THE SELF-COUNTING IDENTITY

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Abstract. This paper presents a curious and interesting analytic identity involving the simple self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$. The shape of this identity is similar to a Lambert series and gives therefore rise to a combinatorial interpretation, which we deduce in the second half of the paper.

1. Introduction

The self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ gives rise to combinatorial questions, for example: let $g(n)$ be the number of ways an integer n can be represented as a sum of elements (repetitions allowed and ignoring the ordering of the summands) from the self-counting sequence. The answer is given by the following identity for the plane partitions $pl(n)$ [1, 2] of n:

$$\sum_{n=0}^{\infty} g(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^{a_k}} = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^k} = \sum_{n=0}^{\infty} pl(n)x^n.$$

Therefore, equating coefficients on both sides, the answer is given by $g(n) = pl(n)$.

In this paper, we will present an analytic identity in the form of a generalized Lambert series [6], which we call the *Self-Counting Identity*. This identity gives rise to another combinatorial partition problem involving the self-counting sequence. For this partition problem, we have to define the three functions $s_1(n)$, $s_2(n)$, and $s_h(n)$ for $h \in \mathbb{N}$, which count special partitions of n into parts involving the self-counting sequence. We will derive explicit representations for these functions using the self-counting sequence itself. At the end, we will generalize the above analytic identity to other functions such as the exponential and logarithmic function.

2. Definitions

Definition 2.1. (*The Self-Counting Sequence*) [7, 3].
The sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ (A002024) consisting of 1 copy of 1, 2 copies of 2, 3 copies of 3, ..., n copies of n, and so on is called the self-counting sequence.

There exist simple formulas for the kth term a_k [7, 3], namely

$$\begin{aligned} a_k &= \left\lfloor \frac{1}{2} + \sqrt{2k} \right\rfloor \\ &= \left\lceil \frac{1}{2}\left(\sqrt{8k+1} - 1\right) \right\rceil. \end{aligned}$$

Using the well-known infinite series identity

$$\frac{x}{1-x}\sum_{k=0}^{\infty} c_k x^k = \left(\sum_{n=1}^{\infty} x^n\right)\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} c_k\right)x^n,$$

we can define the self-counting sequence also by the generating function identity

$$\sum_{k=1}^{\infty} a_k x^k = \frac{x}{1-x} \sum_{k=0}^{\infty} x^{\frac{k(k+1)}{2}} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1,$$

because the self-counting sequence element a_n counts the number of triangular numbers $0, 1, 3, 6, 10, 15, \ldots, T_k := \frac{k(k+1)}{2}, \ldots$ (A000217) [4] that are less than or equal to $n-1$.

Definition 2.2. (*The Self-counting representation functions $s_1(n)$, $s_2(n)$, and $s_h(n)$*). *Furthermore, for every $h \in \mathbb{N}$, we define the self-counting representation functions $s_h(n)$ by*

$$\begin{aligned} s_h(n) :=& \Big| \big\{ (k_1, k_2, \ldots, k_h, m_1, m_2, \ldots, m_h) \in \mathbb{N}^h \times \mathbb{N}_0^h : \\ & n = k_1 + k_2 + \ldots + k_h + m_1 a_{k_1} + m_2 a_{k_2} + \cdots + m_h a_{k_h} \\ & \textit{where } k_1, k_2, \ldots, k_h \in \mathbb{N}, m_1, m_2, \ldots, m_h \in \mathbb{N}_0 \big\} \Big|. \end{aligned}$$

It is important to note that the order of the summands in the definition of $s_h(n)$ is considered to be significant. The special cases $h = 1$ and $h = 2$ give us $s_1(n)$ and $s_2(n)$ defined by

$$s_1(n) := |\{(k, m) \in \mathbb{N} \times \mathbb{N}_0 : n = k + m a_k \textit{ where } k \in \mathbb{N} \textit{ and } m \in \mathbb{N}_0\}|$$

and

$$s_2(n) := \left| \left\{ (k, l, m, r) \in \mathbb{N}^2 \times \mathbb{N}_0^2 : n = k + l + m a_k + r a_l \textit{ where } k, l \in \mathbb{N} \textit{ and } m, r \in \mathbb{N}_0 \right\} \right|.$$

Examples for $s_1(n)$:
For $n = 4$, we have that $s_1(4) = 3$, because
$4 = 1 + 3a_1 = 2 + 1a_2 = 4 + 0a_4$, for $n = 10$, we have that $s_1(10) = 4$, because
$10 = 1 + 9a_1 = 2 + 4a_2 = 4 + 2a_4 = 10 + 0a_{10}$ and for $n = 17$,
we have that $s_1(17) = 6$, because
$17 = 1 + 16a_1 = 3 + 7a_3 = 5 + 4a_5 = 9 + 2a_9 = 12 + 1a_{12} = 17 + 0a_{17}$.

