A NEW COMBINATORIAL INTERPRETATION OF THE FIBONACCI NUMBERS CUBED

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Abstract. We consider the tiling of an n-board (an $n \times 1$ rectangular board) with third-squares ($\frac{1}{3} \times 1$ tiles with the shorter sides always aligned horizontally) and $(\frac{1}{3}, \frac{2}{3})$-fence tiles. A (w, g)-fence tile is composed of two $w \times 1$ subtiles separated by a $g \times 1$ gap. We show that the number of ways to tile an n-board using these types of tiles equals F_{n+1}^3 where F_n is the nth Fibonacci number. We use these tilings to devise straightforward combinatorial proofs of identities relating the Fibonacci numbers cubed to one another, to other combinations of Fibonacci numbers, and to the Pell numbers. Some of these identities appear to be new. We also show that for $p = 2, 3, \ldots$, the number of ways to tile an n-board using either $1/p \times 1$ tiles and $(1/p, 1 - 1/p)$-fences or $(1/2p, 1/2 - 1/2p)$- and $(1/2p, 1 - 1/2p)$-fences is F_{n+1}^p.

1. Introduction

In [6, 7] we showed that the number of ways to tile an n-board using half-squares ($\frac{1}{2} \times 1$ tiles with the shorter sides always aligned with the long direction of the board) and $(\frac{1}{2}, \frac{1}{2})$-fences (where a (w, g)-fence tile is composed of two subtiles (referred to as *posts*) of size $w \times 1$ separated by a gap of size $g \times 1$) is F_{n+1}^2 where F_n is the nth Fibonacci number ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) and then used these tilings to formulate combinatorial proofs of identities relating the Fibonacci numbers squared to themselves and to other number sequences. An obvious question is whether there is an analogous combinatorial interpretation of higher powers of the Fibonacci numbers and whether this can be used to give further identities. In Theorem 3 of [8] we showed that the number of tilings of a pn-board using squares and $(1, p - 1)$-fences is F_{n+1}^p where $p \in \{2, 3, \ldots\}$. Evidently, the same is true if we lengthwise reduce both tiles and board by a factor of p. For convenience, we restate and prove the theorem for the case $p = 3$ in terms of the tiling using third-square tiles ($\frac{1}{3} \times 1$ tiles placed with the shorter sides horizontal and denoted by t) and $(\frac{1}{3}, \frac{2}{3})$-fence tiles (or fences, f). Before doing so, we introduce the concept of a slot which is useful when dealing with tilings in which posts and non-fence tiles have the same width $1/p$ where $p \in \{2, 3, \ldots\}$. A *slot* is part of a cell that can be filled by a post or a non-fence tile and hence there are p slots per cell.

Lemma 1.1. *There is a bijection between the tilings of an n-board using third-squares and $(\frac{1}{3}, \frac{2}{3})$-fences and the tilings of an ordered triple of n-boards using squares and dominoes.*

Proof. If a t (left post of an f) occupies the left slot of cell k, place a square (domino) starting on cell k of the first of the n-boards. Similarly, for each t (left post of an f) occupying the middle slot of a cell, place a square (domino) on the second n-board to be tiled with squares and dominoes. For each t (f) occupying the right slot of a cell, place a square (domino) on the third n-board. Note that each fence occupies two consecutive left, middle, or right slots and corresponds to one domino. The process is clearly reversible and so the mapping is a bijection. □

Theorem 1.2. *Let A_n be the number of ways to tile an n-board using third-squares and $(\frac{1}{3}, \frac{2}{3})$-fences. Then $A_n = F_{n+1}^3$.*

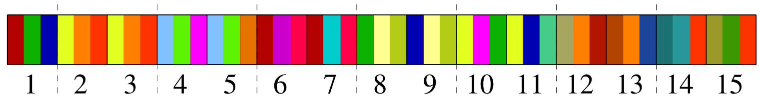

FIGURE 1. A 15-board tiled with all possible metatiles of length less than 3. Dashed lines show boundaries between metatiles.

