GENERATORS OF UNITARY AMICABLE NUMBERS

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1. INTRODUCTION

In this paper, unless otherwise stated, lower-case letters denote positive integers with p and q reserved for primes.

Definition

A divisor d of n is a *unitary divisor* if $(n, n/d) = 1$, denoted by $d \| n$.

The sum of all unitary divisors of n will be denoted $\sigma^*(n)$. If

$$n = p_1^{e_1} p_2^{e_2} \ \ldots \ p_k^{e_k},$$

then

$$\sigma^*(n) = (1 + p_1^{e_1})(1 + p_2^{e_2}) \cdots (1 + p_k^{e_k}).$$

Hence, σ^* is multiplicative. If $\sigma(n)$ is the sum of all divisors of n, then

$$\sigma(n) = \sigma^*(n) \text{ iff } n \text{ is square-free.}$$

Note that

$$\sigma^*(n) = n \text{ iff } n = 1.$$

Hagis [1] defines a pair of positive integers m and n to be *unitary amicable numbers* if $\sigma^*(m) = \sigma^*(n) = m + n$. If m and n are both square-free, then the pair m, n is amicable (see [2]) iff it is unitary amicable. Independently, Wall [3] studies unitary amicable numbers and finds approximately six hundred pairs that are not amicable pairs. Hagis proves some elementary theorems concerning unitary amicable numbers and gives a table of thirty-two unitary amicable pairs that are not amicable pairs. (A thirty-third such pair,

$$11777220 = 2^2 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 719, \quad 12414780 = 2^2 3^2 5 \cdot 7 \cdot 59 \cdot 167,$$

follows from his theorem 4 and was inadvertently omitted from the table.) This paper generalizes Theorems 4 and 5 of [1] and augments Hagis' list of unitary amicable pairs that are not amicable pairs by twenty-five.

2. THE MAIN RESULTS

In this section, we find conditions on a unitary amicable pair which are sufficient to generate another such pair. The main idea is that of a generator.

Definition

The pair (f, k), where f is a rational number not equal to one and k is an integer, is a *generator* if fk is an integer and $\sigma^*(fk) = f\sigma^*(k)$.

Remark: If $k = 1$ in the above definition, then $\sigma^*(f) = f$, which implies that $f = 1$. Thus $k \neq 1$.

Generators, in conjunction with unitary amicable pairs of a specified form, produce new unitary amicable pairs. In what follows, m and n denote a unitary amicable pair.

Theorem 1

If (f, k) is a generator, $m = km_1$, $n = kn_1$, and $(fk, m_1n_1) = (k, m_1n_1) = 1$, then fkm_1, fkn_1 is a unitary amicable pair.

Proof: $\sigma^*(km_1) = \sigma^*(kn_1) = k(m_1 + n_1)$, since m, n is a unitary amicable pair. Thus,

$$\sigma^*(k)\sigma^*(m_1) = \sigma^*(k)\sigma^*(n_1) = k(m_1 + n_1),$$

since $(k, m_1n_1) = 1$. Hence,

$$f\sigma^*(k)\sigma^*(m_1) = f\sigma^*(k)\sigma\ (n_1) = fk(m_1 + n_1),$$

which yields

$$\sigma^*(fk)\sigma^*(m_1) = \sigma^*(fk)\sigma\ (n_1) = fk(m_1 + n_1),$$

since (f, k) is a generator.

Both f, a rational number, and k can be factored uniquely into a product of primes with nonzero (possibly negative) powers. Let $\pi(f)$ and $\pi(k)$ denote the number of primes in the factorization of f and k, respectively. Subsequent results classify all generators with $\pi(f) \leqslant 2$ and $\pi(k) = 1$.

Definition

The numbers f and k are *relatively prime* if their prime factorizations have no common prime.

Lemma 1

If (f, k) is a generator, then f and k are not relatively prime.

