THE FIBONACCI NUMBER OF GENERALIZED PETERSEN GRAPHS

Stephan G. Wagner
Department of Mathematics, Graz University of Technology,
Steyrergasse 30, A-8010 Graz, Austria
e-mail: wagner@finanz.math.tu-graz.ac.at

(Submitted December 2004-Final Revision March 2005)

ABSTRACT

The Fibonacci number $F(G)$ of a graph G is defined as the number of independent vertex subsets of G. It was introduced in a paper of Prodinger and Tichy in 1982. There, they also ask for a formula for the Fibonacci number of a generalized Petersen graph. The aim of the current paper is to solve this problem by deriving a recursion. It will be shown that the Fibonacci number of the generalized Petersen graph with $4n+2$ vertices is asymptotically $\alpha^{n+1/2}$, where $\alpha = 5.6709364838$ is an algebraic number of degree 5.

1. INTRODUCTION

The concept of the Fibonacci number for graphs was introduced in a paper of Prodinger and Tichy [6] in 1982. The Fibonacci number $F(G)$ of a graph G is defined as the number of independent vertex subsets of G, where a set of vertices is said to be independent if it contains no pair of connected vertices. The name is due to the fact that $F(P_n)$, where P_n is a simple path of length n, gives the sequence of Fibonacci numbers. Similarly, $F(C_n)$, where C_n denotes the cycle of length n, gives the Lucas numbers.

Figure 1: The Petersen Graph.

In the original work, the authors could prove that the star has maximal Fibonacci number among all trees, whereas the path has minimal Fibonacci number. They also considered recursions for several classes of graphs. In subsequent papers ([3, 4]), classes of simply generated trees were investigated. The concept of Fibonacci numbers even turned out to be of use in combinatorial chemistry (cf. [5]). One of the problems stated in [6] is the following: we define the generalized Petersen graph (cf. [2]) Pet_n by

$$V(\mathrm{Pet}_n) = \{1, \ldots, 2n+1\} \times \{1, 2\},$$

$$\begin{aligned} E(\mathrm{Pet}_n) = \{((k,1),(k,2)) \,|\, 1 \le k \le 2n+1\} \cup \\ \{((k,1),(k+1,1)) \pmod{2n+1} \,|\, 1 \le k \le 2n+1\} \cup \\ \{((k,2),(k+2,2)) \pmod{2n+1} \,|\, 1 \le k \le 2n+1\}. \end{aligned}$$

The well-known Petersen graph corresponds to the special case $n = 2$ (Figure 1). Now, we ask for a formula for the Fibonacci number of a generalized Petersen graph. This will be solved by means of a recursion.

2. PRELIMINARIES

Lemma 1: Let v be a vertex of a graph G. If $N(v)$ denotes the neighbourhood of v, we have

$$F(G) = F(G \setminus \{v\}) + F(G \setminus (\{v\} \cup N(v))).$$

Proof: Obviously, $F(G \setminus \{v\})$ is the number of independent vertex subsets not containing v, whereas $F(G \setminus (\{v\} \cup N(v)))$ gives the number of independent vertex subsets containing v. $\square$

Next, we draw the generalized Petersen graph in the following way (Figure 2):

Figure 2: Another way to draw a Petersen graph.

The middle row contains the vertices of the form $(k, 1)$, the other rows the vertices of the form $(k, 2)$. Vertices that are marked with the same letter are identified. In that way, the generalized Petersen graph can be seen to be some kind of discretisation of a Möbius strip. Now, by the lemma above (applied consecutively to a, b, c),

$$\begin{aligned} F(\mathrm{Pet}_n) = &F(\mathrm{Pet}_n \setminus \{a, b, c\}) + F(\mathrm{Pet}_n \setminus (\{a, b, c\} \cup N(a))) \\ &+ F(\mathrm{Pet}_n \setminus (\{a, b, c\} \cup N(b))) + F(\mathrm{Pet}_n \setminus (\{a, b, c\} \cup N(c))) \\ &+ F(\mathrm{Pet}_n \setminus (\{a, b, c\} \cup N(a) \cup N(c))) + F(\mathrm{Pet}_n \setminus (\{a, b, c\} \cup N(b) \cup N(c))). \end{aligned}$$

The purpose of this step is to "cut" the Möbius strip in order to establish a recursion. Each of the subgraphs appearing in this decomposition has the form shown in Figure 3 with some of the 12 configurations shown in Figure 4 on the left and right end.

Figure 3: The common form of the subgraphs.

Figure 4: The 12 terminal configurations.

We won't care about the left end of the strip, as the recursion is the same for all the possible configurations. We also note that the same recursion must be true for $F(\mathrm{Pet}_n)$ as well if it holds for all the summands.

