A VARIANT OF PASCAL'S TRIANGLE

H. W. Gould*
West Virginia University, Morgantown, W. Va.

1. INTRODUCTION

Professor Charles A. Halijak has called my attention to the following interesting variant of Pascal's triangle [7]

$$
\text{(1.1)}\qquad
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & 3 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 1 & & 4 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 1 & & 5 & & 4 & & 6 & & 3 & & 1 & & & \\
 & & 1 & & 1 & & 6 & & 5 & & 10 & & 6 & & 4 & & 1 & & \\
 & 1 & & 1 & & 7 & & 6 & & 15 & & 10 & & 10 & & 4 & & 1 & \\
1 & & 1 & & 8 & & 7 & & 21 & & 15 & & 20 & & 10 & & 5 & & 1
\end{array}
$$

The law for formation is evident. One alternately adds together two elements or brings down a single element in order to obtain a new element in the next row. It appears that the elements turn out to be binomial coefficients. More interestingly, it appears that the elements in any row add to give a Fibonacci number: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, etc.

The object of the present note is to verify these observations and to develop some other relations suggested by the array of numbers.

2. RECURRENCE RELATIONS

We may symbolize the array (1.1) as follows:

$$
\begin{array}{ccccc}
 & & A_0^0 & & \\
 & A_0^1 & & A_1^1 & \\
A_0^2 & & A_1^2 & & A_2^2 \\
\end{array}
$$
$$\ldots \text{ etc.}$$

*Supported by National Science Foundation Research Grant GP-482.

If we let A_j^n, $j = 0, 1, 2, \ldots, n$, designate an arbitrary element of the array then we may use the defining recurrence relation (law of formation) to give an inductive definition of the array (1.1). Indeed we may say that the conditions

$$A_{2k+1}^{n+1} = A_{2k}^{n}, \tag{2.1}$$

$$A_{2k}^{n+1} = A_{2k-1}^{n} + A_{2k}^{n}, \tag{2.2}$$

$$A_j^n = 0, \quad j > n \text{ or } j < 0, \tag{2.3}$$

$$A_0^n = 1, \quad n = 0, 1, 2, \ldots, \quad A_1^1 = 1, \tag{2.4}$$

are sufficient to define the array (1.1). We may combine (2.1) and (2.2) into a single recurrence relation

$$A_j^{n+1} = A_{j-1}^{n} + \frac{1 + (-1)^j}{2} A_j^n \tag{2.5}$$

if we desire.

It is not difficult to conjecture (and prove by induction) that

$$A_{2k}^{n} = \binom{n-k}{k}, \tag{2.6}$$

$$A_{2k+1}^{n} = \binom{n-1-k}{k}, \tag{2.7}$$

and, again, these may be expressed in the single formula

$$A_j^n = \binom{n - \left[\frac{1}{2}(j+1)\right]}{\left[\frac{1}{2}(j)\right]} \tag{2.8}$$

where $[x]$ would mean the integral part of x (the "greatest integer in x").

3. FIBONACCI NUMBERS

The Fibonacci numbers, F_n, may be defined by the conditions $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Explicitly it is easy to show that

$$(3.1)\qquad F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} ,$$

and this well-known formula provides the clue to our next results. We have

Theorem 1. For the array (1.1) we have

$$(3.2)\qquad \sum_{j=0}^{n} A_j^n = F_{n+2} , \quad n \geq 0.$$

Proof.

$$\sum_{j=0}^{n} A_j^n = \sum_{k=0}^{[n/2]} A_{2k}^n + \sum_{k=0}^{[(n-1)/2]} A_{2k+1}^n$$

$$= F_{n+1} + F_n = F_{n+2}$$

as desired to show.

Next we may establish

Theorem 2. For the array (1.1) we have

$$(3.3)\qquad \sum_{j=0}^{n} (-1)^j A_j^n = F_{n-1} , \quad n \geq 1.$$

This would also be true for $n = 0$ if we extend the Fibonacci sequence backwards as is usually done. As for the proof, the same steps as used for Theorem 1 give us at once $F_{n+1} - F_n$ or F_{n-1} as claimed.

