ON THE LUCAS CUBES

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1. INTRODUCTION

A Lucas cube $\mathcal{L}_n$ can be defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1. A Lucas cube $\mathcal{L}_n$ is very similar to the Fibonacci cube Γ_n which is the graph defined as $\mathcal{L}_n$ except for the fact that the vertices are binary strings of length n without two consecutive ones. The Fibonacci cube has been introduced as a new topology for the interconnection of parallel multicomputers alternative to the classical one given by the Boolean cube [4]. An attractive property of the Lucas cube of order n is the decomposition, which can be carried out recursively into two disjoint subgraphs isomorphic to Fibonacci cubes of order $n-1$ and $n-3$; on the other hand, the Lucas cube of order n can be embedded in the Boolean cube of order n. This implies that certain topologies commonly used, as the linear array, particular types of meshes and trees and the Boolean cubes, directly embedded in the Fibonacci cube, can also be embedded in the Lucas cube. Thus, the Lucas cube can also be used as a topology for multiprocessor systems.

Among many different interpretations, F_{n+2} can be regarded as the cardinality of the set formed by the subsets of $\{1, \dots, n\}$ which do not contain a pair of consecutive integers; i.e., the set of the binary strings of length n without two consecutive ones, the *Fibonacci strings*.

If C_n is the set of the Fibonacci strings of order n, then $C_{n+2} = 0C_{n+1} + 10C_n$ and $|C_n| = F_{n+2}$.

A Lucas string is a Fibonacci string with the further condition that there is no 1 in the first and in the last position simultaneously. If $\hat{C}_n$ is the set of Lucas strings of order n, then $|\hat{C}_n| = L_n$, where L_n are the Lucas numbers for every $n > 0$. For $n \geq 1$, L_n can be regarded as the cardinality of the family of the subsets of $\{1, \dots, n\}$ without two consecutive integers and without the couple $1, n$. We have

$$L_n = \sum_{k \geq 0} \binom{n-k}{k} \cdot \frac{n}{n-k}. \tag{1}$$

The Fibonacci cube Γ_n of order n is the bipartite graph whose vertices are the Fibonacci strings and two strings are adjacent when their Hamming distance is 1. Based on the decomposition of C_n, a Fibonacci cube of order n can be decomposed into a subgraph Γ_{n-1}, a subgraph Γ_{n-2} and F_{n-2} edges between the two subgraphs; this decomposition is represented by the relation $\Gamma_n = \Gamma_{n-1} \hat{+} \Gamma_{n-2}$. In a similar way, it is easy to decompose the set $\hat{C}_{n+3}$ into the sum $0C_{n+2} + 10C_n0$ and, therefore, to write $\mathcal{L}_n = \Gamma_{n-1} \hat{+} \Gamma_{n-3}$.

In Figure 1, we draw $\mathcal{L}_n$ for the first values of n; the circled vertices denote the vertices in Γ_n that are not in $\mathcal{L}_n$.

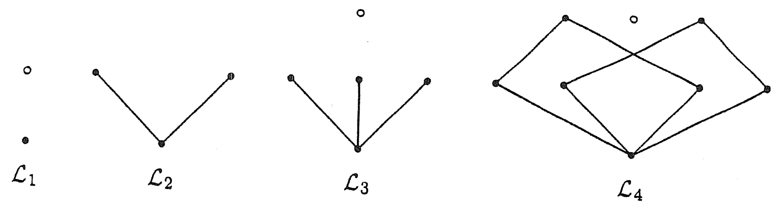

FIGURE 1

In this paper we determine structural and enumerative properties of the Lucas cubes such as the independence numbers of edges and vertices, the radius, the center, the generating function of a sequence of numbers connected to the partite sets, the asymptotic behavior of the ratio of the numbers of edges and vertices. A consequence of the properties on the independence numbers is that $\mathcal{L}_n$ is not Hamiltonian. Moreover, we obtain some identities involving Fibonacci and Lucas numbers which seem to be new. Finally, we introduce the Lucas semilattice and found its characteristic polynomial.