Proof. There are F_{n+1} ways to tile an n-board using squares and dominoes [2]. From Lemma 1.1, A_n is the same as the number of ways to tile an ordered triple of n-boards using squares and dominoes which is F_{n+1}^3. □

Generally, the most helpful way to describe the tilings of n-boards using tiles with gaps or non-integer length tiles is in terms of metatiles. A metatile is a grouping of tiles that exactly covers an integral number of cells and cannot be split to make smaller metatiles [4]. The nature of the identities that can be obtained via combinatorial proof using tilings is determined by the metatiles. In the case of tiling with half-squares and $(\frac{1}{2}, \frac{1}{2})$-fences, there are infinitely many metatiles but they are easily described and there are two of each length for lengths larger than 2 [6]. When tiling with third-squares and $(\frac{1}{3}, \frac{2}{3})$-fences, the metatiles do not follow a simple pattern. However, as we show in the next section, the number of metatiles of each length is simply related to the Pell numbers. This allows us to obtain a number of identities (Section 3). We also present another combinatorial interpretation of F_n^p for $p \in \{2, 3, \dots\}$ in Section 4.

2. METATILES

When tiling with t and f, the simplest metatiles are three third-squares (denoted by t^3) which has length 1, and the *trifence* (f^3) which is three interlocking fences and is of length 2. The remaining metatiles of length 2 are shown in Fig. 1.

A *mixed metatile* is a metatile that contains both t and f. Thus all metatiles are mixed except t^3 and f^3. Let μ_l be the number of mixed metatiles of length l, and $\mu_l^{[\sigma]}$ be the number of mixed metatiles of length l that have slot content σ in the final cell where σ is a length-3 binary string with 1 (0) representing a post (third-square). Note that there are just 6 possible strings for which $\mu_{l\geq 2}^{[\sigma]}$ is non-zero since 000 and 111 would correspond to a t^3 and the end of a trifence, respectively. From the metatiles in cells 4–15 in Fig. 1 taken in order we see that

$$\mu_2^{[110]} = \mu_2^{[101]} = \mu_2^{[011]} = \mu_2^{[100]} = \mu_2^{[010]} = \mu_2^{[001]} = 1. \tag{2.1}$$

Lemma 2.1.

$$\mu_l^{[\sigma]} = \begin{cases} 2\mu_{l-1}^{[\sigma]} + \mu_{l-2}^{[\sigma]} + \delta_{l,2} + \delta_{l,3}, & \sigma \in \{100, 010, 001\}, \\ 2\mu_{l-1}^{[\sigma]} + \mu_{l-2}^{[\sigma]} + \delta_{l,2} - \delta_{l,3}, & \sigma \in \{110, 101, 011\}, \end{cases} \tag{2.2}$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise and $\mu_{l<2}^{[\sigma]} = 0$.

Proof. Given a metatile of length $l-1$ with some t in the final cell, we can create a metatile of length l by replacing one (or more) of the t by, in each instance, the left post of a fence. The corresponding right post will then lie in the lth cell and the metatile is completed by filling any empty slots in that cell with third-squares. It is then easily seen that for $l > 2$,

$$\mu_l^{[100]} = \mu_{l-1}^{[010]} + \mu_{l-1}^{[001]} + \mu_{l-1}^{[011]}, \quad \mu_l^{[010]} = \mu_{l-1}^{[100]} + \mu_{l-1}^{[001]} + \mu_{l-1}^{[101]},$$
$$\mu_l^{[001]} = \mu_{l-1}^{[100]} + \mu_{l-1}^{[010]} + \mu_{l-1}^{[110]}, \quad \mu_l^{[110]} = \mu_{l-1}^{[001]}, \quad \mu_l^{[101]} = \mu_{l-1}^{[010]}, \quad \mu_l^{[011]} = \mu_{l-1}^{[100]}.$$

From these equations, their symmetry, and (2.1), we have $\mu_l^{[100]} = \mu_l^{[010]} = \mu_l^{[001]}$, $\mu_l^{[110]} = \mu_l^{[101]} = \mu_l^{[011]}$, and thus for $l > 3$,

$$\mu_l^{[100]} = 2\mu_{l-1}^{[100]} + \mu_{l-2}^{[100]}.$$

This gives us $\mu_l^{[110]} = 2\mu_{l-1}^{[110]} + \mu_{l-2}^{[110]}$ for $l > 3$ as well. Using symmetry, (2.1), $\mu_3^{[\sigma]} = 3$ for $\sigma \in \{100, 010, 001\}$, $\mu_3^{[\sigma]} = 1$ for $\sigma \in \{110, 101, 011\}$, and the fact that there are no mixed metatiles of length less than 2 leads to the result (2.2). □

Lemma 2.2.

$$\mu_l = 2\mu_{l-1} + \mu_{l-2} + 6\delta_{l,2}, \quad \mu_{l<2} = 0. \tag{2.3}$$