4. A GENERAL POLYNOMIAL

We now define the polynomial $A_n(x)$ by

$$(4.1)\qquad A_n(x) = \sum_{j=0}^{n} A_j^n x^j .$$

In view of (2.6) and (2.7) we have

$$(4.2)\qquad A_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{2k} + \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} x^{2k+1} .$$

The polynomial $A_n(x)$ satisfies a simple recurrence relation which we may find as follows. By means of (2.5) we have

$$\sum_{j=1}^{n+1} A_j^{n+1} x^j = \sum_{j=1}^{n+1} A_{j-1}^n x^j + \frac{1}{2} \sum_{j=1}^{n+1} A_j^n x^j + \frac{1}{2} \sum_{j=1}^{n+1} (-1)^j A_j^n x^j,$$

or

$$\sum_{j=0}^{n+1} A_j^{n+1} x^j = \sum_{j=0}^{n} A_j^n x^{j+1} + \frac{1}{2} \sum_{j=0}^{n} A_j^n x^j + \frac{1}{2} \sum_{j=0}^{n} A_j^n (-x)^j ,$$

or therefore

$$(4.3)\qquad 2 A_{n+1}(x) = (2x+1)A_n(x) + A_n(-x).$$

It would be possible to set down a closed expression for $A_n(x)$ by means of the summation formula

$$(4.4)\qquad \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k = \frac{u^{n+1} - 1}{(u-1)(1+u)^n} , \quad x = \frac{-u}{(1+u)^2} ,$$

but this does not seem to simplify very nicely. It would be of interest to evaluate $A_n(x)$ for values of x other than $x = 1$ and $x = -1$, however. We remark that (4.4) may be written in the alternative form

$$(4.5)\qquad \sum_{k=0}^{[n/2]} \binom{n-k}{k} 2^{n-2k} x^k = \frac{u^{n+1} - v^{n+1}}{u-v}, \quad \begin{cases} u = 1 + \sqrt{x+1} , \\ v = 1 - \sqrt{x+1} . \end{cases}$$

5. LUCAS NUMBER VARIANT OF PASCAL'S TRIANGLE

Using the same law of formation as we imposed to generate rows in (1.1) we may form the array

(5.1)

$$
\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 2 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & 2 & & & & & & & \\
 & & & & & & 1 & & 1 & & 3 & & 2 & & & & & & \\
 & & & & & 1 & & 1 & & 4 & & 3 & & 2 & & & & & \\
 & & & & 1 & & 1 & & 5 & & 4 & & 5 & & 2 & & & & \\
 & & & 1 & & 1 & & 6 & & 5 & & 9 & & 5 & & 2 & & & \\
 & & 1 & & 1 & & 7 & & 6 & & 14 & & 9 & & 7 & & 2 & & \\
 & 1 & & 1 & & 8 & & 7 & & 20 & & 14 & & 16 & & 7 & & 2 & \\
1 & & 1 & & 9 & & 8 & & 27 & & 20 & & 30 & & 16 & & 9 & & 2
\end{array}
$$

where the only difference is that we use a different initial value in the second spot on the second row. Let us symbolize the array by using the notation B_j^n in the same way we discussed A_j^n. We first observe that the rows add to give the Lucas numbers: 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, etc. In other words, we have, evidently, the two relations

$$\sum_{j=0}^{n} B_j^n = L_{n+1} , \tag{5.2}$$

and

$$\sum_{j=0}^{n} (-1)^j B_j^n = L_{n-2} , \tag{5.3}$$

where the Lucas numbers are defined by

$$L_0 = 2, \; L_1 = 1, \; L_{n+1} = L_n + L_{n-1} .$$

Explicitly, we have

$$L_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} . \tag{5.4}$$

The array (5.1) may be specified by the conditions

(5.5) $$B_{2k+1}^{n+1} = B_{2k}^{n},$$

(5.6) $$B_{2k}^{n+1} = B_{2k-1}^{n} + B_{2k}^{n},$$

(5.7) $$B_j^n = 0, \quad j > n \text{ or } j < 0,$$

(5.8) $$B_0^n = 1, \quad n = 0, 1, 2, \ldots, \quad B_1^1 = 2,$$

We may combine (5.5) and (5.6) by writing

(5.9) $$B_j^{n+1} = B_{j-1}^{n} + \frac{1 + (-1)^j}{2} B_j^n,$$

and we conjecture on the basis of (5.4) and the above that

(5.10) $$B_{2k}^{n} = \frac{n}{n-k}\binom{n-k}{k},$$

and

(5.11) $$B_{2k+1}^{n} = \frac{n-1}{n-1-k}\binom{n-1-k}{k}, \quad B_1^1 = 2.$$

The two relations could be combined into a single expression, however, the result is not as simple as was the case with (2.8).

