A GENERALIZATION OF BINET'S FORMULA AND SOME OF ITS CONSEQUENCES

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1. A Generalization of Binet's Formula

We derive a simple generalization of Binet's formula for Fibonacci and Lucas numbers. From the equations

$$F_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right], \tag{1.1}$$

and

$$L_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m, \tag{1.2}$$

we have at once,

$$\frac{L_m + \sqrt{5}F_m}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^m,$$

and

$$\frac{L_m - \sqrt{5}F_m}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^m.$$

Raising both sides to the n^{th} power, and combining the results by means of (1.1) and (1.2), we find

$$F_{nm} = \frac{1}{\sqrt{5}}\left[\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n - \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n\right], \tag{1.3}$$

and

$$L_{nm} = \left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n + \left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n, \tag{1.4}$$

which are the desired generalizations. Equations (1.3) and (1.4) reduce to equations (1.1) and (1.2), respectively, when $m = 1$. Note that, in the right-hand sides of equations (1.3) and (1.4), m and n can be interchanged.

A number of interesting results can be obtained from (1.3) and (1.4). Note, for instance, that one has

$$(L_m + \sqrt{5}F_m)^n = L_m^n + \binom{n}{1}L_m^{n-1}\sqrt{5}F + \binom{n}{2}L_m^{n-2}5F_m^2 + \cdots + (\sqrt{5})^n F_m^n, \tag{1.5}$$

and

$$(L_m - \sqrt{5}F_m)^n = L_m^n - \binom{n}{1}L_m^{n-1}\sqrt{5}F_m + \binom{n}{2}L_m^{n-2}5F_m^2 - \cdots + (-1)^n(\sqrt{5})^n F_m^n. \tag{1.6}$$

If these results are substituted into (1.3), we see that L_m^n cancels out. The remaining terms all have a nonzero power of F_m, and we have found a simple proof of the known result that F_{nm} is divisible by F_m and F_n. For Lucas numbers, we observe that cancellation of the last term in (1.5) and (1.6) will take place only if n is odd. Hence, L_{mn} is divisible by L_m only if n is odd.

With the aid of (1.3) and (1.4), it is possible to obtain some appealing generating functions for Fibonacci and Lucas numbers. We proceed as follows:

$$\sum_{n=0}^{\infty} \frac{F_{nm} t^n}{n!} = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \frac{\left(\frac{L_m + \sqrt{5}F_m}{2}\right)^n t^n}{n!} - \sum_{n=0}^{\infty} \frac{\left(\frac{L_m - \sqrt{5}F_m}{2}\right)^n t^n}{n!}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\exp\left\{\left(\frac{L_m + \sqrt{5}F_m}{2}\right)t - \exp\left(\frac{L_m - \sqrt{5}F_m}{2}\right)t\right\}\right]$$

$$= \frac{2}{\sqrt{5}} \exp\left(\frac{L_m t}{2}\right) \sinh\left(\frac{\sqrt{5}F_m}{2} t\right) \quad (1.7)$$

An identical procedure gives

$$2 \exp\left(\frac{L_m}{2} t\right) \cosh\left(\frac{\sqrt{5}F_m}{2} t\right) = \sum_{n=0}^{\infty} \frac{L_{mn} t^n}{n!}. \quad (1.8)$$

Some curious formulas may be obtained from (1.7) and (1.8). From (1.7), for example, one has

$$F_m t \exp\left(\frac{L_m}{2} t\right) = \frac{\sqrt{5}F_m}{2} t \operatorname{csch}\left(\frac{\sqrt{5}F_m}{2} t\right) \sum_{n=0}^{\infty} \frac{F_{nm} t^n}{n!}. \quad (1.9)$$

Using the expansion [1],

$$z \operatorname{csch} z = \sum_{k=0}^{\infty} -\frac{2(2^{2k-1} - 1)B_{2k}}{(2k)!} z^{2k}, \quad |z| < \pi, \quad (1.10)$$

where the B_{2k} are Bernoulli numbers, and forming the Cauchy product, we have

$$F_m t \exp\left(\frac{L_m}{2} t\right) = -\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{F_{nm} 2(2^{2k-1} - 1)B_{2k} 5^k F_m^{2k} t^{n+2k}}{n!(2k)!2^{2k}}$$

$$= -\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{2(2^{2k-1} - 1)5^k F_m^{2k} F_{(n-2k)m} B_{2k} t^n}{(n - 2k)!(2k)!2^{2k}},$$

where $[n/2]$ designates the greatest integer in $n/2$.