Proof: Suppose that f and k are relatively prime. Then they have distinct primes in their prime factorizations. Since fk is an integer, f is also. Thus,

$$\sigma^*(fk) = \sigma^*(f)\sigma^*(k) = f\sigma^*(k),$$

yielding $\sigma^*(f) = f$, which implies $f = 1$, a contradiction to the definition of a generator.

Theorem 2

There does not exist a generator (f, k) with $\pi(f) = \pi(k) = 1$.

Proof: Suppose that (f, k) is a generator with $\pi(f) = \pi(k) = 1$. By Lemma 1, there is a prime p such that $f = p^a$ and $k = p^b$ for some a and b. Since fk is an integer, $a + b \geqslant 0$. Because $k \neq 1$ in a generator, we must have $b > 0$. Similarly, $f \neq 1$ implies $a \neq 0$.

Case 1: If $a + b = 0$, then

$$\sigma^*(fk) = \sigma^*(p^{a+b}) = \sigma^*(1) = 1$$

and

$$f\sigma^*(k) = p^a\sigma^*(p^b) = p^a(1 + p^b) = p^a + p^{a+b} = p^a + 1.$$

Since $\sigma^*(fk) = f\sigma^*(k)$, we have $1 = p^a + 1$ or $p^a = 0$, a contradiction.

Case 2: If $a + b > 0$, then

$$\sigma^*(fk) = \sigma^*(p^{a+b}) = 1 + p^{a+b}$$

and

$$f\sigma^*(k) = p^a + p^{a+b}.$$

Thus, $1 + p^{a+b} = p^a + p^{a+b}$, which implies $p^a = 1$ or $a = 0$, a contradiction.

Definition

For the positive rational number f, the prime p *divides* f (written $p|f$) if p occurs in the prime factorization of f.

Lemma 2

Let (f, k) be a generator and p be a prime such that $p^a \| k$ and $p \nmid f$. Then (f, kp^{-a}) is a generator.

Proof: Let $k = p^a r$, where $a > 0$ and $(p, r) = 1$. Then $fk = fp^a r$ is an integer. Since $p \nmid f$, it follows that fr is an integer and that $p \nmid fr$. Hence,

$$\sigma^*(fk) = \sigma^*(fp^a r) = (1 + p^a)\sigma^*(fr).$$

Also

$$f\sigma^*(k) = f\sigma^*(p^a r) = f(1 + p^a)\sigma^*(r).$$

Hence, $(1 + p^a)\sigma^*(fr) = (1 + p^a)f\sigma^*(r)$, yielding $\sigma^*(fr) = f\sigma^*(r)$. Thus, (f, r) is a generator.

Therefore, "extraneous" primes may be eliminated from k.

Theorem 3

There does not exist a generator (f, k) with $\pi(f) = 1$ and $\pi(k) = 2$.

Proof: Suppose that (f, k) is a generator with $\pi(f) = 1$ and $\pi(k) = 2$. Then there is a prime p and an integer a with $p^a \| k$ and $p \nmid f$. By Lemma 2, (f, kp^{-a}) is a generator with $\pi(f) = \pi(kp^{-a}) = 1$, a contradiction of Theorem 2.

Theorem 4 characterizes all generators (f, k) with $\pi(f) = 2$ and $\pi(k) = 1$.

Theorem 4

The pair (f, k) is a generator with $\pi(f) = 2$ and $\pi(k) = 1$ iff there are primes p and q and positive integers a, b, and c such that $f = p^b q^c$, $k = p^a$, and $1 + p^{a+b} = q^c(p^b - 1)$.

Proof: Let (f, k) be a generator with $\pi(f) = 2$ and $\pi(k) = 1$. By Lemma 1, there are primes p and q and nonzero integers a, b, and c such that $f = p^b q^c$ and $k = p^a$. Since $k \neq 1$, it follows that $a > 0$. Because fk is an integer, we have $a + b \geqslant 0$ and $c > 0$. We therefore have $fk = p^{a+b}q^c$.