Associated with the Lucas variant of Pascal's triangle we may consider the polynomial

(5.12) $$B_n(x) = \sum_{j=0}^{n} B_j^n x^j .$$

In view of the recurrence (5.9), just as in the case of (2.5), we may show that the companion relation to (4.3) is

(5.13) $$2B_{n+1}(x) = (2x + 1)B_n(x) + B_n(-x).$$

The formula

(5.14) $$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k} x^k = 2\,\frac{u^n + v^n}{u + v},$$

where

$$u = 1 + \sqrt{x+1}\,, \quad v = 1 - \sqrt{x+1}\,,$$

could be used to give a closed form for (5.12).

6. GENERALIZATION

A general array suggested by the two cases we have discussed may be set down as follows:

$$\begin{array}{c}
a \\
a \quad b \\
a \quad a \quad b \\
a \quad a \quad a+b \quad b \\
a \quad a \quad 2a+b \quad a+b \quad b \\
a \quad a \quad 3a+b \quad 2a+b \quad a+2b \quad b \\
a \quad a \quad 4a+b \quad 3a+b \quad 3a+3b \quad a+2b \quad b \\
a \quad a \quad 5a+b \quad 4a+b \quad 6a+4b \quad 3a+3b \quad a+3b \quad b \\
a \quad a \quad 6a+b \quad 5a+b \quad 10a+5b \quad 6a+4b \quad 4a+6b \quad a+3b \quad b
\end{array} \tag{6.1}$$

We may define the array by the following conditions:

$$C_0^0 = 0_0^1 = a, \quad C_1^1 = b, \tag{6.2}$$

$$C_j^n = 0, \text{ if } j > n \text{ or } j < 0, \tag{6.3}$$

$$C_j^{n+1} = C_{j-1}^n + \frac{1 + (-1)^j}{2} C_j^n, \quad n \geq 1, \ j \geq 0. \tag{6.4}$$

For the recurrence (6.4) we have imposed the condition that $n \geq 1$. We do this for the following reason. Choose $C_0^0 = a$. Then, by (6.4), we have $C_0^1 = C_{-1}^0 + C_0^0 = C_0^0$ provided we impose (6.3). But then we have $C_1^1 = C_0^0 + 0 = a$, not b. To avoid this difficulty we may define $C_1^1 = b$. For the next row we have then

$$C_0^2 = C_{-1}^1 + C_0^1 = 0 + a = a,$$

$$C_1^2 = C_0^1 + 0 = a,$$

$$C_2^2 = C_1^1 + C_2^1 = b + 0 = b.$$

Thus a simple condition to attach to the recurrence is that $n \geq 1$. Another way to proceed would be to define $C_0^0 = b$ and $C_0^1 = a$. Everything would be the same except the topmost element, and the recurrence would hold in all cases. However, then the niceness of the array (5.1) would suffer by having $B_0^0 = 2$ which would not fit so well with the Lucas numbers. There is a certain arbitrariness in combining the various properties which seem to be of interest. Because of this, the reader may find it instructive to examine other possible definitions.

From our definition it is easy to show that the row-sums are given by

$$(6.5) \qquad S_n(a, b) = \sum_{j=0}^{n} C_j^n = aF_{n+1} + bF_n, \quad n \geq 0,$$

in terms of the Fibonacci numbers. Thus we find $S_n(1, 1) = F_{n+1} + F_n = F_{n+2}$ as before. Also, $S_n(1, 2) = F_{n+1} + 2F_n = F_{n+1} + F_n + F_n = F_{n+2} + F_n = L_{n+1}$ as before. (It is easily proved that $L_n = F_{n+1} + F_{n-1}$.)

