ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-118 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

Let $F_1 = 1 = F_2$ and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 1$. Show for all $n \ge 1$ that

$$\sum_{k=1}^{n} (F_k/2^k) < 2 .$$

B-119 Proposed by Jim Woolum, Clayton Valley High School, Concord, Calif.

What is the area of an equilateral trapezoid whose bases are F_{n-1} and F_{n+1} and whose lateral side is F_n ?

B-120 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Find a simple function g such that $g(n)$ is an integer when n is an integer and $g(m + n) - g(m) - g(n) = mn$.

B-121 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let $n, q, d,$ and r be integers with $n \ge 0$, $d > 0$, $n = qd + r$, and $0 \le r < d$. Prove that

$$F_n \equiv (F_{d+1})^q F_r \pmod{F_d} .$$

B-122 Proposed by A.J. Montleaf, Univ. of New Mex., Albuquerque, N. Mex.

Show that

$$\sin\left[(2k+1)\theta\right]\Big/\sin\theta = 2\cos\left[2k\theta\right] + 2\cos\left[2(k-1)\theta\right] + 2\cos\left[2(k-2)\theta\right] + \cdots + 2\cos\left[2\theta\right] + 1$$

and obtain the analogous formula for $F_{(2k+1)m}/F_m$ in terms of Lucas numbers.

B-123 (From B-102, Proposed by G. L. Alexanderson, Univ. of Santa Clara, Santa Clara, California.

Show that all the positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ are given by

$$x_n = (P_{n+1})^2 - (P_n)^2 \,;\quad z_n = (P_{n+1})^2 + (P_n)^2 \,;\quad n = 1, 2, \cdots \,;$$

where P_n is the Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$.

SOLUTIONS

A NON-HOMOGENEOUS DIFFERENCE EQUATION

B-100 Proposed by J. A. H. Hunter, Toronto, Canada.

Let $u_{n+2} = u_{n+1} + u_n - 1$, with $u_1 = 1$ and $u_2 = 3$. Find the general solution for u_n.

Solution by F. D. Parker, St. Lawrence University, Canton, N.Y.

The general solution of the difference equation is $u_n = c_1a^n + c_2b^n + 1$, where c_1 and c_2 are arbitrary constants, $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Since $u_1 = 1$ and $u_2 = 3$, we find the particular solution to be

$$u_n = \frac{2}{\sqrt{5}}a^{n-1} - \frac{2}{\sqrt{5}}b^{n-1} + 1 = 2F_{n-1} + 1 \,.$$

Also solved by L. Carlitz; Herta T. Freitag; William T. Jackson; Douglas Lind; William C. Lombard; C.B.A. Peck; Lt. A.G. Shannon, R.A.N; David Zeitlin; and the proposer.

A SEQUENCE OF SEQUENCES

B-101 Proposed by Thomas P. Dence, Bowling Green State Univ., Bowling Green, Ohio.

Let $x_{i,n}$ be defined by $x_{1,n} = 1$, $x_{2,n} = n$, and $x_{i+2,n} = x_{i+1,n} + x_{i,n}$. Express $x_{i,n}$ as a function of F_n and n.

Solution by Douglas Lind, University of Virginia, Charlottesville, Virginia

We claim $x_{i,n} = F_i + (n - 1)F_{i-1}$. This is clearly true for $i = 1, 2$ and all n. Since both expressions obey the same second-order recurrence relation in i and agree in the first two values, they must coincide for all i and n.

Also solved by Gerald Edgar, Herta T. Freitag, William C. Lombard, John W. Milson, F. D. Parker, David Zeitlin, and the proposer.

NOTE: The problem editor misstated the problem as "Express $x_{i,n}$ in terms of F_n and n" instead of "Express $x_{i,n}$ in terms of n and F_i." The proposer intended that F_{i-1} in the solution printed above be expressed in terms of F_i, as one might do, for example, using the result of B-42.

PELL-PYTHAGOREAN TRIPLES

B-102 Proposed by Gerald L. Alexanderson, Univ. of Santa Clara, Santa Clara, Calif.

The Pell sequence $1, 2, 5, 12, 29, \cdots$ is defined by $P_1 = 1$, $P_2 = 2$ and $P_{n+2} = 2P_{n+1} + P_n$. Let $(P_{n+1} + iP_n)^2 = x_n + iy_n$, with x_n and y_n real and let $z_n = |x_n + iy_n|$. Prove that the numbers x_n, y_n, and z_n are the lengths of the sides of a right triangle and that x_n and y_n are consecutive integers for every positive integer n. Are there any other positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ than $(x, z) = (x_n, z_n)$?

Solution by Herta T. Freitag, Hollins, Virginia.

(A) $z_n = |x_n + iy_n| = \sqrt{x_n^2 + y_n^2}$; hence x_n, y_n, and z_n may be interpreted as lengths of the sides of a right triangle.

(B) To show that $y_n - x_n = 1$:
Since $x_n = P_{n+1}^2 - P_n^2$ and $y_n = 2P_{n+1}P_n$, we need to show that

$$\left|2P_{n+1}P_n - P_{n+1}^2 + P_n^2\right| = 1 .$$

Proof by mathematical induction:

(1) $\left|2P_2P_1 - P_2^2 + P_1^2\right| = 1$, hence the statement is correct for $n = 1$.

