ON TRIBONACCI NUMBERS AND 3-REGULAR COMPOSITIONS

NEVILLE ROBBINS

Abstract. Let the sequence $\{U_n\}$ be defined by

$$U_0 = 0, U_1 = 1, U_2 = 2, U_n = U_{n-1} + U_{n-2} + U_{n-3} \text{ for } n \geq 3.$$

We show that U_n, which we call a *Tribonacci* number, counts the number of 3-regular compositions of n, that is, the number of compositions of n into parts not divisible by 3.

1. Introduction

A *composition* of the natural number n is a representation of n as a sum of one or more positive integers, where representations that differ only in the order of terms are considered distinct. It is well-known that the total number of compositions of n is 2^{n-1}. A fact that we will call *Theorem 0* states that the number of compositions of n with odd parts is F_n, the nth Fibonacci number. (The Fibonacci numbers may be defined for $n \geq 0$ by: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.) Theorem 0 appears to have been first stated by Cayley [1]; it appears as an exercise in the monograph by Stanley [5], and it follows immediately from a result in [4], namely a bijection between compositions of n into odd parts and compositions of $n + 1$ into parts exceeding 1.

We will present a proof of Theorem 0, because an adaptation of the argument used in our proof helps to prove our new result, which we state below. First, if the integer $m \geq 2$, define a composition to be *m-regular* if no part is divisible by m. This definition is in analogy with the definition of m-regular partition, that is, partition in which no part is divisible by m. Such partitions play a role in the representation theory of the symmetric group. (See [3], where such partitions are mentioned, although the adjective "regular" is not used.) Our new result states that the number of 3-regular compositions of n is U_n, the nth Tribonacci number of the second kind. A Tribonacci number is an integer, a_n, that satisfies a ternary linear recurrence:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \tag{1}$$

for $n \geq 3$, with suitably chosen a_0, a_1, a_2. In this paper, we are particularly interested in the Tribonacci sequence $\{T_n\}$, where $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, which we call the Tribonacci sequence of the first kind, and the Tribonacci sequence $\{U_n\}$, where $U_0 = 0$, $U_1 = 1$, $U_2 = 2$, which we call the Tribonacci sequence of the second kind. If n is a natural number, we show that the number of 3-regular compositions of n is U_n. The name "Tribonacci sequence" has become associated with sequences of non-negative integers that satisfy (1) when $n \geq 3$. M. Feinberg [2] gave this name to the sequence that we call T_n. The sequence that we call U_n, he called an "intermediate" sequence, noting that $U_n = T_{n+2} - T_{n+1}$. Some additional properties of these two sequences were found by M. Waddill and L. Sachs [6].

The proof of Theorem 0 makes use of what Sills [4] calls the *MacMahon bit sequence* of a composition. Specifically, a composition of $m + 1$ with l parts corresponds bijectively with an m-bit binary sequence with $l - 1$ 1's. Imagine two extra 1's, one each at the extreme left and right. The size of a part is one more than the number of 0's between consecutive 1's.

2. Preliminaries

$$F_m = F_{m-1} + F_{m-2} \text{ for all } m \geq 2. \tag{2}$$

Theorem 0. *Let m be a natural number. Then the number of compositions of m into odd parts is F_m, the mth Fibonacci number.*

Proof. (Induction on m.) The statement is trivially true for $m = 1$. Assuming that it is true for $m - 1$, we will show that it is true for m. The parts of a composition will all be odd if and only if every string of 0's in the MacMahon bit sequence has even length. Call such a bit sequence *good*. Consider the set of all good bit sequences of length m. We claim that (a) the number of good m-bit sequences ending in 1 is F_{m-1}; (b) the number of good m-bit sequences ending in 0 is F_{m-2}; (c) the total number of such bit sequences is F_m. Note that (c) is the conclusion of our theorem, and that in view of equation (2), (c) follows from (a) and (b). Therefore it suffices to prove (a) and (b).

By the induction hypothesis, the total number of good bit sequences of length $m - 1$ is F_{m-1}. If a 1 is added to the right of a good $m-1$-bit sequence, then the result is a good m-bit sequence ending in 1. Conversely, if a good m-bit sequence ends in 1, removing that 1 will produce a good $m - 1$-bit sequence. Therefore the number of good m-bit sequences ending in 1 is F_{m-1}, so that the number of good $m - 1$-bit sequences ending in 1 is F_{m-2}. On the other hand, a good m-bit sequence ending on 0 arises only by removing the last 1 in a good $m - 1$-bit sequence ending in a 1, and replacing it by two 0's. Therefore the number of good $m - 1$-bit sequences ending in 0 is F_{m-2}. We have proven (a) and (b), so we are done. □

3. Tribonacci Identities

Let the Tribonacci sequences of the first and second kinds, namely $\{T_n\}, \{U_n\}$ be defined by:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},\ T_0 = 0,\ T_1 = 1,\ T_2 = 1, \tag{3}$$

$$U_n = U_{n-1} + U_{n-2} + U_{n-3},\ U_0 = 0,\ U_1 = 1,\ U_2 = 2. \tag{4}$$

Then the following identities hold:

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} U_{n-3k} = T_{n+1} \text{ if } n \equiv 1, 2 \pmod 3 \tag{5}$$

$$1 + \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} U_{n-3k} = T_{n+1} \text{ if } n \equiv 0 \pmod 3 \tag{6}$$

$$U_{n-1} + U_{n-2} = T_n + T_{n-2} \text{ for } n \geq 2 \tag{7}$$

$$T_{n+1} - T_{n-2} = U_n \text{ for } n \geq 2 \tag{8}$$

Remark. These identities are easily proved by induction on n. The table below lists T_n and U_n for $0 \leq n \leq 10$.

n	0	1	2	3	4	5	6	7	8	9	10
T_n	0	1	1	2	4	7	13	24	44	81	149
U_n	0	1	2	3	6	11	20	37	68	125	230

4. The Main Result

Theorem 1. *If n is a natural number, and $f(n)$ denotes the number of 3-regular compositions of n, then $f(n) = U_n$, the nth Tribonacci number of the second kind.*

Proof. (Induction on n.) The statement is true by inspection for $1 \leq n \leq 3$. We will use the MacMahon bit sequence of a composition, as in the proof of Theorem 0. That is, each composition of $n+1$ corresponds to an n-bit sequence. No part of the composition is divisible by 3 if and only if no string of zeroes in the corresponding bit sequence has length $\equiv 2 \pmod 3$. Call such a bit sequence *good*. We will show that the number of good n-bit sequences is U_{n+1}.

By induction hypothesis, the number of good bit sequences length of $n-1$ is U_n. If a 1 is added on the right, we obtain a good sequence of length n, ending in 1. Conversely, if we remove the rightmost 1 from a good n-bit sequence ending in 1, we obtain a good $n-1$-bit sequence. Thus, there is a bijection between good bit sequences of length $n-1$, and good bit sequences of length n ending in 1. Therefore U_n is the number of good bit sequences of length n ending in 1. Next, we consider the number of good bit sequences of length n ending in 0. By hypothesis, the number of terminal 0's has the form $3k+1$ or $3k$. By induction hypothesis, the number of such sequences ending in a 1 followed by $3k+1$ 0's is U_{n-3k-1}. If $n \not\equiv 1 \pmod 3$, then, invoking equation (5), and replacing n by $n-1$, the total number of such sequences is:

$$\sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} U_{n-1-3k} = T_n.$$

If $n \equiv 1 \pmod 3$, that is, $n = 3j+1$, then we get an additional good bit sequence, consisting of n zeroes. Therefore, invoking equation (6), and replacing n by $n-1$, the total number of such sequences is

$$1 + \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} U_{n-1-3k} = T_n.$$

We have shown that the total number of good bit sequences of length n ending in $3k+1$ 0's is T_n. Next, we consider the number of good bit sequences of length n ending in $3k$ 0's.

If $n \not\equiv 0 \pmod 3$, then, invoking equations (6) and (8), the total number of such sequences is:

$$\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} U_{n-3k} = T_{n+1} - U_n = T_{n-2}.$$

If $n \equiv 0 \pmod 3$, then we get an additional good bit-sequence consisting of n 0's. Therefore, invoking equations (6) and (8), the total number of such sequences is:

$$1 + \sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} U_{n-3k} = T_{n+1} - U_n = T_{n-2}.$$

We have shown that the total number of good bit sequences of length n ending in $3k$ 0's is T_{n-2}. Now, invoking equations (4) and (7), we see that the total number of good n-bit sequences is:

$$U_n + T_n + T_{n-2} = U_n + U_{n-1} + U_{n-2} = U_{n+1}.$$

□

5. Concluding Conjecture

The work done above suggests the following generalization: Let the integers m, n satisfy $2 \leq m < n$. Let $c(n,m)$ denote the number of m-regular compositions of n. For fixed m, let the sequence $\{a(n,m)\}$ be defined by:

$$a(n,m) = \sum_{i=1}^{m} a(n-i,m) \text{ for } n \geq m.$$

with initial conditions

$$a(0,m) = 0, a(j,m) = 2^{j-1} \text{ for } 1 \leq j \leq m-2, a(m-1,m) = 2^{m-2} - 1.$$

Then $c(n,m) = a(n,m)$.

References

[1] A. Cayley, *Theorems in trigonometry and on partitions*, Collected Mathematical Papers, Vol. 10, p. 16, Cambridge University Press, 2009.
[2] M. Feinberg, *Fibonacci - Tribonacci*, The Fibonacci Quarterly, **1.1** (1963), 71–74.
[3] K. Ono and B. Wilson, *Parity of Fourier coefficients of modular forms*, Illinois J. Math., **41** (1997), 142–150.
[4] A. Sills, *Compositions, partitions, and Fibonacci numbers*, The Fibonacci Quarterly, **49.4** (2011), 348–354.
[5] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Wadsworth, Cambridge, 1986.
[6] M. Waddill and L. Sacks, *Another generalized Fibonacci sequence*, The Fibonacci Quarterly, **5.3** (1967), 209–221.

MSC2010: 11P81

Mathematics Department, San Francisco State University, San Francisco, CA 94132
E-mail address: nrobbins@sfsu.edu