The arbitrariness involved in the first two rows, however, shows up again when we consider the alternating row-sums. We find these are

$$T_n(a, b) = \sum_{j=0}^{n} (-1)^j C_j^n = b,\ a - b,\ b,\ a,\ a + b,\ 2a + b,\ 3a + 2b,\ \ldots$$

and, except for the first such sum, we can show that

$$(6.6) \qquad T_n(a, b) = \sum_{j=0}^{n} (-1)^j C_j^n = aF_{n-2} + bF_{n-3}, \quad n \geq 1.$$

Remark: The usual definition of Fibonacci numbers with negative index is

$$F_{-n} = (-1)^{n-1} F_n$$

so that the doubly infinite sequence of Fibonacci numbers is

$$\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$$

In view of this, the formula (6.7) breaks down for $n = 0$ as it then gives the value $-a + 2b$ instead of the value b. However, for $n \geq 1$ agreement is found. In particular, when $a = 1 = b$, we have $T_n(1, 1) = F_{n-2} + F_{n-3} = F_{n-1}$ as in (3.3) A similar result holds for the Lucas number variant (5.1).

7. FURTHER RELATIONS FOR THE POLYNOMIAL $A_n(x)$

By means of relation (4.2) we may show readily that $A_n(x)$ satisfies the second-order recurrence relation

$$A_{n+2}(x) = A_{n+1}(x) + x^2 A_n(x) \,. \tag{7.1}$$

In fact we have

$$A_{n+1}(x) = \sum_{0 \leq k \leq \frac{n+1}{2}} \binom{n+1-k}{k} x^{2k} + \sum_{0 \leq k \leq \frac{n}{2}} \binom{n-k}{k} x^{2k+1},$$

and

$$x^2 A_n(x) = \sum_{0 \leq k \leq \frac{n}{2}} \binom{n-k}{k} x^{2k+2} + \sum_{0 \leq k \leq \frac{n-1}{2}} \binom{n-1-k}{k} x^{2k+3}$$

$$= \sum_{1 \leq k \leq \frac{n+2}{2}} \binom{n+1-k}{k-1} x^{2k} + \sum_{1 \leq k \leq \frac{n+1}{2}} \binom{n-k}{k-1} x^{2k+1}$$

Using the fact that

$$\binom{p-k}{k} + \binom{p-k}{k-1} = \binom{p+1-k}{k},$$

it then readily follows that

$$A_{n+1}(x) + x^2 A_n(x) = \sum_{0 \le k \le \frac{n+2}{2}} \binom{n+2-k}{k} x^{2k} + \sum_{0 \le k \le \frac{n+1}{2}} \binom{n+1-k}{k} x^{2k+1}$$

$$= A_{n+2}(x).$$

Associated with $A_n(x)$ we may next introduce a related polynomial $K_n(x)$ defined by

$$K_n(x) = x^n A_n(\tfrac{1}{x}) = \sum_{j=0}^{n} A_j^n x^{n-j} . \tag{7.2}$$

Relation (7.1) then becomes

$$K_{n+2}(x) = xK_{n+1}(x) + K_n(x), \text{ with } K_0(x) = 1,\ K_1(x) = x + 1 . \tag{7.3}$$

This recurrence relation is of the same form as one studied by Catalan [4]. This is mentioned by Byrd [3].

It may be of interest to indicate how the Q-matrix technique [1] may be applied to a study of $K_n(x)$. Define

$$Q = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} . \tag{7.4}$$

Then

$$Q^n = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix}, \quad n \ge 1, \tag{7.5}$$

where the f's are Fibonacci polynomials defined by

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad f_0(x) = 0,\ f_1(x) = 1. \tag{7.6}$$

It is easily shown that

$$K_n(x) = f_{n+1}(x) + f_n(x) . \tag{7.7}$$

From (7.7) we have next

$$(-1)^{j+1} K_j(x) = (-1)^{j+1} f_{j+1}(x) - (-1)^j f_j(x)$$

whence

$$\sum_{j=0}^{n} (-1)^j K_j(x) = (-1)^n f_{n+1}(x) \quad , \tag{7.8}$$

so that the Fibonacci polynomials $f_n(x)$ may be expressed in terms of the K or A polynomials very easily.