2. GENERAL PROPERTIES

The following identities hold: $L_n = F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$. For each of them there exists an immediate combinatorial interpretation in terms of Lucas cubes. The first says that the Lucas strings of length n beginning with 0 consist of the element 0 followed by any Fibonacci string of length $n-1$, while the Lucas strings beginning with 1 must start with the couple 10 and end with 0, and have any Fibonacci string of length $(n-3)$ between 10 and 0.

The second equality says that the Lucas n-strings are merely the Fibonacci n-strings not beginning and ending with the couple 10 and 01 simultaneously, and consisting of any Fibonacci $(n-4)$-string between these two extremal couples.

Using the first construction, we notice that the edges of $\mathcal{L}_n$ connecting pairs of vertices of Γ_{n-1} (resp. Γ_{n-3}) are just the edges of Γ_{n-1} (resp. Γ_{n-3}); moreover, for any vertex v of Γ_{n-3} there is exactly one edge connecting it to a vertex of Γ_{n-1}, i.e., the edge connecting $10v0$ to $00v0$. Let f_n and l_n denote the cardinalities of the edge sets of Γ_n and $\mathcal{L}_n$, respectively. Thus,

$$l_n = f_{n-1} + f_{n-3} + F_{n-1}$$

for $n \geq 3$; moreover, by direct computation we have $l_1 = 0$, $l_2 = 2$.

Since $f_n = f_{n-1} + f_{n-2} + F_n$, where $n \geq 2$, $f_0 = 0$, $f_1 = 1$, we have immediately $f_{n-1} < l_n < f_n$.

We will prove the following properties, analogous to the ones proved in [6] for the Fibonacci cubes.

The *eccentricity* of a vertex v in a connected graph G is the maximum distance between v and the other vertices, i.e., the number

$$e(v) := \max_{v \in V(G)} d(u, v);$$

the *diameter* of G is the maximal eccentricity when v runs in G, i.e.,

$$\text{diam}(G) := \max_{u \in V(G)} e(v) = \max_{u,v \in V(G)} d(u, v);$$

the *radius* of G is the minimum eccentricity of the vertices of G, i.e.,

$$\operatorname{rad}(G) := \min_{u \in V(G)} e(v).$$

A vertex v is *central* if $e(v) = \operatorname{rad}(G)$; the *center* $Z(G)$ of G is the set of all central vertices; a string $\alpha = [a_1, \ldots, a_n]$ is said to be *symmetric* if $a_i = a_{n-i}$ for $i = 1, \ldots, n$.

For every n, we have that the diameter of $\mathcal{F}_n$ if equal to n; it is easy to prove that

$$\operatorname{diam}(\mathcal{L}_n) = \begin{cases} n & \text{for } n \text{ even,} \\ n-1 & \text{for } n \text{ odd.} \end{cases}$$

Moreover, we have the following proposition.

Proposition 1: The number of pairs of vertices at distance equal to the diameter is 1 for n even, $n-1$ for n odd.

Proof: Let n be even. The strings having 1 in all the odd or even positions are clearly at distance n and they are the only possible strings at distance n.

Let n be odd. We partition the strings having $\frac{n-1}{2}$ ones into two sets A and B, depending on whether the first element is 1 or 0. Assume that a string starts with 1; then it is possible to decompose it into $\frac{n-1}{2}$ subsequences 10 and one 0. This element 0 can be put after a subsequence 10 into $\frac{n-1}{2}$ ways. Clearly, similar considerations hold for the strings starting with 0. The difference now is that there are $\frac{n-1}{2}$ subsequences 01 and one 0 and the 0 can be put after the subsequences 01 in $\frac{n-1}{2}$ positions and also before the first 01, i.e., into $\frac{n+1}{2}$ positions. In any case, every string contains only one substring 00. A string of the first set has two strings of the second set at distance $n-1$, according to the position of 1 in the subsequence corresponding to 00. Thus, we obtain $2 \cdot \frac{n-1}{2}$ pairs of vertices at distance $n-1$.

Theorem 1: For $n \geq 1$, any Lucas cube $\mathcal{L}_n$ satisfies the following properties:

(i) $\operatorname{rad}(\mathcal{L}_n) = \lfloor \frac{n}{2} \rfloor$.

(ii) $Z(\mathcal{L}_n) = \{\hat{0}\}$.