Examples for $s_2(n)$:
For $n = 2$, we have that $s_2(2) = 1$, because
$2 = 1 + 1 + 0a_1 + 0a_1$, for $n = 3$, we have that $s_2(3) = 4$, because
$3 = 1+1+1a_1+0a_1 = 1+1+0a_1+1a_1 = 1+2+0a_1+0a_2 = 2+1+0a_2+0a_1$ and for $n = 4$, we have that $s_2(4) = 8$, because
$4 = 1 + 1 + 1a_1 + 1a_1 = 1 + 1 + 2a_1 + 0a_1 = 1 + 1 + 0a_1 + 2a_1 = 1 + 2 + 1a_1 + 0a_2$
$= 2 + 1 + 0a_2 + 1a_1 = 2 + 2 + 0a_2 + 0a_2 = 1 + 3 + 0a_1 + 0a_3 = 3 + 1 + 0a_3 + 0a_1$.

3. The Self-Counting Identity

In this section, we prove the Self-Counting Identity, which is interesting because the self-counting sequence appears on both sides of the equal sign. From this identity, we will be able to deduce explicit formulas for $s_h(n)$ for all $h \in \mathbb{N}$.

Theorem 3.1. (*The Self-Counting Identity*).
Let $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ be the self-counting sequence. Then we have that

$$\sum_{k=1}^{\infty} \frac{x^k}{1 - x^{a_k}} = \sum_{k=1}^{\infty} a_k x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1.$$

Proof. This identity follows by the following manipulations

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^k}{1-x^{a_k}} &= \frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^2}+\frac{x^4}{1-x^3}+\frac{x^5}{1-x^3}+\frac{x^6}{1-x^3}+\cdots\\ &= \frac{x}{1-x}(1)+\frac{x^2}{1-x^2}(1+x)+\frac{x^4}{1-x^3}(1+x+x^2)+\cdots\\ &= x\left[\frac{1}{1-x}(1)+\frac{x}{1-x^2}(1+x)+\frac{x^3}{1-x^3}(1+x+x^2)+\cdots\right]\\ &= x\sum_{k=0}^{\infty}\frac{x^{\frac{k^2+k}{2}}}{1-x^{k+1}}\sum_{m=0}^{k}x^m\\ &= x\sum_{k=0}^{\infty}\frac{x^{\frac{k^2+k}{2}}}{1-x^{k+1}}\frac{1-x^{k+1}}{1-x}\\ &= \frac{x}{1-x}\sum_{k=0}^{\infty}x^{\frac{k^2+k}{2}}\\ &= \sum_{k=1}^{\infty}a_kx^k \quad \text{for all } x\in\mathbb{C} \text{ with } |x|<1,\end{aligned}$$

where we have used the infinite series identity

$$\sum_{k=1}^{\infty}a_kx^k = \frac{x}{1-x}\sum_{k=0}^{\infty}x^{\frac{k^2+k}{2}} \quad \text{for all } x\in\mathbb{C} \text{ with } |x|<1$$

from above. □

Remark. It is not difficult to show that the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1,2,2,3,3,3,4,4,4,4,\ldots\}$ is the unique integer sequence which satisfies the analytic identity

$$\sum_{k=1}^{\infty}\frac{x^k}{1-x^{a_k}} = \sum_{k=1}^{\infty}a_kx^k \quad \text{for all } x\in\mathbb{C} \text{ with } |x|<1.$$

Therefore, it is possible to define the self-counting sequence using only the above formula.

The proof of the self-counting identity implies the following corollary.

