GENERALIZED HAPPY NUMBERS

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1. HAPPY NUMBERS

Let $S_2 : \mathbf{Z}^+ \to \mathbf{Z}^+$ denote the function that takes a positive integer to the sum of the squares of its digits. More generally, for $e \geq 2$ and $0 \leq a_i \leq 9$, define S_e by

$$S_e\left(\sum_{i=0}^{n} a_i 10^i\right) = \sum_{i=0}^{n} a_i^e .$$

A positive integer a is a *happy number* if, when S_2 is applied to a iteratively, the resulting sequence of integers (which we will call the S_2*-sequence* of a) eventually reaches 1. Thus a is a happy number if and only if there exists some $m \geq 0$ such that $S_2^m(a) = 1$. For example, 13 is a happy number since $S_2^2(13) = 1$.

Notice that 4 is not a happy number. Its S_2-sequence is periodic with $S_2^8(4) = 4$. It is simple to verify that every positive integer less than 100 either is a happy number or has an S_2-sequence that enters the cyclic S_2-sequence of 4. It can further be shown that, for each positive integer $a \geq 100$, $S_2(a) < a$. This leads to the following well-known theorem. (See [2] for a complete proof.)

Theorem 1: Given $a \in \mathbf{Z}^+$, there exists $n \geq 0$ such that $S_2^n(a) = 1$ or 4.

Generalizing the concept of a happy number, we say that a positive integer a is a *cubic happy number* if its S_3-sequence eventually reaches 1. We note that a positive integer can be a cubic happy number only if it is congruent to 1 modulo 3. This follows immediately from the following lemma.

Lemma 2: Given $a \in \mathbf{Z}^+$, for all m, $S_3^m(a) \equiv a \pmod 3$.

Proof: Let $a = \sum_{i=0}^{n} a_i 10^i$, $0 \leq a_i \leq 9$. Using the fact that, for each i, $a_i^3 \equiv a_i \pmod 3$ and $10^i \equiv 1 \pmod 3$, we get

$$S_3(a) = S_3\left(\sum_{i=0}^{n} a_i 10^i\right) \stackrel{\text{def}}{=} \sum_{i=0}^{n} a_i^3 \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i 10^i = a \pmod 3.$$

Thus, by a simple induction argument, we get that, for all $m \in \mathbf{Z}^+$, $S_3^m(a) \equiv a \pmod 3$. □

The fixed points and cycles of S_3 are characterized in Theorem 3, which can be found without proof in [1].

Theorem 3: The fixed points of S_3 are 1, 153, 370, 371, and 407; the cycles are $136 \to 244 \to 136$, $919 \to 1459 \to 919$, $55 \to 250 \to 133 \to 55$, and $160 \to 217 \to 352 \to 160$. Further, for any positive integer a:

- If $a \equiv 0 \pmod 3$, then there exists an m such that $S_3^m(a) = 153$.
- If $a \equiv 1 \pmod 3$, then there exists an m such that $S_3^m(a) = 1, 55, 136, 160, 370$, or 919.
- If $a \equiv 2 \pmod 3$, then there exists an m such that $S_3^m(a) = 371$ or 407.

Note that the second part of the theorem follows from the first half and Lemma 2. Rather than prove the first part here, we state and prove a generalization of Theorems 1 and 3 in the following section.

2. VARIATIONS OF BASE

By expressing numbers in different bases, we can generalize happy numbers even further.

Fix $b \geq 2$. Let $a = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b-1$. Let $e \geq 2$. We then define the function $S_{e,b}: \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$S_{e,b}(a) = S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If an $S_{e,b}$ sequence reaches 1, we call a an e-power b-happy number.

Theorem 4: For all $e \geq 2$, every positive integer is an e-power 2-happy number.

Proof: Fix e. Let $a = \sum_{i=0}^{n} a_i 2^i$, $0 \leq a_i \leq 1$, $a_n > 0$. Then

$$a - S_{e,2}(a) = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i^e = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i (2^i - 1).$$

Note that none of the terms can be negative. Thus, if $n \geq 1$, $a - S_{e,2}(a) > 0$. So, for $a \neq 1$, $S_{e,2}(a) < a$. With this fact, it is easy to prove by induction that every positive integer is an e-power 2-happy number. □

Again, we ask: What are the fixed points and cycles generated when these functions are iterated? We give the answers for $S_{2,b}$, $2 \leq b \leq 10$, in Table 1 and for $S_{3,b}$, $2 \leq b \leq 10$, in Table 2.