Expanding the exponential and equating corresponding powers of t, we get

$$L_m^{n-1} = -\frac{1}{nF_m} \sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{n-2k}(2^{2k-1} - 1)5^k F_m^{2k} F_{(n-2k)m} B_{2k}, \quad (1.11)$$

which gives powers of Lucas numbers in terms of Fibonacci and Bernoulli numbers.

From (1.8), one has

$$2 \cosh\left(\frac{\sqrt{5}F_m}{2} t\right) = \exp\left(-\frac{L_m}{2} t\right) \sum_{n=0}^{\infty} \frac{L_{mn} t^n}{n!}.$$

Expanding the exponential term, forming the Cauchy product, and separating the even part, since the left-hand side is even, one finds

$$F_m^{2n} = \frac{2^{2n-1}}{5^n} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k 2^{-k} L_m^k L_{m(2n-k)}, \quad (1.12)$$

which gives even powers of Fibonacci numbers in terms of Lucas numbers.

In (1.8), change t to $-t$ and add the result to (1.8) to obtain

$$2\cosh\left(\frac{L_m}{2}t\right)\cosh\left(\frac{\sqrt{5}F_m}{2}t\right) = \sum_{n=0}^{\infty}\frac{L_{2nm}t^{2n}}{(2n)!},$$

which may be written

$$2\cosh\left(\frac{\sqrt{5}F_m}{2}t\right) = \operatorname{sech}\left(\frac{L_m}{2}t\right)\sum_{n=0}^{\infty}\frac{L_{2nm}t^{2n}}{(2n)!}.$$

Using the expansion [1],

$$\operatorname{sech} z = \sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}z^{2n}, \qquad |z| < \tfrac{1}{2}\pi$$

where the E_{2n} are Euler numbers, we find

$$F^{2n} = \frac{1}{5^n}\sum_{k=0}^{n}\binom{2n}{2k}2^{2k-1}L_{2km}L_m^{2n-2k}E_{2n-2k}, \tag{1.13}$$

which gives even powers of Fibonacci numbers in terms of Lucas and Euler numbers.

Byrd [4], [5] has obtained expressions for Fibonacci and Lucas numbers which bear some resemblance to the expressions obtained by the author.

Now, observe that

$$\left(\frac{L_m + \sqrt{5}F_m}{2}\right)\left(\frac{L_m - \sqrt{5}F_m}{2}\right) = (-1)^m. \tag{1.14}$$

This relation can be used to advantage to obtain sums of reciprocals of Fibonacci and Lucas numbers. For this purpose, it is convenient to introduce the abbreviations:

$$a_m = \frac{L_m + \sqrt{5}F_m}{2}, \quad a_1 = a = \frac{1+\sqrt{5}}{2}, \tag{1.15}$$

$$b_m = \frac{L_m - \sqrt{5}F_m}{2}, \quad b_1 = b = \frac{1-\sqrt{5}}{2}. \tag{1.16}$$

We define the *Lambert series* as

$$L(\beta) = \sum_{n=1}^{\infty}\frac{\beta^n}{1-\beta^n}, \qquad |\beta| < 1, \tag{1.17}$$

and note that

$$L(\beta) - L(\beta^2) = \sum_{n=1}^{\infty}\frac{\beta^n}{1-\beta^n} - \sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\beta^{2n}} = \sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\beta^{2n}}. \tag{1.18}$$

We will make use of *Jacobi's identity* [2]

$$\theta_3^2(q) = 1 + 4\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}, \qquad |q| < 1, \tag{1.19}$$

where

$$\theta_3(q) = \sum_{n=-\infty}^{\infty}q^{n^2},$$

is a special case of the *third theta function* of Jacobi. Jacobi's *second theta function*:

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2},$$

is related to the third theta function through the identity

$$\theta_3^2(q^2) + \theta_2^2(q^2) = \theta_3^2(q). \tag{1.20}$$

Recalling (1.14), observe that we have

$$\sum_{n=1}^{\infty} \frac{1}{a_m^n - b_m^n} = \sum_{n=1}^{\infty} \frac{b_m^n}{(-1)^{nm} - b_m^{2n}},$$

which, for even m, gives at once

$$\sum_{n=1}^{\infty} \frac{1}{F_{2nm}} = \sqrt{5}[L(b_{2m}) - L(b_{2m}^2)], \tag{1.21}$$

while, for odd m, remembering that b_{2m+1} is negative, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{(2n+1)(2m+1)}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-b_{2m+1})^{2n+1}}{1 + (-b_{2m+1})^{4n+2}}$$

$$= \frac{\sqrt{5}}{4}[\theta_3^2(-b_{2m+1}) - \theta_3^2(b_{2m+1}^2)] = \frac{\sqrt{5}}{4}\,\theta_2^2(b_{2m+1}^2). \tag{1.22}$$

Equations (1.21) and (1.22) are generalizations of results obtained by Landau [8].

For Lucas numbers, one has

$$\sum_{n=1}^{\infty} \frac{1}{a_m^n + b_m^n} = \sum_{n=1}^{\infty} \frac{b_m^n}{(-1)^{nm} + b_m^{2n}},$$

which, for even m, gives

$$\sum_{n=1}^{\infty} \frac{1}{L_{2nm}} = \frac{1}{4}[\theta_3^2(b_{2m}) - 1], \tag{1.23}$$

while, for odd m, we get

$$\sum_{n=0}^{\infty} \frac{1}{L_{(2n+1)(2m+1)}} = \sum_{n=0}^{\infty} \frac{(-b_{2m+1})^{2n+1}}{1 - (-b_{2m+1})^{4n+2}}$$

$$= L(-b_{2m+1}) - 2L(b_{2m+1}^2) + L(b_{2m+1}^4). \tag{1.24}$$

The last equality above is established in a manner wholly analogous to equation (1.18).

Many more relations can be established by simply imitating the procedures used for ordinary Fibonacci and Lucas numbers. The only change is to replace a and b by a_m and b_m. In particular, Borwein & Borwein [2], and Bruckman [3] give a host of such relations.

2. A Class of Series for the Arc Tangent

In reference [6], we made use of Chebyshev polynomials of the first and second kinds

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n], \tag{2.1}$$

$$U_n(x) = \frac{1}{2}\left[\frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{\sqrt{x^2 - 1}}\right] \tag{2.2}$$

to establish, with $x = \sqrt{5}/2$, the relations

$$F_{2n} = \frac{1}{\sqrt{5}} U_{2n-1}\left(\frac{\sqrt{5}}{2}\right), \; n \geq 1, \tag{2.3}$$

$$F_{2n+1} = \frac{2}{\sqrt{5}} T_{2n+1}\left(\frac{\sqrt{5}}{2}\right). \tag{2.4}$$

$$L_{2n} = 2T_{2n}\left(\frac{\sqrt{5}}{2}\right), \tag{2.5}$$

$$L_{2n+1} = U_{2n}\left(\frac{\sqrt{5}}{2}\right). \tag{2.6}$$

Equations (2.5) and (2.6) were given in a different guise.

In reference [6], we also established the two series for the arc tangent:

$$\tan^{-1}\alpha = \frac{2}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{(-1)^n T_{2n+1}(x) t^{2n+1}}{5^n(2n + 1)}, \tag{2.7}$$

with

$$t = \frac{\sqrt{5}\alpha}{x + \sqrt{x^2 + \alpha^2}}, \tag{2.8}$$

and

$$\tan^{-1}\alpha = 4 \sum_{n=0}^{\infty} \frac{(-1)^n T_{2n+1}^2(x)}{(2n + 1)(t + \sqrt{t^2 + 1})^{2n+1}}, \tag{2.9}$$