Corollary 3.2. (*Generating function for the triangular numbers*). *We have the following formula*

$$\sum_{k=0}^{\infty}x^{\frac{k^2+k}{2}} = \sum_{k=0}^{\infty}\frac{x^k}{1+x+x^2+x^3+\cdots+x^{a_{k+1}-1}} \quad \textit{for all } x\in\mathbb{C} \textit{ with } |x|<1.$$

4. Another Self-Counting Partition Problem

Let us now study the self-counting representation functions $s_h(n)$ for $h\in\mathbb{N}$.

In the next Corollary, we will see that the function $s_2(n)$ is the convolution of the function $s_1(n)$ with itself.

Corollary 4.1. (*Explicit formulas for $s_1(n)$ [7, 3] and $s_2(n)$ [5]*).
We have that

$$s_1(n) = a_n \quad \textit{for all } n \in \mathbb{N},$$
$$s_2(n) = \sum_{k=1}^{n-1} a_k a_{n-k} \quad \textit{for all } n \in \mathbb{N},$$

where $\{a_k\}_{k=1}^{\infty}$ is the self-counting sequence.

Proof. By the Self-Counting Identity, we have that

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{k=1}^{\infty} \frac{x^k}{1-x^{a_k}} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.$$

Expanding the function

$$\frac{1}{1-x^{a_k}} = \sum_{m=0}^{\infty} x^{ma_k} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1$$

in a geometric series, we get that

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{k=1}^{\infty} \frac{x^k}{1-x^{a_k}} = \sum_{k=1}^{\infty} x^k \sum_{m=0}^{\infty} x^{ma_k} = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} x^{k+ma_k} = \sum_{n=1}^{\infty} s_1(n) x^n.$$

Equating coefficients, we see that $s_1(n) = a_n$ must hold.

For the second formula, we square both sides of the Self-Counting Identity to get

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)^2 = \left(\sum_{k=1}^{\infty} \frac{x^k}{1-x^{a_k}}\right)^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^{k+l}}{(1-x^{a_k})(1-x^{a_l})} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.$$

Expanding the functions

$$\frac{1}{1-x^{a_k}} = \sum_{m=0}^{\infty} x^{ma_k} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1$$

and

$$\frac{1}{1-x^{a_l}} = \sum_{r=0}^{\infty} x^{ra_l} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1$$

again in two geometric series, we get that

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)^2 = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{x^{k+l}}{(1-x^{a_k})(1-x^{a_l})} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} x^{k+l+ma_k+ra_l} = \sum_{n=1}^{\infty} s_2(n) x^n.$$

If we also use the fact that

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)^2 = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} a_k a_{n-k}\right) x^n,$$

we obtain in total that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} a_k a_{n-k}\right) x^n = \sum_{n=1}^{\infty} s_2(n) x^n \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.$$

Equating coefficients, we see that

$$s_2(n) = \sum_{k=1}^{n-1} a_k a_{n-k}.$$

□

Similarly, we can prove the following.

Corollary 4.2. (*Recursion formula for $s_h(n)$*).
For $s_h(n)$ we have the following recursion formula

$$s_1(n) = a_n \quad \textit{for all } n \in \mathbb{N},$$

$$s_{h+1}(n) = \sum_{k=1}^{n-1} a_k s_h(n-k) \quad \textit{for all } n \in \mathbb{N} \textit{ if } h \geq 1,$$

where $\{a_k\}_{k=1}^{\infty}$ is the self-counting sequence.

5. Generalizations to Other Functions

Using the identity

$$\frac{1-x^{k+1}}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k = \sum_{m=0}^{k} x^m,$$

for finite geometric summation, we can also prove (just like in the proof of the Self-Counting Identity) the following

Theorem 5.1. (*Series identities for some analytic functions*).
We have that

$$\frac{1}{1-x}\sum_{k=0}^{\infty} x^{k^2} = \sum_{k=0}^{\infty} \frac{1}{1-x^{2k+1}} \sum_{n=0}^{2k} x^{k^2+n} \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1,$$

$$\frac{1}{1-x}\sum_{k=1}^{\infty} x^{\frac{k^2+k}{2}} = \sum_{k=1}^{\infty} \frac{1}{1-x^{k}} \sum_{n=0}^{k-1} x^{\frac{k^2+k}{2}+n} \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1,$$

and that

$$\frac{1}{(1+x)(1-x)^3} = \sum_{k=0}^{\infty} \frac{x^k}{1-x^{k+1}} \left(\sum_{n=0}^{k} x^n\right)^2 \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1.$$