Proof: ***(i)*** The distance $d(v, \hat{0})$ is the number of the elements 1 in the string v; hence, $e(\hat{0}) = \frac{n}{2}$ if n is even and $e(\hat{0}) = \frac{n-1}{2}$ if n is odd.

If $v \neq \hat{0}$, let k denote the number of the elements 1 in the string v. The set of the 0's (with the order induced by v) can be regarded as a Lucas string of length $n-k$ and precisely as the $\hat{0} \in \mathcal{L}_{n-k}$. In order to prove that $e(v) \geq \lfloor \frac{n}{2} \rfloor$, we consider the string $v*$ obtained by replacing the k elements 1 with 0 and the set of the 0's with a Lucas string of length $n-k$ at maximal distance from $\hat{0}$. Then $v* \in \mathcal{L}_n$ and we have

$$d(v, v*) = k + \left\lfloor \frac{n-k}{2} \right\rfloor = \begin{cases} \frac{n+k}{2} > \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor & n-k \text{ even,} \\ \frac{n+k-1}{2} \geq \frac{n}{2} \geq \lfloor \frac{n}{2} \rfloor & n-k \text{ odd.} \end{cases}$$

(ii) The previous construction of $v*$ shows that $e(v) > e(\hat{0})$ for $k > 1$ or for n odd. If $k = 1$ and n is even, we replace $v*$ with the string $v**$ defined in the following way: let h be the number of 0's on the left of the element 1 and l the number of 0's on the right. Without loss of generality, we can assume that h is even and l is odd. Let us replace the l-sequence of 0's, regarded as $\hat{0} \in \Gamma_l$, by a Fibonacci string β with maximal distance from $\hat{0}$ and replace the h-sequence of 0's, regarded as $\hat{0} \in \mathscr{L}_h$, by the Lucas string α obtained by concatenating $\frac{h}{2}$ couples 01.

The sequence $v^{**} = (\alpha 0 \beta)$ is again a Lucas string whose distance from v is greater than $\lfloor \frac{n}{2} \rfloor$. Indeed,

$$d(v, v^{**}) = \frac{h}{2} + 1 + \left\lceil \frac{l}{2} \right\rceil = \frac{h}{2} + 1 + \frac{l+1}{2} = \frac{h+l+1}{2} + 1 = \frac{n}{2} + 1. \ \square$$

We have already noticed that the distance $d(v, \hat{0})$ is the number of the 1's in the string v. Thus, the summands in equality (1) can be regarded as the cardinalities of the sets of the n-strings at distance k from $\hat{0}$. Now, if n is odd, in $\mathscr{L}_n$ there are $\frac{n-1}{2}$ strings starting with 1 and $\frac{n+1}{2}$ strings starting with 0 at maximum distance $\frac{n-1}{2}$ from $\hat{0}$. Hence, the number N of Lucas strings of order n odd having maximal eccentricity is

$$N = \frac{n-1}{2} + \frac{n+1}{2} = n.$$

Then, in equality (1), the summand for $k = \frac{n-1}{2}$ becomes

$$\binom{\frac{n+1}{2}}{\frac{n-1}{2}} \cdot \frac{n}{\frac{n+1}{2}} = n$$

and we obtain a new combinatorial interpretation of the well-known identity

$$\binom{\frac{n+1}{2}}{\frac{n-1}{2}} = \frac{n+1}{2}.$$

Theorem 2: The number of symmetric Lucas strings of $\mathscr{L}_n$ is sim $\mathscr{L}_n = F_{\lfloor \frac{n}{2} \rfloor + 1 - (-1)^n}$.

Proof: Let n be even. In this case, we will write $n = 2m+2$, $m > 0$. Any symmetric string must begin and end with 0 and have in its center a couple 00; hence, sim $\mathscr{L}_{2m+2} = F_{m+1}$. Now let n be odd, $n = 2m+3$, $m > 0$. The symmetric strings having at the center 1 must have as center the triple 010 and two other 0's as extremal. The symmetric strings having at the center 0 satisfy the only condition of having two 0's as extremals; hence, sim $\mathscr{L}_{2m+3} = F_{m+1} + F_{m+2} = F_{m+3}$. In both cases, the statement holds. $\square$