Proof. Sum (2.2) over the 6 possible σ. □

The Pell numbers, $P_{n\geq 0} = 0, 1, 2, 5, 12, 29, \ldots$, obey

$$P_n = 2P_{n-1} + P_{n-2} + \delta_{n,1} \tag{2.4}$$

with $P_{n<1} = 0$. Thus on comparing (2.4) with (2.3) we obtain

$$\mu_l = 6P_{l-1}, \quad l > 0. \tag{2.5}$$

From (2.4),

$$P_n \pm P_{n-1} = 2(P_{n-1} \pm P_{n-2}) + (P_{n-2} \pm P_{n-3}) + \delta_{n,1} \pm \delta_{n,2}.$$

Then from (2.2),

$$\mu_l^{[\sigma]} = \begin{cases} P_{l-1} + P_{l-2}, & \sigma \in \{100, 010, 001\}, \\ P_{l-1} - P_{l-2}, & \sigma \in \{110, 101, 011\}. \end{cases} \tag{2.6}$$

3. Identities

Lemma 3.1. *For all non-negative integers n,*

$$A_n = \delta_{n,0} + A_{n-1} + 7A_{n-2} + \sum_{l=3}^{n} \mu_l A_{n-l}, \tag{3.1}$$

where $A_n = 0$ for $n < 0$.

Proof. Following [1, 5, 6], we condition on the last metatile. If the last metatile is of length l there will be A_{n-l} ways to tile the remaining $n - l$ cells. There is one metatile of length 1 (t^3), seven of length 2, and μ_l metatiles of length l for each $l \geq 3$. If $n = l$ there is exactly one tiling corresponding to that final metatile so we make $A_0 = 1$. There is no way to tile an n-board if $n < l$ and so $A_{n<0} = 0$. □

Identity 3.2. *For all non-negative integers n,*

$$F_n^3 = \delta_{n,1} + F_{n-1}^3 + F_{n-2}^3 + 6\sum_{l=2}^{n-1} P_{l-1} F_{n-l}^3. \tag{3.2}$$

Proof. It follows from Lemma 3.1, (2.5), P_1 equalling 1, and Theorem 1.2. □

Identity 3.3. *For $n \geq 0$,*

$$F_n^3 = \delta_{n,1} - 2\delta_{n,2} - \delta_{n,3} + 3F_{n-1}^3 + 6F_{n-2}^3 - 3F_{n-3}^3 - F_{n-4}^3. \tag{3.3}$$

Proof. Representing (3.1) by $E(n)$, in the equation $E(n) - 2E(n-1) - E(n-2)$ we re-index two of the sums and rearrange to give

$$A_n = \delta_{n,0} - 2\delta_{n,1} - \delta_{n,2} + 3A_{n-1} + 6A_{n-2} - 3A_{n-3} - A_{n-4} + \sum_{l=5}^{n}(\mu_l - 2\mu_{l-1} - \mu_{l-2})A_{n-l}.$$

The sum vanishes by virtue of (2.3) and after changing n to $n-1$, (3.3) follows from (2.5) and Theorem 1.2. □

Identity 3.4. *For $n \geq 0$ and $j = 0, 1$,*

$$F^3_{2n+j+1} = 1 + \sum_{k=1}^{n}\left\{F^3_{2k+j} + 6\sum_{i=1}^{2k} P_{2k+j-i}F_i^3\right\}. \tag{3.4}$$

Proof. How many ways are there to tile a $(2n+j)$-board using at least one t? *Answer 1*: $A_{2n+j} - \delta_{j,0}$ since only the all-trifence tiling has no t and this only occurs for even-length boards. *Answer 2*: the final t must lie on an even (odd) cell if j is 0 (1) since the cells after this, if any, must be filled with trifences (which are each two cells long). Condition on the location of the final t. Suppose it is in cell $2k+j$ ($k = \delta_{j,0}, \ldots, n$). Either it is part of t^3 and so there are A_{2k+j-1} ways to tile the remaining cells, or it is part of a mixed metatile and so there are $\mu_2 A_{2k+j-2} + \mu_3 A_{2k+j-3} + \cdots + \mu_{2k+j}A_0$ ways to tile the remaining cells. In the latter case, evidently, k cannot be zero. Hence, equating the answers,

$$A_{2n+j} - \delta_{j,0} = \sum_{k=\delta_{j,0}}^{n} A_{2k+j-1} + \sum_{k=1}^{n}(\mu_{2k+j}A_0 + \mu_{2k+j-1}A_1 + \cdots + \mu_2 A_{2k-2+j}).$$