For the exponential function, we have for all $x \in \mathbb{C}$ where x is not a root of unity

$$\frac{e^x}{1-x} = \sum_{k=0}^{\infty} \left(\sum_{n=\lfloor \frac{k+3}{2} \rfloor}^{k+1} \frac{1}{(n-1)!\,(1-x^n)} \right) x^k,$$

which is after multiplying both sides with $(1-x)$ equivalent to

$$e^x = \sum_{k=0}^{\infty} \left(\sum_{n=\lfloor \frac{k+3}{2} \rfloor}^{k+1} \frac{1}{(n-1)!\,(x^{n-1}+x^{n-2}+\cdots+x^2+x+1)} \right) x^k$$

and for the logarithmic function

$$\frac{\log(1-x)}{1-x} = \sum_{k=1}^{\infty} \left(\sum_{n=\lfloor \frac{k+2}{2} \rfloor}^{k} \frac{1}{n\left(x^n - 1\right)} \right) x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1,$$

which is after multiplying both sides with $(1-x)$ *equivalent to*

$$\log(1-x) = -\sum_{k=1}^{\infty} \left(\sum_{n=\lfloor \frac{k+2}{2} \rfloor}^{k} \frac{1}{n\left(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1\right)} \right) x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1.$$

Moreover, we also have that

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\lfloor \frac{k+2}{2} \rfloor} \frac{1}{1 - x^{n + \lfloor \frac{k+1}{2} \rfloor}} \right) x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1,$$

which is, after multiplying both sides with $(1-x)$*, equivalent to*

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\lfloor \frac{k+2}{2} \rfloor} \frac{1}{x^{n+\lfloor \frac{k+1}{2} \rfloor - 1} + x^{n+\lfloor \frac{k+1}{2} \rfloor - 2} + \cdots + x^2 + x + 1} \right) x^k \quad \textit{for all } x \in \mathbb{C} \textit{ with } |x| < 1.$$

Proof. All these formulas are obtained similarly using the formula

$$\frac{1 - x^{k+1}}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^k = \sum_{m=0}^{k} x^m,$$

for finite geometric summation. The first formula is proved by the following calculation

$$\begin{aligned}
\frac{1}{1-x} \sum_{k=0}^{\infty} x^{k^2} &= \frac{1}{1-x} \left[1 + x + x^4 + x^9 + \cdots + x^{k^2} + \cdots \right] \\
&= \frac{1}{1-x} + \frac{x}{1-x} + \frac{x^4}{1-x} + \frac{x^9}{1-x} + \cdots + \frac{x^{k^2}}{1-x} + \cdots \\
&= \frac{1}{1-x}\left(\frac{1-x}{1-x}\right) + \frac{x}{1-x^3}\left(\frac{1-x^3}{1-x}\right) + \frac{x^4}{1-x^5}\left(\frac{1-x^5}{1-x}\right) + \cdots \\
&\quad + \frac{x^{k^2}}{1-x^{2k+1}}\left(\frac{1-x^{2k+1}}{1-x}\right) + \cdots \\
&= \frac{1}{1-x} + \frac{x}{1-x^3}(1 + x + x^2) + \frac{x^4}{1-x^5}(1 + x + x^2 + x^3 + x^4) + \cdots \\
&= \frac{1}{1-x} + \frac{1}{1-x^3}(x + x^2 + x^3) + \frac{1}{1-x^5}(x^4 + x^5 + x^6 + x^7 + x^8) + \cdots \\
&= \sum_{k=0}^{\infty} \frac{1}{1-x^{2k+1}} \sum_{n=0}^{2k} x^{k^2+n} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.
\end{aligned}$$