3. ENUMERATIVE PROPERTIES

In [6] we denoted by E_n and O_n the sets of Fibonacci strings having an even or odd number of 1's, the partite sets of Γ_n, and by e_n, o_n their cardinalities. Now we use analogous notations. Thus, we denote by $\hat{E}_n$ and $\hat{O}_n$ the sets of vertices of $\mathscr{L}_n$ having an even or odd number of ones. Their cardinalities $\hat{e}_n$ and $\hat{o}_n$ are

$$\hat{e}_n := |\hat{E}_n| = \sum_{k \geq 0} \binom{n-2k}{2k} \frac{n}{n-2k}$$

and (2)

$$\hat{o}_n := |\hat{O}_n| = \sum_{k \geq 0} \binom{n-2k-1}{2k+1} \frac{n}{n-2k-1},$$

where $n \geq 2$ and obviously $\hat{e}_n + \hat{o}_n = L_n$.

Remark: The Lucas cubes $\mathcal{L}_n$ are defined properly only for $n \geq 1$; however, we shall define also $\hat{C}_0$ as the set formed by the string of length 0, i.e., the empty set. Since in an empty string there are no 1's, we set $\hat{e}_0 = 1$ and $\hat{o}_0 = 0$.

Using the construction related to the equality $L_n = F_{n+1} + F_{n-1}$, we see that the even (odd) vertices of Γ_{n-1} remain even (resp. odd) also in $\mathcal{L}_n$. In fact, by adjoining 0 before the strings of Γ_{n-1}, the number of 1's is not changed. On the contrary, the vertices of Γ_{n-3} becoming vertices of $\mathcal{L}_n$ change parity, because one element 1 is adjoined to their strings.

Furthermore, we have immediately the following relations:

$$\hat{e}_n = e_{n-1} + o_{n-3} \quad \text{and} \quad \hat{o}_n = o_{n-1} + e_{n-3}. \tag{3}$$

In [6] it was proved that

$$h_{n+2} = h_{n+1} - h_n, \; h_{n+3} = -h_n, \text{ and } h_{n+6} = h_n. \tag{4}$$

Consider $\hat{h}_n := \hat{e}_n - \hat{o}_n$. From (3) and (4), it follows immediately that $\hat{h}_{n+3} = h_{n+2} - h_n = -\hat{h}_n$ and $\hat{h}_n = h_{n-1} - h_{n-3} = \hat{h}_{n-1} - \hat{h}_{n-2}$. Moreover, we have the following theorem.

Theorem 3: The sequence $\{\hat{h}_n\}$ satisfies the properties:

(i) $\hat{h}_{n+6} = \hat{h}_n$, $n \geq 1$, and the repeated values are $1, -1, -2, -1, 1, 2$.

(ii) The generating function of $\hat{h}_n$ is $\hat{H}(x) = \frac{1}{1-x+x^2}$.

Proof: ***(i)*** $\hat{h}_{n+6} = -\hat{h}_{n+3} = \hat{h}_n$. By direct computation, we have: $\hat{e}_1 = 1$, $\hat{o}_1 = 0$; thus, $\hat{h}_1 = 1$. $\hat{e}_2 = 1$, $\hat{o}_2 = 2$; thus, $\hat{h}_2 = -1$. $\hat{e}_3 = 1$, $\hat{o}_3 = 3$; thus, $\hat{h}_3 = -2$. Also, $\hat{h}_4 = -\hat{h}_1 = -1$, $\hat{h}_5 = -\hat{h}_2 = 1$, and $\hat{h}_6 = -\hat{h}_3 = 2$. From the settings in the Remark, we have $\hat{h}_0 = 1$.