with

$$t = \frac{x^2}{\alpha}(1 + \sqrt{1 + [\alpha^2(2x^2 - 1)/x^4]}). \tag{2.10}$$

These series give, with $x = \sqrt{5}/2$, the results

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} t^{2n+1}}{5^n(2n + 1)}, \tag{2.11}$$

with

$$t = \frac{2\alpha}{1 + \sqrt{1 + (4\alpha^2/5)}}, \tag{2.12}$$

and

$$\tan^{-1}\alpha = 5 \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n + 1)(t + \sqrt{t^2 + 1})^{2n+1}}, \tag{2.13}$$

with

$$t = \frac{5}{4\alpha}(1 + \sqrt{1 + (24\alpha^2/25)}). \tag{2.14}$$

To generalize these equations, we need an analogue of (1.3) for Chebyshev polynomials of the first kind.

We know that

$$T_m(\cos\theta) = \cos m\theta. \tag{2.15}$$

Let $m\theta = \phi$, and we have

$$T_n[T_m(\cos\theta)] = T_n(\cos\phi) = \cos n\phi = \cos nm\theta,$$

or

$$T_n[T_m(z)] = T_{nm}(z), \tag{2.16}$$

which is the desired relation.

For n and m odd and $z = \sqrt{5}/2$, (2.16) gives

$$T_{2n+1}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \frac{\sqrt{5}}{2}F_{(2n+1)(2m+1)}, \tag{2.17}$$

while, for even m, we get

$$T_n(\tfrac{1}{2}L_{2m}) = \tfrac{1}{2}L_{2nm}. \tag{2.18}$$

Similar relations may be obtained directly from (2.1) and (2.2):

$$T_{2n+1}\left(i\frac{\sqrt{5}}{2}F_{2m}\right) = \frac{(-1)^n i\sqrt{5}}{2}F_{(2n+1)2m}, \tag{2.19}$$

$$T_n(\tfrac{1}{2}iL_{2m+1}) = \tfrac{1}{2}i^n L_{n(2m+1)}, \tag{2.20}$$

$$T_{2n}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \tfrac{1}{2}L_{2n(2m+1)}, \tag{2.21}$$

$$U_{2n-1}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \sqrt{5}\,\frac{F_{2n(2m+1)}}{L_{2m+1}}, \tag{2.22}$$

$$U_{2n-1}\left(i\frac{\sqrt{5}}{2}F_{2m}\right) = (-1)^{n+1}i\sqrt{5}\,\frac{F_{4nm}}{L_{2m}}, \tag{2.23}$$

$$U_{n-1}(\tfrac{1}{2}L_{2m}) = \frac{F_{2nm}}{F_{2m}}, \tag{2.24}$$

$$U_{n-1}(\tfrac{1}{2}iL_{2m+1}) = i^{n-1}\frac{F_{n(2m+1)}}{F_{2m+1}}, \tag{2.25}$$

$$U_{2n}\left(\frac{\sqrt{5}}{2}F_{2m+1}\right) = \frac{L_{(2n+1)(2m+1)}}{L_{2m+1}}. \tag{2.26}$$

In equations (2.19), (2.20), (2.23), and (2.25), i is the imaginary unit.

Changing x to $T_{2m+1}(x)$ in (2.7) through (2.10), letting $x = \sqrt{5}/2$, and using (2.16), (2.4), and (2.17), we find

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n F_{(2n+1)(2m+1)}t^{2n+1}}{5^n(2n+1)}, \tag{2.27}$$

with

$$t = \frac{2\alpha}{F_{2m+1} + \sqrt{F_{2m+1}^2 + (4/5)\alpha^2}}, \tag{2.28}$$

and

$$\tan^{-1}\alpha = 5\sum_{n=0}^{\infty}\frac{(-1)^n F_{(2n+1)(2m+1)}^2}{(2n+1)(t+\sqrt{t^2+1})^{2n+1}}, \tag{2.29}$$

with

$$t = \frac{5F_{2m+1}^2}{4\alpha}\left(1 + \sqrt{1 + [(4\alpha/5)^2((5/2)F_{2m+1}^2 - 1)/F_{2m+1}^4]}\right). \tag{2.30}$$

Observe that the right-hand sides of (2.27) and (2.29) are independent of m. Equations (2.27) through (2.30) reduce to (2.11) through (2.14) when $m = 0$.