Case 1: If $a + b = 0$, then

$$\sigma^*(fk) = \sigma^*(q^c) = 1 + q^c$$

and

$$f\sigma^*(k) = p^b q^c \sigma^*(p^a) = p^b q^c(1 + p^a) = p^b q^c + p^{a+b}q^c = p^b q^c + q^c.$$

Thus, $1 + q^c = p^b q^c + q^c$, which implies $p^b q^c = 1$. Thus, $b = c = 0$, a contradiction.

Case 2: If $a + b > 0$, then

$$\sigma^*(fk) = \sigma^*(p^{a+b}q^c) = (1 + p^{a+b})(1 + q^c) = 1 + p^{a+b} + q^c + p^{a+b}q^c$$

and

$$f\sigma^*(k) = p^b q^c \sigma^*(p^a) = p^b q^c(1 + p^a) = p^b q^c + p^{a+b}q^c.$$

Therefore,

$$1 + p^{a+b} + q^c + p^{a+b}q^c = p^b q^c + p^{a+b}q^c,$$

yielding

$$1 + p^{a+b} + q^c = p^b q^c.$$

Since $1 + p^{a+b} + q^c$ is an integer, $p^b q^c$ is an integer and, hence, $b \geqslant 0$. If $b = 0$, then $k = p^a$, $f = q^c$, and $(f, k) = 1$, a contradiction of Lemma 1. Thus, $b > 0$ and $1 + p^{a+b} = q^c(p^b - 1)$.

If p and q are primes, and a, b, and c are positive integers such that $f = p^b q^c$, $k = p^a$, and $1 + p^{a+b} = q^c(p^b - 1)$, then clearly fk is an integer. Also

$$\begin{aligned}\sigma^*(fk) = \sigma^*(p^{a+b}q^c) &= (1 + p^{a+b})(1 + q^c) = 1 + p^{a+b} + q^c + p^{a+b}q^c\\ &= q^c(p^b - 1) + q^c + p^{a+b}q^c = p^b q^c + p^{a+b}q^c\\ &= p^b q^c(1 + p^a) = f\sigma^*(k).\end{aligned}$$

Therefore, (f, k) is a generator.

Theorem 5

The equation

$$1 + p^{a+b} = q^c(p^b - 1)$$

has a solution only if $p = 2$ and $b = 1$ or $p = 2$ and $b = 2$ or $p = 3$ and $b = 1$.

Proof: Suppose that $1 + p^{a+b} = q^c(p^b - 1)$ has a solution. Then,

$$p^b - 1 \mid p^{a+b} + 1 \quad \text{or} \quad p^{a+b} = -1 \text{ in } Z(p^b - 1),$$

the ring of integers modulo $p^b - 1$. Since $p^b = 1$ in $Z(p^b - 1)$, we have

$$p^{a+b} = p^a p^b = p^a \text{ in } Z(p^b - 1).$$

Hence,

$$p^a = -1 = p^b - 2 \text{ in } Z(p^b - 1).$$

Since

$$(p,\ p^b - 1) = (p^b - 2,\ p^b - 1) = 1,$$

we see that p and $p^b - 2$ belong to $U(p^b - 1)$, the group of units of $Z(p^b - 1)$. Thus, $p^a = p^b - 2$ in $U(p^b - 1)$. Also, there exist a and b such that

$$p^a = p^b - 2 \text{ iff } p^b - 2 \in \langle p \rangle,$$

the cyclic subgroup generated by p in $U(p^b - 1)$. If $c < b$, then

$$p^c - 1 < p^b - 1 \quad \text{and} \quad p^b - 1 \nmid p^c - 1,$$

so $p^c \neq 1$ in $U(p^b - 1)$. Since $p^b = 1$ in $U(p^b - 1)$, the order of p in $U(p^b - 1)$ is b and $\langle p \rangle = \{1, p, p^2, \ldots, p^{b-1}\}$. Note that

$$\begin{aligned} p^{b-1} < p^b - 2 \ &\text{iff } p^b - p^{b-1} > 2 \\ &\text{iff } p^{b-1}(p - 1) > 2 \\ &\text{iff } p^{b-1} > \frac{2}{p - 1} \\ &\text{iff } b - 1 > \log_p \frac{2}{p - 1} \\ &\text{iff } b > 1 + \log_p \frac{2}{p - 1}. \end{aligned}$$