TABLE 1. Fixed points and cycles of $S_{2,b}$, $2 \leq b \leq 10$

Base	Fixed Points and Cycles
2	1
3	1, 12, 22 2 → 11 → 2
4	1
5	1, 23, 33 4 → 31 → 20 → 4
6	1 32 → 21 → 5 → 41 → 25 → 45 → 105 → 42→ 32
7	1, 13, 34, 44, 63 2 → 4 → 22 → 11 → 2 16 → 52 → 41 → 23 → 16
8	1, 24, 64 4 → 20 → 4 5 → 31 → 12 → 5 15 → 32 → 15
9	1, 45, 55 58 → 108 → 72→ 58 82 → 75 → 82
10	1 4 → 16 → 37 → 58 → 89 → 145 → 42 → 20 → 4

TABLE 2. Fixed points and cycles of $S_{3,b}$, $2 \le b \le 10$

Base	Fixed Points and Cycles
2	1
3	1, 122 2 → 22 → 121 → 101 → 2
4	1, 20, 21, 203, 313, 130, 131, 223, 332
5	1, 103, 433 14 → 230 → 120 → 14
6	1, 243, 514, 1055 13 → 44 → 332 → 142 → 201 → 13
7	1, 12, 22, 250, 251, 305, 505 2 → 11 → 2 13 → 40 → 121 → 13 23 → 50 → 236 → 506 → 665 → 1424 → 254 → 401 → 122 → 23 51 → 240 → 132 → 51 160 → 430 → 160 161 → 431 → 161 466 → 1306 → 466 516 → 666 → 1614 → 552 → 516
8	1, 134, 205, 463, 660, 661 662 → 670 → 1057 → 725 → 734 → 662
9	1, 30, 31, 150, 151, 570, 571, 1388 38 → 658 → 1147 → 504 → 230 → 38 152 → 158 → 778 → 1571 → 572 → 578 → 1308 → 660 → 530 → 178 → 1151 → 152 638 → 1028 → 638 818 → 1358 → 818
10	1, 153, 371, 407, 370 55 → 250 → 133 → 55 136 → 244 → 136 160 → 217 → 352 → 160 919 → 1459 → 919

It is easy to verify that each entry in the tables above is, indeed, a fixed point or cycle. Theorem 5 asserts that the tables are, in fact, complete.

Theorem 5: Tables 1 and 2 give all of the fixed points and cycles of $S_{2,b}$ and $S_{3,b}$, respectively, for $2 \le b \le 10$.

The proof of Theorem 5 uses the same techniques as the proof of Theorem 1 given in [2]. First, we find a value N for which $S_{e,b}(a) < a$ for all $a \ge N$. This implies that, for each $a \in \mathbf{Z}^+$, there exists some $m \in \mathbf{Z}^+$ such that $S_{e,b}^m(a) < N$. Then a direct calculation for each $a < N$ completes the process and Theorem 5 is proven. Lemma 6 provides an N for $e = 2$ and all bases $b \ge 2$ while Lemma 8 does the same for $e = 3$.

Lemma 6: If $b \ge 2$ and $a \ge b^2$, then $S_{2,b}(a) < a$.

Proof: Let $a = \sum_{i=0}^n a_i b^i$. We have

$$a - S_{2,b}(a) = \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^2 = \sum_{i=0}^{n} a_i (b^i - a_i).$$

Every term in the final sum is positive with the possible exception of the $i = 0$ term which is at least $(b-1)(1-(b-1))$. It is not difficult to show that the $i = n$ term is minimal if $a_n = 1$. From

$a \geq b^2 = 100_{(b)}$, it follows that $n \geq 2$. So the $i = n$ term is at least $1(b^2 - 1)$. Thus, $a - S_{2,b}(a) > b^2 - 1 + (b-1)(1-(b-1)) = 3b - 3 > 0$, since $b \geq 2$. Hence, for all $a \geq b^2$, $S_{2,b}(a) < a$. □

Using induction, Corollary 7 is immediate.