(ii) Let $\hat{H}(x) := \sum_{n=0}^{\infty} \hat{h}_n x^n$. We have

$$x\hat{H}(x) = \sum_{n=0}^{\infty} \hat{h}_n x^{n+1} \quad \text{and} \quad x^2 \hat{H}(x) = \sum_{n=0}^{\infty} \hat{h}_n x^{n+2}.$$

Then it follows that

$$(1 - x + x^2)\hat{H}(x) = \hat{h}_0 + (\hat{h}_1 - \hat{h}_0)x + \sum_{n=2}^{\infty} (\hat{h}_n - \hat{h}_{n-1} + \hat{h}_{n-2})x^n = \hat{h}_0 + (\hat{h}_1 - \hat{h}_0)x = 1. \quad \square$$

The first values of these sequences are

n	0	1	2	3	4	5	6	7	8	9	10
F_n		1	1	2	3	5	8	13	21	34	55
L_n		1	3	4	7	11	18	29	47	76	123
$\hat{e}_n$	(1)	1	1	1	3	6	10	15	23	37	61
$\hat{o}_n$	(0)	0	2	3	4	5	8	14	24	39	62
$\hat{h}_n$	(1)	1	−1	−2	−1	1	2	1	−1	−2	−1

Remark: A standard argument enables us to obtain identities concerning positive integers starting from generating functions. In fact, we have, identically,

$$\frac{1}{1-x+x^2}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x},$$

where

$$\alpha=\frac{1+\sqrt{-3}}{2},\ \beta=\frac{1-\sqrt{-3}}{2},\ A=\frac{3-\sqrt{-3}}{6},\ B=\frac{3+\sqrt{-3}}{6},$$

which implies

$$\begin{aligned}\frac{1}{1-x+x^2}&=A(1+\alpha x+\alpha^2x^2+\cdots)+B(1+\beta x+\beta^2x^2+\cdots)\\&=1+(A\alpha+B\beta)x+(A\alpha^2+B\beta^2)x^2+\cdots;\end{aligned}$$

hence, $\hat{h}_n=A\alpha^n+B\beta^n$ for all n. Thus, for any $n\in\mathbf{N}$, we have

$$\begin{aligned}&A\alpha^{6m+1}+B\beta^{6m+1}=1,\ A\alpha^{6m+2}+B\beta^{6m+2}=-1,\ A\alpha^{6m+3}+B\beta^{6m+3}=-2,\\&A\alpha^{6m+4}+B\beta^{6m+4}=-1,\ A\alpha^{6m+5}+B\beta^{6m+5}=1,\ A\alpha^{6m}+B\beta^{6m}=2\end{aligned}$$

(in accord with the fact that $\alpha^3=\beta^3=-1$). Combining the equalities $\hat{h}_n=\hat{e}_n-\hat{o}_n$ and $\hat{e}_n+\hat{o}_n=L_n$, we obtain

$$\begin{cases}\hat{e}_n=\frac{L_n+\hat{h}_n}{2},\\ \hat{o}_n=\frac{L_n-\hat{h}_n}{2}.\end{cases}\tag{5}$$

From (2) and (5), we have immediately the following identities concerning the Lucas numbers.

Proposition 2:

$$L_n=2\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k}-\hat{h}_n;\quad L_n=2\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}+\hat{h}_n;$$

$$L_n=2\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k}-(A\alpha^n+B\beta^n);\ L_n=2\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}+A\alpha^n+B\beta^n.$$

In [6] it was proved that

$$h_n=2e_n-F_{n+2}.^*\tag{6}$$

Furthermore, from (4) and (6) we can obtain the following proposition.

Proposition 3:

(i) $$F_{n+2}=\sum_{k\ge0}\binom{n+3-2k}{2k}\frac{n+3}{n+3-2k}-\sum_{k\ge0}\binom{n+3-2k}{2k}+\sum_{k\ge0}\binom{n+1-2k}{2k}.$$

(ii) $$F_{n+2}=\sum_{k\ge0}\binom{n-2k}{2k+1}\frac{n+1}{n-2k}+\sum_{k\ge0}\binom{n+1-2k}{2k}-\sum_{k\ge0}\binom{n-1-2k}{2k}.$$

Proof: Let

$$\Sigma=\sum_{k\ge0}\binom{n-2k}{2k}\frac{n}{n-2k},\quad \Sigma'=\sum_{k\ge0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1}.$$

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* Indeed in [6] this equality is written $h_n=2e_n-F_n$ because in [6] the Fibonacci numbers are defined by the recurrence $F_0=1,\ F_1=2,\ F_{n+2}=F_{n+1}+F_n$.