3. DERIVATION OF THE RECURSION

Definition 1: Let the configuration C on the left end of the strip be fixed. By $G_i(n)$, we denote the graph whose left-end-configuration is C, whose right-end-configuration is the $i-th$ of the 12 configurations in our list, and whose number of vertices in the uppermost row is n. Furthermore, let $a_i(n) = F(G_i(n))$.

Figure 5: The vertices used in the recurrences.

For all i, we choose a vertex v (indicated by a surrounding circle in Figure 5) and apply Lemma 1 to obtain the following system of linear recursions:

$$\begin{aligned} a_1(n) &= a_6(n) + a_5(n), & a_7(n) &= a_9(n) + a_1(n),\\ a_2(n) &= a_5(n) + a_6(n-1), & a_8(n) &= a_{10}(n) + a_2(n-1),\\ a_3(n) &= a_1(n) + a_6(n), & a_9(n) &= a_3(n) + a_{12}(n),\\ a_4(n) &= a_2(n) + a_5(n), & a_{10}(n) &= a_4(n-1) + a_{11}(n-2),\\ a_5(n) &= a_1(n-1) + a_7(n-2), & a_{11}(n) &= a_3(n) + a_1(n),\\ a_6(n) &= a_2(n) + a_8(n), & a_{12}(n) &= a_4(n) + a_2(n). \end{aligned}$$

Now, we simplify in the following way:

$$\begin{aligned} a_6(n) &= a_2(n) + a_8(n) = a_5(n) + a_6(n-1) + a_8(n),\\ a_5(n+1) &= a_1(n) + a_7(n-1) = a_6(n) + a_5(n) + a_7(n-1)\\ &= 2a_5(n) + a_6(n-1) + a_7(n-1) + a_8(n),\\ a_8(n+1) &= a_{10}(n+1) + a_2(n) = a_4(n) + a_{11}(n-1) + a_2(n)\\ &= 2a_2(n) + a_5(n) + a_{11}(n-1) = 3a_5(n) + 2a_6(n-1) + a_{11}(n-1),\\ a_7(n) &= a_9(n) + a_1(n) = a_3(n) + a_{12}(n) + a_6(n) + a_5(n)\\ &= a_1(n) + 2a_6(n) + a_4(n) + a_2(n) + a_5(n) = 3a_5(n) + 3a_6(n) + 2a_2(n)\\ &= 8a_5(n) + 5a_6(n-1) + 3a_8(n),\\ a_{11}(n) &= a_3(n) + a_1(n) = 2a_1(n) + a_6(n) = 3a_6(n) + 2a_5(n)\\ &= 5a_5(n) + 3a_6(n-1) + 3a_8(n). \end{aligned}$$

So we have a system of recursions in $a_5(n), a_6(n), a_7(n), a_8(n)$ and $a_{11}(n)$ only. We write this in matrix form: with

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 8 & 5 & 0 & 3 & 0 \\ 3 & 2 & 0 & 0 & 1 \\ 5 & 3 & 0 & 3 & 0 \end{pmatrix} \qquad \text{and} \qquad b(n) = \begin{pmatrix} a_5(n+1) \\ a_6(n) \\ a_7(n) \\ a_8(n+1) \\ a_{11}(n) \end{pmatrix},$$

we have $b(n) = Mb(n-1)$. The characteristic polynomial of M is $x^5 - 3x^4 - 15x^3 - 3x^2 + 13x - 4$, so $a_i(n)$ satisfies the recurrence relation

$$a_i(n) - 3a_i(n-1) - 15a_i(n-2) - 3a_i(n-3) + 13a_i(n-4) - 4a_i(n-5) = 0$$

(cf. [1]). This is clear for $i = 5, 6, 7, 8, 11$, and all others can be written as linear combinations:

$$\begin{aligned}
a_1(n) &= a_6(n) + a_5(n), \\
a_2(n) &= a_5(n) + a_6(n-1), \\
a_3(n) &= a_1(n) + a_6(n) = a_5(n) + 2a_6(n), \\
a_4(n) &= a_2(n) + a_5(n) = 2a_5(n) + a_6(n-1), \\
a_9(n) &= a_3(n) + a_{12}(n) = a_2(n) + a_3(n) + a_4(n) = 4a_5(n) + 2a_6(n) + 2a_6(n-1), \\
a_{10}(n) &= a_4(n-1) + a_{11}(n-2) = 2a_5(n-1) + a_6(n-2) + a_{11}(n-2), \\
a_{12}(n) &= a_4(n) + a_2(n) = 3a_5(n) + 2a_6(n-1).
\end{aligned}$$

Since we know that $F(\mathrm{Pet}_n)$ can also be written as a linear combination of such sequences, it must satisfy the same recursive relation. The initial values can be computed directly – $F(\mathrm{Pet}_n)$ is thus uniquely determined.