If $p = 2$, then

$$\log_p \frac{2}{p - 1} = \log_2 \frac{2}{2 - 1} = \log_2 2 = 1.$$

Then

$$\begin{aligned} b > 2 \ &\text{iff } p^{b-1} < p^b - 2 \\ &\text{iff } p^b - 2 \notin \langle p \rangle, \end{aligned}$$

a contradiction. Thus, if $b > 2$, there does not exist a solution to (1).

If $p = 3$, then

$$\log_p \frac{2}{p - 1} = \log_3 1 = 0.$$

Then $b > 1$ iff $p^{b-1} < p^b - 2$. Hence, if $b > 1$, there does not exist a solution to (1).

Also

$$\begin{aligned} \log_p \frac{2}{p - 1} < 0 \ &\text{iff } \log_p 2 - \log_p(p - 1) < 0 \\ &\text{iff } \log_p 2 < \log_p(p - 1) \\ &\text{iff } 2 < p - 1 \\ &\text{iff } p > 3. \end{aligned}$$

Thus, if $p > 3$, then

$$1 + \log_p \frac{2}{p - 1} < 1,$$

which yields

$$b > 1 + \log_p \frac{2}{p - 1} \quad \text{for all } b.$$

Hence, $p^{b-1} < p^b - 2$ and there does not exist a solution to (1).

A computer-assisted search for solutions to (1) for a restricted range of values of a yields Table 1, which also lists the sixteen generators associated with these solutions. When these sixteen generators are applied, iteratively, to the table of thirty-three unitary amicable pairs that are not amicable pairs in [1], the result is the collection of twenty-five pairs in Table 2. Although not in [1], all but the 12th, 17th, and 18th pairs are found in [3].

Table 1

	a	c	q	k	f
$p = 2,\ b = 1,\ 1 \leqslant a \leqslant 31$	1	1	5	2	$2 \cdot 5$
	2	2	3	2^2	$2 \cdot 3$
	3	1	17	2^3	$2 \cdot 17$
	7	1	257	2^7	$2 \cdot 257$
	15	1	65537	2^{15}	$2 \cdot 65537$
$p = 2,\ b = 2,\ 1 \leqslant a \leqslant 30$	1	1	3	2	$2^2 3$
	3	1	11	2^3	$2^2 11$
	5	1	43	2^5	$2^2 43$
	9	1	683	2^9	$2^2 683$
	11	1	2731	2^{11}	$2^2 2731$
	15	1	43691	2^{15}	$2^2 43691$
	17	1	174763	2^{17}	$2^2 173763$
	21	1	2796203	2^{21}	$2^2 2796203$
$p = 3,\ b = 1,\ 1 \leqslant a \leqslant 19$	1	1	5	3	$3 \cdot 5$
	3	1	41	3^3	$3 \cdot 41$
	15	1	21523361	3^{15}	$3 \cdot 21523361$

Table 2. Unitary Amicable Pairs

(1) $1707720 = 2^3 3 \cdot 5 \cdot 7 \cdot 19 \cdot 107$
$2024760 = 2^3 3 \cdot 5 \cdot 47 \cdot 359$

(2) $3951990 = 2 \cdot 3^4 5 \cdot 7 \cdot 17 \cdot 41$
$4974858 = 2 \cdot 3^4 7 \cdot 41 \cdot 107$

(3) $6940890 = 2 \cdot 3^4 5 \cdot 11 \cdot 19 \cdot 41$
$7937190 = 2 \cdot 3^4 5 \cdot 41 \cdot 239$

(4) $29656530 = 2 \cdot 3^4 5 \cdot 19 \cdot 41 \cdot 47$
$29855790 = 2 \cdot 3^4 5 \cdot 29 \cdot 31 \cdot 41$