Series (2.27) and (2.29) converge most rapidly for $m = 0$. The reader should have no trouble showing that, as m increases without bound, (2.27) and (2.29) degenerate into Gregory's series:

$$\tan^{-1}\alpha = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{2n+1}, \quad |\alpha| \le 1. \tag{2.13}$$

Equations (2.27) through (2.30) provide a class of series for the arc tangent whose convergence lies between those of series (2.11) and (2.13) and that of series (2.31).

3. Some Series for π

The series we obtained in the previous section can be used to obtain some curious expressions for π.

For instance, (2.27) through (2.30) give, with $\alpha = 1$,

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n F_{(2n+1)(2m+1)}}{5^n(2n+1)} \frac{2^{2n+1}}{(F_{2m+1} + \sqrt{F_{2m+1}^2 + (4/5)})^{2n+1}}, \tag{3.1}$$

and

$$\frac{\pi}{4} = 5 \sum_{n=0}^{\infty} \frac{(-1)^n F_{(2n+1)(2m+1)}^2}{(2n+1)(t + \sqrt{t^2+1})^{2n+1}}, \tag{3.2}$$

with

$$t = \frac{5F_{2m+1}^2}{4}\left(1 + \sqrt{1 + [(4/5)^2((5/2)F_{2m+1}^2 - 1)/F_{2m+1}^4]}\right). \tag{3.3}$$

For $m = 0$, (3.1) and (3.2) become

$$\frac{\pi}{4} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} 2^{2n+1}}{(2n+1)(3+\sqrt{5})^{2n+1}}, \tag{3.4}$$

and

$$\frac{\pi}{4} = 5 \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n+1)(3+\sqrt{10})^{2n+1}} \tag{3.5}$$

Series (3.4) and (3.5) were published by the author in [6].

Note that, as m increases, series (3.1) and (3.20 will go from equations (3.4) and (3.5) to the limiting case of Leibniz's series

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots . \tag{3.6}$$

An explicit evaluation of series (3.4) and (3.5) requires a rapid algorithm for the numerical determination of $\sqrt{5}$ and $\sqrt{10}$. The interested reader may use the series

$$\sqrt{5} = \frac{1117229}{499640} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k 9559^k}{20^{3k} 12491^{2k} k!}, \tag{3.7}$$

and

$$\sqrt{10} = \frac{790269}{249905} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-1)^k 8444^k}{10^{3k} 49981^{2k} k!}, \tag{3.8}$$

where

$(\alpha)_k = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+k-1), \ \alpha \ne 0,$

$(\alpha)_0 = 1$ is Pochhammer's symbol.

Either series taken to $k = 12$ gives one hundred decimal places of the corresponding root.

Series (3.7) and (3.8) are special cases of the following general expression

$$\sqrt[N]{s/t} = \frac{n}{smt^{r-1}} \sum_{k=0}^{\infty} \frac{(1/N)_k \alpha^k}{s^{(N+1)k} t^{[(r-1)N-1]k} m^{Nk} k!}, \tag{3.9}$$

where s, t, and N are positive integers, n is the positive integer nearest

$$\frac{s}{t}\, t^r m \left(\frac{s}{t}\right)^{1/N}, \quad r > 1, \tag{3.10}$$

determined with a calculator, and r and m are arbitrary positive integers (m may even be a positive rational). α is an integer, positive or negative, that satisfies the equation

$$\frac{s^{N+1}}{t^{N+1}}\, t^{rN} m^N - \alpha = n^N. \tag{3.11}$$

Equation (3.9) is simply an identity found by expanding the expression

$$\left(1 - \frac{\alpha}{[(s^{N+1})/(t^{N+1})]t^{rN}m^N}\right)^{-1/N} \tag{3.12}$$

in two different ways: (1) by putting the quantity inside parentheses under a common denominator and using (3.11) and (2) by expanding (3.12) by Newton's binomial theorem:

$$(1 - z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!}, \quad |z| < 1. \tag{3.13}$$

Generally, the larger the m and r are, the more rapidly converging the series is.