We have $L_n = 2\Sigma - h_{n-1} + h_{n-3} = 2\Sigma - 2e_{n-1} + F_{n+1} + 2e_{n-3} - F_{n-1}$,

$$F_{n-1} = \Sigma - e_{n-1} + e_{n-3} = \sum_{k\geq 0}\binom{n-2k}{2k}\frac{n}{n-2k} - \sum_{k\geq 0}\binom{n-2k}{2k} + \sum_{k\geq 0}\binom{n-2-2k}{2k}$$

and so the first statement is proved; moreover, we have $L_n = 2\Sigma' + h_{n-1} - h_{n-3} = 2\Sigma' + 2e_{n-1} - F_{n+1} - 2e_{n-3} + F_{n-1}$, thus we have

$$F_{n+1} = \Sigma' + e_{n-1} - e_{n-3} = \sum_{k\geq 0}\binom{n-2k-1}{2k+1}\frac{n}{n-2k-1} + \sum_{k\geq 0}\binom{n-2k}{2k} - \sum_{k\geq 0}\binom{n-2-2k}{2k},$$

hence the second statement is proved. □

4. INDEPENDENCE NUMBERS

Recall that in a connected graph the *vertex independence number* $\beta_0(G)$ is the maximum among all cardinalities of independent sets of vertices of G, the *edge independence number* $\beta_1(G)$ is the maximum among all cardinalities of independent sets of edges of G. We have the following theorem.

Theorem 4: Let $\beta_1(\mathcal{L}_n)$ be the edge independence number of $\mathcal{L}_n$. Then

$$\beta_1(\mathcal{L}_n) = \left\lfloor \frac{L_n - 1}{2} \right\rfloor.$$

Proof: Let L_n be odd. Since $L_n = F_{n+1} + F_{n-1}$, then F_{n+1} and F_{n-1} have different parities. In [5] it was proved that the Fibonacci cubes have a Hamiltonian cycle in the case of an even number of vertices and a cycle containing all the vertices but one in the odd case [5]. Thus, it is possible to determine $\frac{L_n-1}{2}$ independent edges; since this is the maximum, the result holds.

When L_n is even, it follows from the sequences of Fibonacci and Lucas numbers that F_{n+1} and F_{n-1} are both odd. In this case, the Fibonacci cubes Γ_{n-1} and Γ_{n-3} have cycles of length $F_{n+1}-1$ and $F_{n-1}-1$, respectively, and we can find $\frac{L_n-2}{2}$ independent edges. By Theorem 3, we have $|\hat{e}_n - \hat{o}_n| = 2$ when L_n is even. Then the order of one of the partite sets is $\frac{L_n-2}{2}$, which coincides with the maximal number of independent edges. Thus, the maximum number of independent edges is exactly $\frac{L_n-2}{2}$. □

We immediately have the following.

Corollary 1: $\mathcal{L}_n$ is not Hamiltonian.

Proof: It is obvious in the case of L_n odd. In the even case, it follows from Theorem 4 that the maximum number of independent edges is $\frac{L_n-2}{2}$. This excludes that $\mathcal{L}_n$ is Hamiltonian. □

Corollary 2: $\beta_1(\mathcal{L}_n) = \min(\hat{e}_n, \hat{o}_n)$.

Proof: From Theorem 3, it follows that $|\hat{e}_n - \hat{o}_n|$ is equal to 1 or 2, depending on whether L_n is odd or even. Since $L_n = \hat{e}_n + \hat{o}_n$, $\lfloor\frac{L_n-1}{2}\rfloor$ coincides with $\min(\hat{e}_n, \hat{o}_n)$. The result follows from Theorem 4. □

We are now able to prove the following theorem, analogous to the one in [6].

Theorem 5: Let $\beta_0(\mathcal{L}_n)$ be the vertex independence number of $\mathcal{L}_n$. Then $\beta_0(\mathcal{L}_n) = \max(\hat{e}_n, \hat{o}_n)$.