(2) Assume the formula correct for $n = k$, i.e., assume that:

$$\left|2P_{k+1}P_k - P_{k+1}^2 + P_k^2\right| = 1 .$$

Then,

$$\begin{aligned}\left|2P_{k+2}P_{k+1} - P_{k+2}^2 + P_{k+1}^2\right| &= \left|2(2P_{k+1} + P_k)P_{k+1} - (2P_{k+1} + P_k)^2 + P_{k+1}^2\right| \\ &= \left|P_{k+1}^2 - 2P_kP_{k+1} - P_k^2\right| = 1 .\end{aligned}$$

This, however, means that correctness of the statement for $n = k$ causes its correctness for $n = k + 1$, and the query is settled.

(C) No, there are no other positive integral solutions of $x^2 + (x \pm 1)^2 = z^2$ than $(x, z) = (x_n, z_n)$. This, however, is only a hunch; I was unable to establish the unicity.

Also solved by the proposer.

NOTE: See proposed problem B-123.

AN INCREASING SEQUENCE

B-103 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Va.

Let

$$a_n = \sum_{d|n} F_d \quad (n \geq 1),$$

where the sum is over all divisors d of n. Prove that $\{a_n\}$ is a strictly increasing sequence. Also show that

$$\sum_{n=1}^{\infty} \frac{F_n x^n}{1 - x^n} = \sum_{n=1}^{\infty} a_n x^n .$$

Solution by Gerald Edgar, Boulder, Colorado.

For $n \geq 1$, we have

$$a_{n+1} = \sum_{d|(n+1)} F_d \geq F_1 + F_{n+1} > F_{n+1} .$$

Observe that

$$\begin{aligned} a_1 &= 1 = F_2 , \\ a_2 &= 2 = F_3 , \\ a_3 &= 3 = F_4 , \end{aligned}$$

and that for $n > 3$, since $(n - 1) \nmid n$ and $(n - 2) \nmid n$,

$$a_n = \sum_{d|n} F_d \leq F_n + \sum_{i=1}^{n-3} F_i = F_n + F_{n-1} - 1 < F_{n+1}$$

so that in all cases for $n \geq 1$, we have $a_n \leq F_{n+1}$.

Therefore, for all $n \geq 1$, $a_n < a_{n+1}$, so that $\{a_n\}$ is a strictly increasing sequence. Also, we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} \left(\sum_{d|n} F_d \right) x^n \\ &= \sum_{d=1}^{\infty} \left(\sum_{d|n}^{\infty} x^n \right) F_d \quad \text{(rearranging terms)} \\ &= \sum_{d=1}^{\infty} \left(\sum_{i=1}^{\infty} x^{id} \right) F_d \\ &= \sum_{d=1}^{\infty} \left(\frac{x^d}{1 - x^d} \right) F_d \\ &= \sum_{d=1}^{\infty} \frac{F_d x^d}{1 - x^d} \end{aligned}$$

Also solved by the proposer.

TELESCOPING SERIES

B-104 Proposed by H. H. Ferns, Victoria, British Columbia.

Show that

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \frac{1}{3} ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

Solution by L. Carlitz, Duke University, Durham, N.C.

It is easily verified that

$$F_{n+1}L_{n+2} - F_{n+2}L_n = F_{2n+1} .$$

Thus

$$\sum_{n=1}^{N} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \sum_{n=1}^{N} \left(\frac{F_{n+1}}{L_n L_{n+1}} - \frac{F_{n+2}}{L_{n+1} L_{n+2}} \right) = \frac{F_2}{L_1 L_2} - \frac{F_{N+2}}{L_{N+1} L_{N+2}}$$

and therefore

$$\sum_{n=1}^{\infty} \frac{F_{2n+1}}{L_n L_{n+1} L_{n+2}} = \frac{1}{3} .$$

Also solved by Douglas Lind, F. D. Parker, Lt. A. G. Shannon, David Zeitlin, and the proposer.

A PERIODIC SEQUENCE

B-105 Proposed by Phil Mana, University of New Mex., Albuquerque, New Mex.

Let g_n be the number of finite sequences $c_1, c_2, \cdots, c_n$, with $c_1 = 1$, each c_i in $\{0,1\}$, (c_i, c_{i+1}) never $(0,0)$, and (c_i, c_{i+1}, c_{i+2}) never $(0,1,0)$.

Prove that for every integer $s > 1$ there is an integer t with $t \leq s^3 - 3$ and g_t an integral multiple of s.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

Acceptable sequences of length n can be produced by appending a "1" to all sequences of length $n - 1$, and a "110" to those of length $n - 3$. Then all n-sequences not included are not acceptable since they violate the given restraints. It follows that $g_n = g_{n-1} + g_{n-3}$. Put $I_k = (g_k, g_{k+1}, g_{k+2})$. Each I_k determines the entire sequence g_n by using the above recurrence relation. Thus modulo $s > 1$, if $I_j \equiv I_k$, then $\{g_n\}$ is periodic with period $\leq |j - k|$. Now there are $(s - 1)^3$ possible distinct triplets (a, b, c) modulo s such that $a, b, c \not\equiv 0 \pmod{s}$. Also $(s - 1)^3 < s^3 - 5$ for $s > 1$. Thus either one of $I_1, I_2, \cdots, I_{s-5}$ contains a 0, in which case there is a $t \leq s^3 - 3$ such that $g_t \equiv 0 \pmod{s}$, or $I_j \equiv I_k \pmod{s}$ for some $j, k \leq s^3 - 3$ with $j \neq k$. But then $\{g_n\}$ has period $t = |j - k| > 0$, so $g_t \equiv g_0 = 0 \pmod{s}$, where here $t < s^3 - 3$.

Also solved by Robert L. Mercer and the proposer.

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All subscription correspondence should be addressed to Brother U. Alfred, St. Mary's College, Calif. All checks ($4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.

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