We next observe that (7.5) and (7.7) yield

$$Q^n + Q^{n-1} = \begin{pmatrix} K_n(x) & K_{n-1}(x) \\ K_{n-1}(x) & K_{n-2}(x) \end{pmatrix} . \tag{7.9}$$

From this result it is possible to evaluate the determinant of the K's as follows. To begin with, $|Q^n| = |Q|^n = (-1)^n$. Then we find that

$$\begin{aligned} \begin{vmatrix} K_n(x) & K_{n-1}(x) \\ K_{n-1}(x) & K_{n-2}(x) \end{vmatrix} = |Q^n + Q^{n-1}| &= |Q^{n-1}(Q + I)| \,, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \\ &= |Q^{n-1}| \cdot |Q + I| \\ &= (-1)^{n-1} x \quad . \end{aligned}$$

We may state the result more elegantly in the form

$$\begin{vmatrix} K_{n+1}(x) & K_n(x) \\ K_n(x) & K_{n-1}(x) \end{vmatrix} = (-1)^n x \quad . \tag{7.10}$$

This may be compared with the relation

$$\begin{vmatrix} F_{n+a} & F_{n+a+b} \\ F_n & F_{n+b} \end{vmatrix} = (-1)^n F_a F_b \tag{7.11}$$

for the ordinary Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$) which was posed as a problem in the American Mathematical Monthly[8]. In particular, this raises the question about a similar generalization of the determinant (7.10). Indeed, we shall now prove by induction that

$$(7.12)\qquad \begin{vmatrix} K_{n+a}(x) & K_{n+a+b}(x) \\ K_n(x) & K_{n+b}(x) \end{vmatrix} = (-1)^n \begin{vmatrix} K_a & K_{a+b} \\ K_0 & K_b \end{vmatrix} = (-1)^n(K_aK_b - K_{a+b}).$$

This will be true for all integers if we define

$$(7.13)\qquad K_{-n}(x) = K_{n-1}(-x)$$

as is suggested by recurrence relation (7.3).

As for the proof of (7.12), we may first show that (as is obvious for $n = 0$)

$$(7.14)\qquad K_{n+1}K_{n+b} + K_nK_{n+b+1} = (-1)^n(K_1K_b - K_0K_{1+b}) \quad ,$$

where, for brevity, we omit writing x which will remain unchanged. Now, in fact, by means of (7.3) we have

$$\begin{aligned}(-1)^n\left[K_{n+1}K_{n+b} - K_nK_{n+b+1}\right] &= (-1)^n\left[K_{n+1}(K_{n+b+2} - xK_{n+b+1}) - K_nK_{n+b+1}\right] \\ &= (-1)^n\left[K_{n+1}K_{n+b+2} - (xK_{n+1} + K_n)K_{n+b+1}\right] \\ &= (-1)^{n+1}\left[K_{n+2}K_{n+b+1} - K_{n+1}K_{n+b+2}\right] ,\end{aligned}$$

so that the expression is unchanged when n is replaced by $n + 1$. By induction, then, relation (7.14) follows.

In the same way, we could show that (7.12) holds for $a = 2$, that is,

$$(7.15)\qquad K_{n+2}K_{n+b} - K_nK_{n+b+2} = (-1)^n(K_2K_b - K_0K_{2+b}) \quad .$$

We may complete the argument by an induction on a. Suppose that (7.12) holds for fixed n, b and up to a certain value of $a(\geq 1)$. Then

$$K_{n+a}K_{n+b} - K_nK_{n+a+b} = (-1)^n\ K_aK_b - K_0K_{a+b}$$

and

$$K_{n+a-1}K_{n+b} - K_nK_{n+a-1+b} = (-1)_n\ K_{a-1}K_b - K_0K_{a-1+b} \quad ,$$

and if we multiply the first of these by x, add to the second, and recall the basic recurrence relation (7.3), we obtain precisely

$$K_{n+a+1}K_{n+b} - K_nK_{n+a+1+b} = (-1)^n K_{a+1}K_b - K_0K_{a+1+b} \quad ,$$

so that the induction goes through. This proof is nothing more than a variant of a similar proof for Problem E 1396, relation (7.11) above, suggested by Mr. John H. Biggs who was then a graduate student at West Virginia University. Clearly the same technique may be used in other cases where a recurrence relation of a suitable sort is presupposed. Thus (7.12) also holds for $f_n(x)$ in place of $K_n(x)$.