For series (3.7), we searched for a value of m in the neighborhood of 100,000 for which n would differ from an integer by not more than ±0.01. This makes α small and improves convergence. The parameter r, of course, plays no part when $t = 1$.

For $m = 99928$, we found

$$5 \cdot 99928 \cdot \sqrt{5} = 1117229.00427,$$

so we take $n = 1117229$ and find, using (3.11),

$$5^3 \cdot 99928^2 - \alpha = 1117229^2,$$

which gives $\alpha = 9559$.

For the series (3.8), we found $m = 99962$, and

$$10 \cdot 99962 \cdot \sqrt{10} = 3161075.99464,$$

which gives $n = 3161076$, and

$$10^3 \cdot 99962^2 - \alpha = 3161076^2,$$

which gives $\alpha = -33776$.

These sets of values, when substituted in (3.9), give series (3.7) and (3.8).

It can be shown [7] that, if p_n/q_n is a convergent in the expansion of a real number x as a continued fraction, then there does not exist any rational number a/b with $b \le q_n$ that approximates x better than p_n/q_n. Hence, a sensible way to make (3.10) nearly an integer is to choose m as the denominator of a high enough convergent in the expansion of the N^{th} root of s/t as a continued fraction.

For the case of square roots, the identity

$$\sqrt{a} - b = (a - b^2)/(2b + (\sqrt{a} - b)),$$

due to Michel Rolle (1652-1719), *Mémoires de mathématiques et de physiques*, vol. 3, p. 24 (Paris, 1692), gives at once the continued fraction

$$\sqrt{a} = b + \frac{a - b^2}{2b} + \frac{a - b^2}{2b} + \frac{a - b^2}{2b} + \cdots \tag{3.14}$$

from which we can obtain suitable continued fraction expansions by giving appropriate values to a and b.

For instance, $a = 5$, $b = 2$ gives

$$\sqrt{5} = 2 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots \tag{3.15}$$

and $a = 10$, $b = 3$ gives

$$\sqrt{10} = 3 + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \cdots. \tag{3.16}$$

Note that $a = 5/4$, $b = 1/2$, gives the well-known result

$$\frac{\sqrt{5} + 1}{2} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots.$$

Let p_n/q_n be the n^{th} convergent in the expansion of a real number $\sqrt{D}$ in a continued fraction. Consider the following identity

$$\sqrt{D} = \frac{p_n}{q_n}\left(\frac{p_n^2}{Dq_n^2}\right)^{-\frac{1}{2}} \quad \frac{p_n}{q_n}\left(1 + \frac{p_n^2 - Dq_n^2}{Dq_n^2}\right)^{-\frac{1}{2}} \tag{3.17}$$

Now, it is known that, for an appropriate value of n, the expression $p_n^2 - Dq_n^2$ will be either +1 or -1, a fact intimately bound up with the properties of Pell's equation. These n's occur in cycles; hence, we can make the second term in parentheses in (3.17) as small as we please by choosing a sufficiently large value of n.

For the continued fraction (3.16), we choose the convergent

$$9238605483/2921503573,$$

and from (3.17) find the series

$$\sqrt{10} = \frac{9238605483}{2921503573}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k}{10^k 2921503573^{2k}k!}, \tag{3.18}$$

which picks up about twenty decimals per term, i.e., series (3.18) carried to $k = 5$ gives one hundred decimal places of the square root of ten.

For the square root of five, we can use (3.15), but we can do better if we remember that $L_n/F_n \to \sqrt{5}$ as n increases. Using the same idea exemplified in (3.17), we obtain

$$\sqrt{5} = \frac{L_n}{F_n}\left(1 + \frac{L_n^2 - 5F_n^2}{5F_n^2}\right)^{-\frac{1}{2}}. \tag{3.19}$$

Since $L_n^2 - 5F_n^2 = 4(-1)^n$, we see that the numerator in the fraction inside parentheses is $4(-1)^n$ and the corresponding series will give any number of decimal places per term by choosing n large enough. It is desirable to choose n as a multiple of 3, because then F_n is even and the factor of 4 cancels out. In that case, (3.19) becomes identical to (3.17) with $D = 5$.

