{(n+1)/2}, (6s-s^2)^{(n+3)/2},\dots,(6s-s^2)^n$, and in this case the coefficient of s^n in $g(s)$ is

$$6^n-\binom{n-1}{1}6^{n-2}+\binom{n-2}{2}6^{n-4}-\cdots+(-1)^{(n-1)/2}\binom{\frac{n+1}{2}}{\frac{n-1}{2}}6.\tag{15}$$

It is clear that (14) represents the right-hand side of (12) when n is even and (15) represents the right-hand side of (12) when n is odd. This completes the proof of Theorem 6.1.

7. ANOTHER NONRECURSIVE FORM FOR BALANCING NUMBERS

In Section 6 we obtained a nonrecursive form for B_n, $n=0,1,2,\dots$, using the generating function. In this section we shall obtain another nonrecursive form for B_n by solving the recurrence relation (9) as a difference equation.

We rewrite (9) in the form

$$B_{n+1}-6B_n+B_{n-1}=0,\tag{16}$$

which is a second-order linear homogeneous difference equation whose auxiliary equation is

$$\lambda^2-6\lambda+1=0.\tag{17}$$

The roots $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$ of (17) are real and unequal. Thus,

$$B_n=A\lambda_1^n+B\lambda_2^n,\tag{18}$$

where A and B are determined from the values of B_0 and B_1. Substituting $B_0=1$ and $B_1=6$ into (18), we get

$$A+B=1,\tag{19}$$

$$A\lambda_1+B\lambda_2=6.\tag{20}$$

Solving (19) and (20) for A and B, we obtain

$$A=\frac{\lambda_2-6}{\lambda_2-\lambda_1}=\frac{\lambda_1}{\lambda_1-\lambda_2}; \quad B=\frac{6-\lambda_1}{\lambda_2-\lambda_1}=-\frac{\lambda_2}{\lambda_1-\lambda_2}.$$

Substituting these values into (18), we get

$$B_n=\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}, \quad n=0,1,2,\ldots.$$

Theorem 7.1: If B_n is the n^{th} balancing number, then

$$B_n=\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}, \quad n=0,1,2,\ldots,$$

where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$.

8. LIMIT OF THE RATIO OF THE SUCCESSIVE TERMS

The Fibonacci numbers ([1], p. 6) are defined as follows: $F_0=1, F_1=1,\ F_n=F_{n-1}+F_{n-2}$ for $n=2,3,\ldots.$ It is well known that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2},$$

which is called the *golden ratio* [1]. We prove a similar result concerning balancing numbers.

Theorem 8.1: If B_n is the n^{th} balancing number, then

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n}=3+\sqrt{8}.$$

Proof: From the recurrence relation (9), we have

$$\frac{B_{n+1}}{B_n}+\frac{B_{n-1}}{B_n}=6. \tag{21}$$

Putting $\lambda=\lim_{n\to\infty}\frac{B_{n+1}}{B_n}$ in (21), we get $\lambda^2-6\lambda+1=0$, i.e., $\lambda=3\pm\sqrt{8}$. Since $B_{n+1}>B_n$, we must have $\lambda\ge 1$. Thus, $\lambda=3+\sqrt{8}$. This completes the proof of Theorem 8.1.

An alternative proof of Theorem 8.1 can be obtained by considering the relation

$$B_{n+1}=3B_n+\sqrt{8B_n^2+1}$$

and using the fact that $B_n\to\infty$ as $n\to\infty$.

It is important to note that the limit ratio $3+\sqrt{8}$ represents the *simple periodic continued fraction* ([4], Ch. X)

$$[\dot{6},-\dot{6}]=6+\cfrac{1}{-6+\cfrac{1}{6+\cfrac{1}{-6+\cdots}}}, \tag{22}$$

and from Theorem 178 ([4], p. 147) it follows that, if C_n is the n^{th} *convergent* of (22), then

$$C_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1^{n+1} - \lambda_2^{n+1}},$$

where $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$. An application of Theorem 7.1 shows that $C_n = \frac{B_{n+1}}{B_n}$; thus, $B_0 = 1$ and $B_{n+1} = B_n C_n$, $n = 0, 1, 2, \ldots$.

9. AN APPLICATION OF BALANCING NUMBERS TO A DIOPHANTINE EQUATION

It is quite well known that the solutions of the Diophantine equation

$$x^2 + y^2 = z^2, \quad x, y, z \in \mathbf{Z}^+ \tag{23}$$

are of the form

$$x = u^2 - v^2, \quad y = 2uv, \quad z = u^2 + v^2,$$

where $u, v \in \mathbf{Z}^+$ and $u > v$ ([3], [4], [7]). The solution (x, y, z) is called a *Pythagorean triplet*. We consider the solutions of (23) in a particular case, namely,

$$x^2 + (x+1)^2 = y^2. \tag{24}$$

In this section we relate the solutions of (24) with balancing numbers.

Let (x, y) be a solution of (24). Hence, $2y^2 - 1 = (2x+1)^2$. Thus,

$$\frac{(2y^2-1)\cdot 2y^2}{2} = y^2 \cdot (2y^2 - 1)$$

is a triangular number as well as a perfect square. Therefore,

$$B = \sqrt{y^2(2y^2-1)} \tag{25}$$

is an odd balancing number (since y^2 and $2y^2 - 1$ are odd). Since $y^2 \geq 1$, it follows from (25) that

$$y^2 = \frac{1+\sqrt{8B^2+1}}{4}. \tag{26}$$

Again, since y is positive by assumption, we have

$$y = \frac{1}{2}\sqrt{1+\sqrt{8B^2+1}}.$$

From (24) and (26), we obtain

$$2x^2 + 2x + 1 = \frac{1+\sqrt{8B^2+1}}{4}.$$

Since x is positive, it follows that

$$x = \frac{\sqrt{\frac{1}{2}\left(\sqrt{8B^2+1}-1\right)} - 1}{2}.$$

For example, if we take $B = 35$ (an odd balancing number), then we have

$$x = \frac{\sqrt{\frac{1}{2}\left(\sqrt{8\cdot 35^2+1}-1\right)} - 1}{2} = 3,$$

$$y = \frac{1}{2}\sqrt{1+\sqrt{8\cdot 35^2+1}} = 5,$$

and

$$3^2 + (3+1)^2 = 5^2,$$

i.e.,

$$x^2 + (x+1)^2 = y^2.$$

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