(5) $58062480 = 2^4 3 \cdot 5 \cdot 7 \cdot 17 \cdot 19 \cdot 107$
$68841840 = 2^4 3 \cdot 5 \cdot 17 \cdot 47 \cdot 359$

(6) $72696690 = 2 \cdot 3^4 5 \cdot 11 \cdot 41 \cdot 199$
$76084110 = 2 \cdot 3^4 5 \cdot 29 \cdot 41 \cdot 79$

(7) $75139680 = 2^5 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 107$
$89089440 = 2^5 3 \cdot 5 \cdot 11 \cdot 47 \cdot 359$

(8) $491170680 = 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 29 \cdot 47$
$553923720 = 2^3 3^2 5 \cdot 7 \cdot 19 \cdot 23 \cdot 503$

Table 2—continued

(9)	1476394920	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 71 \cdot 241$
	6479522280	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 23 \cdot 10163$
(10)	5530444920	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 103 \cdot 149$
	5791411080	$= 2^3 3^2 5 \cdot 7 \cdot 13 \cdot 17 \cdot 10399$
(11)	6365038680	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 1039$
	7221188520	$= 2^3 3^2 5 \cdot 7 \cdot 13 \cdot 53 \cdot 4159$
*(12)	12924024960	$= 2^7 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 43 \cdot 107$
	15323383680	$= 2^7 3 \cdot 5 \cdot 11 \cdot 43 \cdot 47 \cdot 359$
(13)	16699803120	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 47$
	18833406480	$= 2^4 3^2 5 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 503$
(14)	74555240760	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 10889$
	83515287240	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 83 \cdot 36299$
(15)	88962742748880	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 43 \cdot 131 \cdot 1289$
	95916546799920	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 17 \cdot 43 \cdot 139 \cdot 17027$
(16)	209173484520	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 13499$
	221927955480	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 29 \cdot 359 \cdot 769$
*(17)	214910193960	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 19 \cdot 53 \cdot 7699$
	216191246040	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 17 \cdot 149 \cdot 3079$
*(18)	408774005640	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 191 \cdot 5939$
	418940759160	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 19 \cdot 307 \cdot 2591$
(19)	2534878185840	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 10899$
	2839519766160	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 17 \cdot 83 \cdot 36299$
(20)	2616551257320	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 131 \cdot 1289$
	2821074905880	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 43 \cdot 139 \cdot 17027$
(21)	6642948829440	$= 2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 43 \cdot 107 \cdot 257$
	7876219211520	$= 2^8 3 \cdot 5 \cdot 7 \cdot 11 \cdot 43 \cdot 47 \cdot 257 \cdot 359$
(22)	7111898473680	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 43 \cdot 13499$
	7545550486320	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 17 \cdot 29 \cdot 359 \cdot 769$
(23)	13898316191760	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 191 \cdot 5939$
	14243985811440	$= 2^4 3^2 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 307 \cdot 2591$
(24)	32583815704440	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 13 \cdot 181 \cdot 499559$
	33402225434760	$= 2^3 3^2 5 \cdot 7 \cdot 13 \cdot 17 \cdot 181 \cdot 229 \cdot 1447$
(25)	106595643389918760	$= 2^3 3^2 5 \cdot 7 \cdot 11 \cdot 19 \cdot 61 \cdot 853 \cdot 3889679$
	106934121830433240	$= 2^3 3^2 5 \cdot 7 \cdot 17 \cdot 19 \cdot 37 \cdot 61 \cdot 853 \cdot 68239$

3. CONJECTURES

A preliminary investigation of generators in which $\pi(f) \geqslant 2$ and $\pi(k) \geqslant 2$ suggests the following.

Conjecture 1

The only generator (f, k) with $\pi(f) = \pi(k) = 2$ is $(3/2, 12)$.

Conjecture 2

There are no generators (f, k) with $\pi(f) > 2$ or $\pi(k) > 2$.

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