Choosing $n = 48$, we have $F_{48} = 4807526976$ and

$$\sqrt{5} = \frac{5374978561}{2403763488} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (-1)^k}{5^k 2403763488^{2k} k!}, \tag{3.20}$$

which also picks up about twenty decimal places per term.

Needless to say, one may also use the computer to search for a good value of m, and then use this value to construct an appropriate series. Note, for instance, the values found in this manner:

$$5 \cdot \sqrt{5} \cdot 83204 = 930248.9999994625, \tag{3.21}$$

$$10 \cdot \sqrt{10} \cdot 777526 = 24587531.00000079. \tag{3.22}$$

For (3.21), we searched for a value of m in the neighborhood of 100,000. The corresponding series gives about twelve decimal places per term. For (3.22), we searched for a value in the neighborhood of 750,000. The corresponding series picks up about seventeen decimal places per term.

By way of comparison, the values we used in series (3.18) and (3.20) give

$$10 \cdot \sqrt{10} \cdot 2921503573 = 92386054830.00000000057353236,$$

$$5 \cdot \sqrt{5} \cdot 2403763488 = 26874892804.999999999624625216.$$

4. Some Identities for Fibonacci Numbers

Equations (2.17) through (2.26) provide many interesting relations for Fibonacci and Lucas numbers.

The identity [1]

$$2(x^2 - 1) \sum_{m=0}^{n} U_{2m}(x) = T_{2n+2}(x) - 1 \tag{4.1}$$

gives, with $x = \frac{1}{2}L_{2k}$ and use of (2.24) and (2.18), the result

$$\sum_{m=0}^{n} F_{(2m+1)2k} = \frac{L_{(2n+2)2k} - 2}{5F_{2k}} \tag{4.2}$$

The identity [1]

$$2(1 - x^2) \sum_{m=1}^{n} U_{2m-1}(x) = x - T_{2n+1}(x) \tag{4.3}$$

gives, with $x = \frac{1}{2}L_{2k}$ and use of (2.24) and (2.18), the result

$$\sum_{m=1}^{n} F_{(2m)2k} = \frac{L_{(2n+1)2k} - L_{2k}}{5F_{2k}}. \tag{4.4}$$

Equations (4.2) and (4.4) can be combined to give

$$\sum_{m=1}^{n} F_{m2k} = \frac{L_{(n+1)2k} + L_{n2k} - L_{2k} - 2}{5F_{2k}}. \tag{4.5}$$

Equation (4.3), with $x = (\sqrt{5}/2)F_{2k+1}$ and use of (2.17) and (2.22) gives

$$\sum_{m=1}^{n} F_{2m(2k+1)} = \frac{F_{(2n+1)(2k+1)} - F_{2k+1}}{L_{2k+1}}. \tag{4.6}$$

The identity [1]

$$\sum_{m=0}^{n-1} T_{2m+1}(x) = \tfrac{1}{2}U_{2n-1}(x) \tag{4.7}$$

gives, with $x = (\sqrt{5}/2)F_{2k+1}$ and the use of (2.17) and (2.22), the result

$$\sum_{m=0}^{n-1} F_{(2m+1)(2k+1)} = \frac{F_{2n(2k+1)}}{L_{2k+1}}. \tag{4.8}$$

Equations (4.6) and (4.8) combine to give the expression

$$\sum_{m=1}^{n} F_{m(2k+1)} = \frac{F_{(n+1)(2k+1)} + F_{n(2k+1)} - F_{2k+1}}{L_{2k+1}}. \tag{4.9}$$

Equations (4.5) and (4.9) are generalizations of well-known results.

The reader should note that these formulas, once established, may be verified by induction.

5. Other Numerical Sequences Associated with Classical Polynomials

Much of the success we have had in obtaining properties of Fibonacci and Lucas numbers has depended largely on our being able to associate a recurrent sequence of numbers with the set of Chebyshev polynomials. The question naturally arises as to whether other such sequences of positive integers exist associated with other classical polynomials. Surprisingly, such sequences do exist in a number of important cases.