Proof: By Theorem 3, $\hat{e}_n$ and $\hat{o}_n$ are always distinct. Without loss of generality, we can assume $\hat{e}_n < \hat{o}_n$. Thus, by Theorem 4 and Corollary 2, $\mathcal{L}_n$ contains $\hat{e}_n$ independent edges and every vertex $v \in \hat{E}_n$ can be paired with a vertex $v' \in \hat{O}_n$. This implies that a set A of independent vertices cannot have cardinality greater than $\hat{o}_n$, because both v and v' cannot belong to A. □

5. ASYMPTOTIC BEHAVIOR

For the applications, it seems to be useful to consider the indices

$$i(\Gamma_n) := \frac{f_n}{F_{n+2}}, \quad i(\mathcal{L}_n) := \frac{l_n}{L_n}$$

and their asymptotic behavior. In order to prove that $\lim_{n\to\infty} i(\mathcal{L}_n) = +\infty$, it is convenient to express f_n and l_n in a direct way instead of by recurrence, for instance, by writing

Proposition 4: The following equalities hold:

(i) $f_n = \dfrac{nF_{n+1} + 2(n+1)F_n}{5}$ for $n \ge 2$;

(ii) $l_n = nF_{n-1}$ for $n \ge 3$.

Proof: ***(i)*** Indeed,

$$\frac{2F_3 + 2(2+1)F_2}{5} = 2 = f_2 \quad \text{and} \quad \frac{3F_4 + 2(3+1)F_3}{5} = 5 = f_3.$$

Now assume by induction that

$$f_{n+1} = \frac{(n-1)F_n + 2nF_{n-1}}{5} \quad \text{and} \quad f_{n-2} = \frac{(n-2)F_{n-1} + 2(n-1)F_{n-2}}{5}.$$

Then

$$f_n = f_{n-1} + f_{n-2} + F_n = \frac{(n+4)F_n + nF_{n-1} + (2n-2)(F_{n-1} + F_{n-2})}{5} = \frac{(2n+2)F_n + n(F_n + F_{n-1})}{5}.$$

(ii)
$$\begin{aligned} l_n &= f_{n-1} + f_{n-3} + F_{n-1} \\ &= \frac{(n-1)F_n + 2nF_{n-1} + (n-3)F_{n-2} + 2(n-2)F_{n-3} + 5F_{n-1}}{5} \\ &= \frac{(3n+4)F_{n-1} + (2n-4)F_{n-2} + (2n-4)F_{n-3}}{5} = nF_{n-1}. \ \square \end{aligned}$$

Furthermore, we recall that

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \quad \text{and} \quad L_n = \phi^n + \hat{\phi}^n, \tag{7}$$

(where $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$). Then we have

Theorem 6:

(i) $i(\Gamma_{n-1}) < i(\mathcal{L}_n) < i(\Gamma_n)$.

(ii) $\lim_{n\to\infty} i(\mathcal{L}_n) = +\infty$.

Proof: ***(i)*** We have to prove that $f_{n-1}L_n < l_nF_{n+1}$ and $l_nF_{n+2} < f_nL_n$ and that these inequalities are part of an increasing sequence of positive integers:

$$\dots f_{n-1}L_n < l_nF_{n+1} < l_nF_{n+2} < f_nL_n < f_nL_{n+1} < l_{n+1}F_{n+2} < l_{n+1}F_{n+3} < f_{n+1}L_{n+1}\cdots.$$

Now let $a_n := f_nF_{n+1} - f_{n-1}F_{n+2}$. We begin by showing that $a_n > 0$. Indeed, by direct computation we have $a_1 = f_1F_2 - f_0F_3 = 1$, $a_2 = f_2F_3 - f_1F_4 = 1$, and for $n \geq 3$,

$$a_n = f_{n-2}F_{n+1} + F_{n+1}F_n - f_{n-1}F_n = f_{n-2}F_{n-1} - f_{n-3}F_n + F_n^2 = a_{n-2} + F_n^2 > a_{n-2}.$$

In order to prove the first inequality, we have

$$\begin{aligned} l_nF_{n+1} - f_{n-1}L_n &= (f_{n-1} + f_{n-3} + F_{n-1})F_{n+1} - f_{n-1}(F_{n+1} + F_{n-1}) \\ &= (f_{n-3} + F_{n-1})F_{n+1} - f_{n-1}F_{n-1} \\ &= (f_{n-3} + F_{n-1})(F_n + F_{n-1}) - (f_{n-2} + f_{n-3} + F_{n-1})F_{n-1} \\ &= (f_{n-3} + F_{n-1})F_n - f_{n-2}F_{n-1} = a_{n-1} > 0. \end{aligned}$$