The second formula follows by the manipulations

$$\begin{aligned}
\frac{1}{1-x}\sum_{k=1}^{\infty} x^{\frac{k^2+k}{2}} &= \frac{1}{1-x}\left[x + x^3 + x^6 + x^{10} + \cdots + x^{\frac{k^2+k}{2}} + \cdots\right] \\
&= \frac{x}{1-x} + \frac{x^3}{1-x} + \frac{x^6}{1-x} + \frac{x^{10}}{1-x} + \cdots + \frac{x^{\frac{k^2+k}{2}}}{1-x} + \cdots \\
&= \frac{x}{1-x}\left(\frac{1-x}{1-x}\right) + \frac{x^3}{1-x^2}\left(\frac{1-x^2}{1-x}\right) + \frac{x^6}{1-x^3}\left(\frac{1-x^3}{1-x}\right) + \cdots \\
&\quad + \frac{x^{\frac{k^2+k}{2}}}{1-x^k}\left(\frac{1-x^k}{1-x}\right) + \cdots \\
&= \frac{x}{1-x} + \frac{x^3}{1-x^2}(1+x) + \frac{x^6}{1-x^3}\left(1+x+x^2\right) \\
&\quad + \frac{x^{10}}{1-x^4}\left(1+x+x^2+x^3\right) + \cdots \\
&= \frac{x}{1-x} + \frac{1}{1-x^2}\left(x^3+x^4\right) + \frac{1}{1-x^3}\left(x^6+x^7+x^8\right) \\
&\quad + \frac{1}{1-x^4}\left(x^{10}+x^{11}+x^{12}+x^{13}\right) + \cdots \\
&= \sum_{k=1}^{\infty}\frac{1}{1-x^k}\sum_{n=0}^{k-1} x^{\frac{k^2+k}{2}+n} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.
\end{aligned}$$

Using that

$$\left(\sum_{n=0}^{k} x^n\right)^2 = \frac{\left(1-x^{k+1}\right)^2}{(1-x)^2},$$

we can prove the third identity as follows

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{x^k}{1-x^{k+1}}\left(\sum_{n=0}^{k} x^n\right)^2 &= \sum_{k=0}^{\infty}\frac{x^k}{1-x^{k+1}}\cdot\frac{\left(1-x^{k+1}\right)^2}{(1-x)^2} \\
&= \frac{1}{(1-x)^2}\sum_{k=0}^{\infty} x^k\left(1-x^{k+1}\right) \\
&= \frac{1}{(1-x)^2}\left(\sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty} x^{2k+1}\right) \\
&= \frac{1}{(1-x)^2}\left(\frac{1}{1-x} - \frac{x}{1-x^2}\right) \\
&= \frac{1}{(1-x)^2}\cdot\frac{1}{1-x^2} \\
&= \frac{1}{(1+x)(1-x)^3} \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.
\end{aligned}$$

We get the proof of the above identity for the exponential function in the following way

$$\begin{aligned}
\frac{e^x}{1-x} &= \frac{1}{1-x}\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\cdots\right] \\
&= \frac{1}{1-x}+\frac{x}{1!(1-x)}+\frac{x^2}{2!(1-x)}+\frac{x^3}{3!(1-x)}+\frac{x^4}{4!(1-x)}+\frac{x^5}{5!(1-x)}+\frac{x^6}{6!(1-x)}+\cdots \\
&= \frac{1}{1-x}\left(\frac{1-x}{1-x}\right)+\frac{x}{1!(1-x^2)}\left(\frac{1-x^2}{1-x}\right)+\frac{x^2}{2!(1-x^3)}\left(\frac{1-x^3}{1-x}\right) \\
&\quad +\frac{x^3}{3!(1-x^4)}\left(\frac{1-x^4}{1-x}\right)+\cdots \\
&= \frac{1}{1-x}+\frac{x}{1!(1-x^2)}(1+x)+\frac{x^2}{2!(1-x^3)}\left(1+x+x^2\right) \\
&\quad +\frac{x^3}{3!(1-x^4)}\left(1+x+x^2+x^3\right)+\cdots \\
&= \frac{1}{1-x}+\frac{1}{1!(1-x^2)}\left(x+x^2\right)+\frac{1}{2!(1-x^3)}\left(x^2+x^3+x^4\right) \\
&\quad +\frac{1}{3!(1-x^4)}\left(x^3+x^4+x^5+x^6\right)+\cdots \\
&= \sum_{n=1}^{\infty}\frac{1}{(n-1)!(1-x^n)}\sum_{m=n-1}^{2n-2}x^m \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=\left\lfloor\frac{k+3}{2}\right\rfloor}^{k+1}\frac{1}{(n-1)!\,(1-x^n)}\right)x^k \quad \text{for all } x\in\mathbb{C} \text{ where } x \text{ is not a root of unity,}
\end{aligned}$$

where, in the last step, we have used that

$$(k+1)-1 \le k \le 2\cdot\left\lfloor\frac{k+3}{2}\right\rfloor-2 \quad \text{for all } k\in\mathbb{N}_0,$$

but

$$2\cdot\left(\left\lfloor\frac{k+3}{2}\right\rfloor-1\right)-2<k<(k+2)-1 \ \text{ for all } k\in\mathbb{N}_0.$$