Then, after simplifying, (3.4) follows from (2.5) and Theorem 1.2. □

Identity 3.5. *For $n \geq 0$,*

$$F^3_{n+3} - 1 = \sum_{k=0}^{n}\left\{F^3_{k+1} + 6\sum_{i=0}^{k} P_{k+1-i}F^3_{i+1}\right\}. \tag{3.5}$$

Proof. How many ways are there to tile an $(n+2)$-board using at least 1 fence? *Answer 1*: $A_{n+2} - 1$ since this corresponds to all tilings except the all-t tiling. *Answer 2*: condition on the location of the last fence. Suppose this fence lies on cells $k+1$ and $k+2$ ($k = 0, \ldots, n$). Either there is a trifence covering these cells and so there are A_k ways to tile the remaining cells, or the cells are at the end of a mixed metatile and so there are $\mu_2 A_{k+2-2} + \mu_3 A_{k+2-3} + \cdots + \mu_{k+2}A_0$ ways to tile the remaining cells. Hence, equating the two answers,

$$A_{n+2} - 1 = \sum_{k=0}^{n}\left\{A_k + \mu_{k+2}A_0 + \mu_{k+1}A_1 + \cdots + \mu_3 A_{k-1} + \mu_2 A_k\right\}.$$

The identity then follows from (2.5) and Theorem 1.2. □

In the following identity we use the fact that the number of ways to tile an n-board using only t^3 and f^3 is F_{n+1} since this is equivalent to tiling an n-board with squares and dominoes [3, 2].

Identity 3.6. *For $n \geq 0$,*

$$F^3_{n+1} = F_{n+1} + 6\sum_{k=0}^{n-2}\sum_{r=2}^{n-k} P_{r-1}F_{k+1}F^3_{n+1-k-r}. \tag{3.6}$$

Proof. How many ways are there to tile an n-board using at least 1 mixed metatile? *Answer 1*: $A_n - F_{n+1}$ since F_{n+1} is the number of ways to tile an n-board without using mixed metatiles. *Answer 2*: condition on the position of the first mixed metatile. If it lies on cells $k+1$ to $k+r$ where $k = 0, \ldots, n-r$ and $r = 2, \ldots, n-k$, there are $F_{k+1}\mu_r A_{n-k-r}$ ways to tile the board. Summing over all possible k and r and equating to Answer 1 gives

$$A_n - F_{n+1} = \sum_{\substack{k\geq 0,\, r\geq 2,\\ k+r\leq n}} F_{k+1}\mu_r A_{n-k-r}.$$

After re-expressing the right-hand side as a double sum, the identity follows from (2.5) and Theorem 1.2. □

Before proving the remaining identities we need the following lemma.

Lemma 3.7. *There are $F_n^q F_{n-1}^{3-q}$ ways to tile an n-board if the final cell contains q third-squares where $0 \leq q \leq 3$.*

Proof. We use the bijection described in the proof of Lemma 1.1. For each final cell slot containing a third-square (post), there corresponds a square-domino tiled n-board that ends in a square (domino) for which there remain F_n (F_{n-1}) possible tilings. □

Identity 3.8. *For $n > 0$,*

$$F_n^2 F_{n-1} = \sum_{k=1}^{n-1} (P_k + P_{k-1}) F_{n-k}^3.$$

Proof. How many ways are there to tile an n-board that ends with the right post of a fence which is immediately preceded by two third-squares? *Answer 1*: as the final cell contains 2 third-squares, by Lemma 3.7, there are $F_n^2 F_{n-1}$ ways. *Answer 2*: the number of possible final metatiles of length l is $\mu_l^{[001]}$. Hence if the final metatile has length l, there are $\mu_l^{[001]} A_{n-l}$ ways to tile the board. Summing over all possible $l = 2, \ldots, n$ and equating to Answer 1 gives

$$F_n^2 F_{n-1} = \sum_{l=2}^{n} \mu_l^{[001]} A_{n-l}.$$

Replacing l by $k+1$ and then using (2.6) and Theorem 1.2 gives the identity. □

Identity 3.9. *For $n > 0$,*

$$F_n F_{n-1}^2 = \sum_{k=1}^{n-1} (P_k - P_{k-1}) F_{n-k}^3.$$

Proof. How many ways are there to tile an n-board that ends with the right post of a fence which is immediately preceded by a third-square which is itself preceded by another right post? *Answer 1*: as the final cell contains 1 third-square, by Lemma 3.7, there are $F_n F_{n-1}^2$ ways. *Answer 2*: the number of possible final metatiles of length l is $\mu_l^{[101]}$ and so the number of ways to tile the board is $\sum_{l=2}^{n} \mu_l^{[101]} A_{n-l}$. Replacing l by $k+1$, equating the answers, and then using (2.6) and Theorem 1.2 gives the identity. □