We should like to mention still another interesting relation involving the polynomial $K_n(x)$. The reader may find it worthwhile to carry out an inductive proof that

$$K_n(x) + (-1)^a K_{n+2a}(x) + xK_{n+a}(x) = 0 \quad . \tag{7.16}$$

When $a = 1$ this becomes again (7.3). It is possible to base a proof of (7.12) on this relation. The idea traces back as far as George Boole [2], and may have further unsuspected possibilities. Under miscellaneous propositions, in Chapter XII, pp. 229-231, Boole uses an invariance technique which may be of interest. By (7.16) we have (omitting x for brevity)

$$K_n + (-1)^a K_{n+2a} = -xK_{n+a} \quad .$$

This relation being true for all integers n, a, we next replace n by $n + b$, and we have, for arbitrary n, a, b,

$$K_{n+b} + (-1)^a K_{n+2a+b} = -xK_{n+a+b} \quad .$$

Here, $-x$ plays the part of the number p in Boole's argument. We may eliminate $-x$ from the last two relations by multiplying the former by K_{n+a+b}, the latter by K_{n+a}, and equating the resulting left-hand members. This yields

$$K_{n+a}K_{n+b} + (-1)^a K_{n+a}K_{n+2a+b} = (-1)^a K_{n+2a}K_{n+a+b} + K_nK_{n+a+b} \quad .$$

Multiplying through by $(-1)^n$ we have, transposing terms,

$$(-1)^n \left[K_{n+a}K_{n+b} - K_nK_{n+a+b}\right] = (-1)^{n+a} \left[K_{n+2a}K_{n+a-b} - K_{n+a}K_{n+2a+b}\right]. \tag{7.17}$$

Call the left-hand member of this $F(n)$. Then the crux of Boole's argument would be that (7.17) asserts that $F(n) = F(n + a)$. This being so for a perfectly arbitrary integer a, as we supposed to begin with, then it follows that $F(n)$ is invariant with respect to n. Hence we have only to set $n = 0$, and we find that

$$F(n) = F(0) = K_a K_b - K_0 K_{a+b}$$

and this of course is precisely what we claimed in relation (7.12).

The beauty of Boole's method is that one may oftentimes begin with a non-linear recurrence relation (difference equation), such as (7.12) is indeed, and relate this back to a linear relation, as (7.16) actually is. The method is especially useful in the study of determinants of polynomials which satisfy suitable recurrence relations.

The relations (7.11) and (7.12) may be called Turán relations, and the reader is referred to [5, 6] for pertinent journal references and some variations. A detailed bibliography on the Turán expressions (and Turán inequalities) would contain over 110 references to journal articles and books according to the author's current file on the literature.

REFERENCES

1. S. L. Basin and V. E. Hoggatt, A primer on the Fibonacci sequence, Part II, Fibonacci Quarterly, 1(1963), No. 2, 61-68.
2. George Boole, Calculus of Finite Differences, Fourth Edition, Chelsea Reprint, 1958, N. Y.
3. Paul F. Byrd, Expansion of analytic functions in polynomials associated with Fibonacci numbers, Fibonacci Quarterly, 1(1963), No. 1, 16-29.
4. E. Catalan, Notes sur la théorie des fractions continues et sur certaines séries, Mem. Acad. R. Belgique, 45(1883), 1-82.
5. H. W. Gould, A general Turán expression for the Zeta function, Canadian Mathematical Bulletin, 6(1963), 359-366.
6. H. W. Gould, The functional operator $Tf(x) = f(x+a)f(x+b) - f(x)\cdot f(x+a+b)$, Mathematics Magazine, 37(1964), pp. 38-46, January issue.

7. P. C. Parks, A new proof of the Hurwitz stability criterion by the second method of Liapunov, with applications to "optimum" transfer functions. Proceedings, Joint Automatic Control Conference, University of Minnesota, June, 1963. American Institute of Chemical Engineers, N. Y. Note page 475.

8. Problem E 1396, American Mathematical Monthly, 67(1960), 81-82; and solution, ibid., 67(1960), 694.

xxxxxxxxxxxxxxx