Corollary 7: For each $a \in \mathbf{Z}^+$, there is an $m \in \mathbf{Z}^+$ such that $S_{2,b}^m(a) < b^2$.

This completes the argument for $e = 2$. Now we consider $e = 3$.

Lemma 8: If $b \geq 2$ and $a \geq 2b^3$, then $S_{3,b}(a) < a$.

Proof: The proof of Theorem 4 gives an even stronger result for $b = 2$, so we will assume $b > 2$. Using the notation from above, we have

$$a - S_{3,b}(a) = \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^3 = \sum_{i=0}^{n} a_i(b^i - a_i^2).$$

The $i = 0$ term is at least $(b-1)(1-(b-1)^2)$ and the $i = 1$ term is at least $(b-1)(b-(b-1)^2)$. The remaining terms are all nonnegative. Since $a \geq 2b^3 = 2000_{(b)}$, $n \geq 3$ and if $n = 3$, then $a_3 \geq 2$. So, if $n = 3$, the a_n term is at least $2(b^3 - 4)$. If $n > 3$, then the a_n term is at least $b^4 - 1 > 2(b^3 - 4)$. Thus,

$$\begin{aligned} a - S_{3,b}(a) &\geq a_n(b^3 - a_n^2) + a_1(b - a_1^2) + a_0(1 - a_0^2) \\ &\geq 2(b^3 - 4) + (b-1)(b-(b-1)^2) + (b-1)(1-(b-1)^2) \\ &= 7b^2 - 6b - 7 > 0 \end{aligned}$$

since $b > 2$. Hence, for all $a \geq 2b^3$, $S_{3,b}(a) < a$. □

Corollary 9: For each $a \in \mathbf{Z}^+$, there is an $m \in \mathbf{Z}^+$ such that $S_{3,b}^m(a) < 2b^3$.

Theorem 5 now follows from a direct calculation of the $S_{2,b}$-sequences for all $a < b^2$ and the $S_{3,b}$-sequences for all $a < 2b^3$. These calculations are easily completed with a computer.

We conclude with two general theorems concerning congruences. If, for given e, b, and d, $S_{e,b}^m(a) \equiv a \pmod{d}$ for all a and m, then, as in Lemma 2, all e-power b-happy numbers must be congruent to 1 modulo d. Thus, the following theorems yield a great deal of information concerning generalized happy numbers. In particular, bounds on the densities of the numbers are immediate.

Theorem 10: Let p be prime and let $b \equiv 1 \pmod{p}$. Then, for any $a \in \mathbf{Z}^+$ and $m \in \mathbf{Z}^+$, $S_{p,b}^m(a) \equiv a \pmod{p}$.

Proof: Let $a = \sum_{i=0}^{n} a_i b^i$. By Fermat, $a_i^p \equiv a_i \pmod{p}$ for all i. Thus,

$$S_{p,b}(a) = S_{p,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^p \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i b^i = a \pmod{p}.$$

Using induction, we see that, for all $m \in \mathbf{Z}^+$, $S_{p,b}^m(a) \equiv a \pmod{p}$. □

Corollary 11: If a is a (2-power) b-happy number with b odd, then a must be odd. In general, if a is a p-power b-happy number with $b \equiv 1 \pmod{p}$ for some prime p, then $a \equiv 1 \pmod{p}$.

Theorem 12: Let $b \equiv 1 \pmod{\gcd(6, b-1)}$. Then, for any $a \in \mathbf{Z}^+$ and $m \in \mathbf{Z}^+$, $S_{3,b}^m(a) \equiv a$ $\pmod{\gcd(6, b-1)}$.

Proof: Let $a = \sum_{i=0}^n a_i b^i$ and $d = \gcd(6, b-1)$. If $d = 1$, then the theorem is vacuous. For $d = 2$, note that $a^3 \equiv a \pmod 2$. Since $b \equiv 1 \pmod 2$, we have

$$S_{3,b}(a) = S_{3,b}\left(\sum_{i=0}^n a_i b^i\right) = \sum_{i=0}^n a_i^3 \equiv \sum_{i=0}^n a_i \equiv \sum_{i=0}^n a_i b^i = a \pmod 2,$$

and induction completes the argument. The case $d = 3$ is immediate from Theorem 10. Finally, $d = 6$ follows from the cases $d = 2$ and $d = 3$. □

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