The proof of the identity for the logarithmic function follows by these manipulations

$$\begin{aligned}
&\frac{\log(1-x)}{1-x} = -\frac{1}{1-x}\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}+\cdots\right] \\
&= \frac{x}{x-1}+\frac{x^2}{2(x-1)}+\frac{x^3}{3(x-1)}+\frac{x^4}{4(x-1)}+\frac{x^5}{5(x-1)}+\frac{x^6}{6(x-1)}+\cdots \\
&= \frac{x}{x-1}\left(\frac{x-1}{x-1}\right)+\frac{x^2}{2(x^2-1)}\left(\frac{x^2-1}{x-1}\right)+\frac{x^3}{3(x^3-1)}\left(\frac{x^3-1}{x-1}\right)+\frac{x^4}{4(x^4-1)}\left(\frac{x^4-1}{x-1}\right)+\cdots
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x}{x-1} + \frac{x^2}{2(x^2-1)}(1+x) + \frac{x^3}{3(x^3-1)}\left(1+x+x^2\right) + \frac{x^4}{4(x^4-1)}\left(1+x+x^2+x^3\right) + \cdots \\
&= \frac{x}{x-1} + \frac{1}{2(x^2-1)}\left(x^2+x^3\right) + \frac{1}{3(x^3-1)}\left(x^3+x^4+x^5\right) + \frac{1}{4(x^4-1)}\left(x^4+x^5+x^6+x^7\right) + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{n(x^n-1)} \sum_{m=n}^{2n-1} x^m \\
&= \sum_{k=1}^{\infty} \left(\sum_{n=\left\lfloor \frac{k+2}{2} \right\rfloor}^{k} \frac{1}{n\left(x^n-1\right)} \right) x^k \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1.
\end{aligned}
$$

In the last step above, we have used that

$$
k \le k \le 2 \cdot \left\lfloor \frac{k+2}{2} \right\rfloor - 1 \quad \text{for all } k \in \mathbb{N}_0, \text{ but } 2 \cdot \left(\left\lfloor \frac{k+2}{2} \right\rfloor - 1 \right) - 1 < k < k+1 \quad \text{for all } k \in \mathbb{N}_0.
$$

The last identity is obtained by the calculation

$$
\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{1}{1-x}\left[1+x+x^2+x^3+x^4+x^5+x^6+\cdots\right] \\
&= \frac{1}{1-x} + \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} + \frac{x^4}{1-x} + \frac{x^5}{1-x} + \frac{x^6}{1-x} + \cdots \\
&= \frac{1}{1-x}\left(\frac{1-x}{1-x}\right) + \frac{x}{1-x^2}\left(\frac{1-x^2}{1-x}\right) + \frac{x^2}{1-x^3}\left(\frac{1-x^3}{1-x}\right) + \frac{x^3}{1-x^4}\left(\frac{1-x^4}{1-x}\right) + \cdots \\
&= \frac{1}{1-x} + \frac{x}{1-x^2}(1+x) + \frac{x^2}{1-x^3}\left(1+x+x^2\right) + \frac{x^3}{1-x^4}\left(1+x+x^2+x^3\right) + \cdots \\
&= \frac{1}{1-x} + \frac{1}{1-x^2}\left(x+x^2\right) + \frac{1}{1-x^3}\left(x^2+x^3+x^4\right) + \frac{1}{1-x^4}\left(x^3+x^4+x^5+x^6\right) + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{1-x^n} \sum_{m=n-1}^{2n-2} x^m \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=\left\lfloor \frac{k+3}{2} \right\rfloor}^{k+1} \frac{1}{1-x^n} \right) x^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\left\lfloor \frac{k+2}{2} \right\rfloor} \frac{1}{1-x^{n+\left\lfloor \frac{k+1}{2} \right\rfloor}} \right) x^k \quad \text{for all } x \in \mathbb{C} \text{ with } |x| < 1,
\end{aligned}
$$