The second inequality is immediate for $n < 4$; for $n > 4$ we have

$$\begin{aligned} f_nL_n - l_nF_{n+2} &= -f_{n-1}F_{n-2} + f_{n-2}(F_{n+1} + F_{n-1}) - f_{n-3}F_{n+2} + F_{n+1}F_{n-2} \\ &= -(f_{n-2} + f_{n-3} + F_{n-1})F_{n-2} + f_{n-2}(3F_{n-1} + F_{n-2}) - f_{n-3}(3F_{n-1} + 2F_{n-2}) + F_{n+1}F_{n-2} \\ &= (-3f_{n-3}F_{n-2} - 3f_{n-3}F_{n-1}) + 3f_{n-2}F_{n-1} + F_{n+1}F_{n-2} - F_{n-1}F_{n-2} \\ &= 3(f_{n-2}F_{n-1} - f_{n-3}F_n) + F_nF_{n-2} = 3a_{n-2} + F_nF_{n-2} > 0. \end{aligned}$$

(ii) From (7) it follows that

$$i(\mathscr{L}_n) = \frac{l_n}{L_n} = \frac{nF_{n-1}}{L_n} \sim \frac{n}{\sqrt{5}\phi}. \quad \square$$

6. LUCAS SEMILATTICES

In [3] we studied a poset connected to Γ_n. In a similar way, the set of Lucas strings can be partially ordered with respect to the relation $\leq$ defined by $[a_1, \dots, a_n] \leq [b_1, \dots, b_n]$ if and only if $a_i \leq b_i$ for $i = 1, \dots, n$ for all Lucas strings $[a_1, \dots, a_n]$, $[b_1, \dots, b_n]$. Moreover,

$$[a_1, \dots, a_n] \vee [b_1, \dots, b_n] = [c_1, \dots, c_n],$$

where $c_1 = \max(a_i, b_i)$ for $i = 1, \dots, n$ if $[c_1, \dots, c_n]$ exists. The minimal element is $\hat{0} = [0, \dots, 0]$. The poset $(\hat{C}_n, \leq)$ is closed under *inf*, where $[a_1, \dots, a_n] \wedge [b_1, \dots, b_n] = [\min(a_1, b_1), \dots, \min(a_n, b_n)]$ and $\hat{0} = [0, \dots, 0]$. Thus, $(\hat{C}_n, \leq)$ is a meet-semilattice L_n.

By Theorem 1, the height of L_n, i.e., the maximum number of 1's in a Lucas string of length n is $\lfloor \frac{n}{2} \rfloor$.

Recall that in a semilattice S an *atom* is an element covering $\hat{0}$; the set of atoms is denoted by $\text{Atom}(S)$. A semilattice is *atomic* if for each $x \in S$ there exists a subset $A \subseteq \text{Atom}(S)$ such that $x = \vee A$; it is *strictly atomic* when for each element $x \in S$ there exists a unique $A \subseteq \text{Atom}(S)$ such that $x = \vee A$.

A semilattice is *simplicial* where every interval is isomorphic to a Boolean lattice. In [3] we proved that a finite semilattice S with $\hat{0}$ is strictly atomic if and only if it is simplicial. Moreover, every finite strictly atomic semilattice S is ranked, where the *rank* is the function $r : S \to \mathbf{N}$

defined by $r(x) = |A|$ if and only if $x = \vee A$. Finally, we proved that the characteristic polynomial of a finite strictly atomic semilattice S is $\chi(S, x) = \Sigma(-1)^k W_k(S) \cdot x^{h(S)-k}$, where W_k is a Whitney number of the second kind (i.e., the number of elements of S of rank k) and $h(S)$ is the height of S. All the properties of the Fibonacci semilattices also hold in this case. The difference concerns W_k and the height. Now it is

$$W_k(L_n) = \binom{n-k}{k} \cdot \frac{n}{n-k} \quad \text{and} \quad h(L_n) = \left\lfloor \frac{n}{2} \right\rfloor;$$

then we have

$$\chi(L_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \cdot \frac{n}{n-k} \cdot (-1)^k \cdot x^{\lfloor \frac{n}{2} \rfloor - k}.$$

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