Theorem 2: *The Fibonacci number $F(Pet_n)$ of the generalized Petersen graph Pet_n is given by its initial values*

$$F(Pet_1) = 13, \; F(Pet_2) = 76, \; F(Pet_3) = 435, \; F(Pet_4) = 2461, \; F(Pet_5) = 13971$$

and the recursion

$$F(\mathrm{Pet}_n) = 3F(\mathrm{Pet}_{n-1}) + 15F(\mathrm{Pet}_{n-2}) + 3F(\mathrm{Pet}_{n-3}) - 13F(\mathrm{Pet}_{n-4}) + 4F(\mathrm{Pet}_{n-5}).$$

The largest root of the characteristic polynomial is $\alpha = 5.6709364838$. Therefore, we also obtain the formula $F(\mathrm{Pet}_n) = \beta \cdot \alpha^n + O(|\alpha'|^n)$, where α' is the second-largest root. By solving the recursion explicitly, one obtains that $\beta = \sqrt{\alpha} = 2.3813728150$; β is a root of the polynomial $x^5 - x^4 - x^3 - 3x^2 - 5x - 2$.

Remark: The Petersen graph can be generalized further in the following way (cf. [2]): if m, r are integers with $m \geq 3$ and $r < m/2$, we define the generalized Petersen graph $\mathrm{Pet}_{m,r}$ by

$$V(\mathrm{Pet}_{m,r}) = \{1, \ldots, m\} \times \{1, 2\},$$

$$\begin{aligned}E(\mathrm{Pet}_{m,r}) = &\{((k,1),(k,2))\,|\,1 \le k \le m\} \cup \\ &\{((k,1),(k+1,1)) \pmod m\,|\,1 \le k \le m\} \cup \\ &\{((k,2),(k+r,2)) \pmod m\,|\,1 \le k \le m\}.\end{aligned}$$

Here, we only considered the case of odd m and $r = 2$. The method can be generalized to any other fixed r, but the calculations are very cumbersome. However, it seems to be an interesting problem how

$$c_r := \lim_{m\to\infty} \sqrt[m]{F(\mathrm{Pet}_{m,r})}$$

behaves in terms of r. For $r = 1$, it is easy to obtain $c_1 = 1 + \sqrt{2}$; the case $r = 2$ was only solved for odd m; however, the argument for even m is exactly the same and gives the same recursion. One can even show that $x_m := F(\mathrm{Pet}_{m,2})$ satisfies the recurrence formula

$$x_m = x_{m-1} + x_{m-2} + 3x_{m-3} + 5x_{m-4} + 2x_{m-5}$$

from which we also obtain the asymptotics $F(\mathrm{Pet}_{m,2}) \sim \beta^m$. Therefore, we know that $c_2 = \beta = 2.3813728150$. Eventually, we state the following problem:

Problem: Does the limit $\lim_{r\to\infty} c_r$ exist, and if so, what is its value?

ACKNOWLEDGMENT

The author was supported by Austrian Science Fund project no. S-8307-MAT.

REFERENCES

[1] Ravi P. Agarwal. "Difference Equations and Inequalities." *Monographs and Textbooks in Pure and Applied Mathematics*, Volume 228, Second edition, Marcel Dekker Inc., New York, (2000).

[2] D. A. Holton and J. Sheehan. "The Petersen Graph." *Australian Mathematical Society Lecture Series*, Volume 7, Cambridge University Press, Cambridge, (1993).

[3] Peter Kirschenhofer, Helmut Prodinger and Robert F. Tichy. "Fibonacci Numbers of Graphs. II." *Fibonacci Quart.* **21.3** (1983): 219-229.

[4] Peter Kirschenhofer, Helmut Prodinger and Robert F. Tichy. "Fibonacci Numbers of Graphs. III. Planted Plane Trees." *Fibonacci Numbers and their Applications (Patras, 1984)*, Volume 28 of Math. Appl., Reidel, Dordrecht, (1986), 105-120.

[5] R. E. Merrifield and H. E. Simmons. *Topological Methods in Chemistry*, Wiley, New York, (1989).

[6] Helmut Prodinger and Robert F. Tichy. "Fibonacci Numbers of Graphs." *Fibonacci Quart.* **20.1** (1982): 16-21.

AMS Classification Numbers: 05C30, 05C69, 11B37

✠ ✠ ✠