Our final more aesthetically pleasing identity can be obtained by summing the previous two although a combinatorial proof can also be obtained by asking how many ways there are to tile an n-board that ends with the right post of a fence which is immediately preceded by a third-square.

Identity 3.10. *For $n > 0$,*

$$F_{n+1}F_nF_{n-1} = 2\sum_{k=1}^{n-1} P_k F_{n-k}^3.$$

4. Tiling with $(1/2p, 1/2-1/2p)$- and $(1/2p, 1-1/2p)$-fences

In [7] we showed that enumerating the tilings of an n-board using $(\frac{1}{4}, \frac{1}{4})$- and $(\frac{1}{4}, \frac{3}{4})$-fences also gives the Fibonacci numbers squared. Here we generalize the result.

Theorem 4.1. *For $p = 2, 3, \ldots$, the number of ways to tile an n-board using $(1/2p, 1/2-1/2p)$- and $(1/2p, 1-1/2p)$-fences is F_{n+1}^p.*

Proof. We construct a bijection between the tiling of an n-board using $(1/2p, 1/2-1/2p)$-fences (denoted by φ) and $(1/2p, 1-1/2p)$-fences (F) and the tiling of an n-board with $1/p \times 1$ tiles (r, always aligned so that the shorter sides are horizontal) and $(1/p, 1-1/p)$-fences (f). When tiling with φ and F, in which case there are $2p$ slots per cell, whatever fills the left half of a cell determines what will appear in the other half. Clearly, if a left post of a φ occurs in the left half of a cell, the right post will occur p slots later in the right half. If a left post of an F occurs in the left half of a cell, the pth slot after this must be filled by the left post of another F, since if the left post of a φ were placed there, its right post would coincide with the right post of the first F. As a result, F's always occur in pairs, distance $\frac{1}{2}$ apart. If a right post of an F appears in the left half of a cell, it must be the first of a pair of fences and hence there will be another right post of an F p slots later. The bijection is then as follows. If an r appears in slot s (where $s = 1, \ldots, p$) of a cell, place (the left post of) a φ in slot s in the same cell of the φ-F board. If the left post of an f is in slot s, place (the left post of) an F in slot s and another F in slot $s+p$ of the same cell of the φ-F board. Then a right post of an f in slot s of a cell corresponds to two F right posts in slots s and $s+p$ in the same cell of the φ-F board (see Fig. 2 for an example). If we expand an r-f tiling lengthwise by a factor of p, we obtain a tiling of a pn-board using squares and $(1, p-1)$-fences. By Theorem 3 of [8] there are F_{n+1}^p such tilings. □

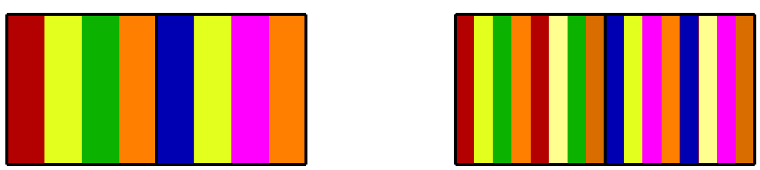

Figure 2. An example of the bijection between the tiling of a 2-board with r and f (left, in this instance, 4 r and 2 f) and the tiling of a 2-board with φ and F (right, in this instance, 4 φ and 4 F) when $p = 4$.

5. Discussion

The approach used here can be generalized to give identities involving higher integer powers of the Fibonacci numbers. The number of tilings of an n-board with $1/p \times 1$ rectangles (placed with the shorter sides horizontal) and $(1/p, 1-1/p)$-fences is F_{n+1}^p. As a cell contains p slots, there will be $p-1$ sets of equations giving the number of metatiles with q posts in the final cell for $q = 1, \ldots, p-1$. Combining these will give a $(p-1)$th order recursion relation for the number of metatiles.

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