where we have again used the relations

$$
(k+1) - 1 \le k \le 2 \cdot \left\lfloor \frac{k+3}{2} \right\rfloor - 2 \quad \text{for all } k \in \mathbb{N}_0,
$$

but

$$
2 \cdot \left(\left\lfloor \frac{k+3}{2} \right\rfloor - 1 \right) - 2 < k < (k+2) - 1 \quad \text{for all } k \in \mathbb{N}_0.
$$

□

Using the same method, one can derive a lot of other similar series identities.

6. Conclusion

We have seen and proved the interesting Self-Counting Identity, which describes an unexpected property of the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ and we have used it to deduce explicit formulas for the self-counting representation functions $s_h(n)$ for all $h \in \mathbb{N}$. At the end of the paper, we have obtained related series identities for other functions and integer sequences. Using the techniques and formulas presented in this paper, one can obtain much more interesting series identities.

For example, substituting $x := \frac{1}{e^s}$ into the Self-Counting Identity, we obtain the following formula

$$\sum_{k=1}^{\infty} \frac{a_k}{e^{ks}} = \sum_{k=1}^{\infty} \frac{e^{a_k s}}{e^{ks}\left(e^{a_k s} - 1\right)} \quad \text{for all } s > 0,$$

where $\{a_k\}_{k=1}^{\infty}$ is again the self-counting sequence. This is proved by the following calculation

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{a_k}{e^{ks}} &= \sum_{k=1}^{\infty} a_k x^k \bigg|_{x=\frac{1}{e^s}} \\
&= \sum_{k=1}^{\infty} \frac{x^k}{1 - x^{a_k}} \bigg|_{x=\frac{1}{e^s}} \\
&= \sum_{k=1}^{\infty} \frac{1}{\frac{1}{x^k} - x^{a_k - k}} \bigg|_{x=\frac{1}{e^s}} \\
&= \sum_{k=1}^{\infty} \frac{1}{e^{ks} - e^{(k-a_k)s}} \\
&= \sum_{k=1}^{\infty} \frac{e^{a_k s}}{e^{ks}\left(e^{a_k s} - 1\right)} \quad \text{for all } s > 0.
\end{aligned}$$

Similarly, substituting $x := \frac{1}{e^s}$ into the generating function identity of the self-counting sequence on page 2, we deduce that

$$\sum_{k=1}^{\infty} \frac{a_k}{e^{ks}} = \frac{1}{e^s - 1} \sum_{k=0}^{\infty} \frac{1}{e^{\frac{k^2+k}{2}s}} \quad \text{for all } s > 0.$$

Substituting $x := \frac{1}{e^s}$ into

$$\sum_{k=1}^{\infty} H_k x^k = -\frac{\log(1-x)}{1-x} = \sum_{k=1}^{\infty} \left(\sum_{n=\left\lfloor \frac{k+2}{2} \right\rfloor}^{k} \frac{1}{n\left(x^n - 1\right)} \right) x^k \ \text{ for all } x \in \mathbb{C} \text{ with } |x| < 1,$$

we get that

$$\sum_{k=1}^{\infty} \frac{H_k}{e^{ks}} = \sum_{k=1}^{\infty} \left(\sum_{n=\left\lfloor \frac{k+2}{2} \right\rfloor}^{k} \frac{e^{ns}}{n\left(e^{ns} - 1\right)} \right) \frac{1}{e^{ks}} \quad \text{for all } s > 0,$$

where we have used the identity [8]

$$\sum_{k=1}^{\infty} H_k x^k = -\frac{\log(1-x)}{1-x}$$

for the Harmonic numbers $H_n := \sum_{k=1}^{n} \frac{1}{k}$ [8].

Finally, we have the following formula

$$\frac{e^{e^x}}{e^x - 1} = \sum_{k=0}^{\infty} \left(\sum_{n=\left\lfloor \frac{k+3}{2} \right\rfloor}^{k+1} \frac{1}{(n-1)!\,(e^{nx}-1)} \right) e^{kx} \quad \text{for all } x > 0$$

for the double exponential function.

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