For example, if $P_n(x)$ designates Legendre polynomials, the expression

$$b_n = 2^n i^{-n} P_n(i) \tag{5.1}$$

gives a recurrent sequence of positive integers associated with Legendre polynomials. We have, explicitly,

$$b_n = \sum_{k=0}^{[n/2]} \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}. \tag{5.2}$$

The pure recurrence relation for Legendre polynomials

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x), \; n \geq 2, \tag{5.3}$$

together with (5.1) gives

$$nb_n = 2(2n-1)b_{n-1} + 4(n-1)b_{n-2}, \; n \geq 2, \; b_0 = 1, \; b_1 = 2, \tag{5.4}$$

which defines the b_n recurrently. The first few are

$$b_0 = 1, \; b_1 = 2, \; b_2 = 8, \; b_3 = 32, \; b_4 = 136, \; b_5 = 592, \text{ etc.}$$

Similarly, if $L_n(x)$ designates the simple Laguerre polynomials, the expression

$$c_n = n!L_n(-1) \tag{5.5}$$

gives a recurrent sequence of positive integers associated with simple Laguerre polynomials. We have, explicitly,

$$c_n = n! \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k!}. \tag{5.6}$$

The pure recurrence relation for simple Laguerre polynomials

$$nL_n(x) = (2n - 1 - x)L_{n-1}(x) - (n - 1)L_{n-2}(x),\ n \ge 2, \tag{5.7}$$

together with (5.5) gives

$$c_n = 2nc_{n-1} - (n - 1)^2 c_{n-2},\ n \ge 2,\ c_0 = 1,\ c_1 = 2, \tag{5.8}$$

which defines the c_n recurrently. The first few are

$$c_0 = 1,\ c_1 = 2,\ c_2 = 7,\ c_3 = 34,\ c_4 = 209,\ c_5 = 1546, \text{ etc.}$$

Using the known generating function for simple Laguerre polynomials

$$(1 - t)^{-1}\exp\left(\frac{-xt}{1 - t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n,$$

we obtain at once

$$(1 - t)^{-1}\exp\left(\frac{t}{1 - t}\right) = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!}.$$

Now, replacing $t/(1 - t)$ by x, we find the interesting expansion

$$e^x = \sum_{n=0}^{\infty} \frac{c_n}{n!}\, \frac{x^n}{(1 + x)^{n+1}},\ x > -\tfrac{1}{2}. \tag{5.9}$$

Another curious series for the exponential is found from the expression

$$e^{xt} = \left(\frac{t}{2}\right)^{-\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\nu + n)I_{\nu+n}(t)C_n^{\nu}(x), \tag{5.10}$$

due to Gegenbauer, where $I_k(t)$ are modified Bessel functions of the first kind [9], given by

$$I_k(t) = \frac{(\frac{1}{2}t)^k}{\Gamma(k + 1)}\ {}_0F_1(-;\ 1 + k;\ \tfrac{1}{4}t^2),$$

and $C_n^{\nu}(x)$ are ultraspherical polynomials [9] defined by

$$C_n^{\nu}(x) = \frac{(2\nu)_n P_n^{(\nu-\frac{1}{2},\ \nu-\frac{1}{2})}(x)}{(\nu + \frac{1}{2})_n},$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials.

In terms of ultraspherical polynomials, Chebyshev polynomials are given by

$$U_n(x) = C_n^1(x), \tag{5.11}$$

$$T_n(x) = \lim_{\nu \to 0}\frac{C_n^{\nu}(x)}{C_n^{\nu}(1)}. \tag{5.12}$$

With appropriate substitutions in (5.10) and making use of (2.3) and (2.6), we have

$$e^x = \frac{\sqrt{5}}{x}\sum_{n=1}^{\infty} n\left[\frac{1 + (-1)^n}{2}\sqrt{5}F_n + \frac{1 - (-1)^n}{2}L_n\right]I_n(2x/\sqrt{5}). \tag{5.13}$$

If $H_n(x)$ designates Hermite polynomials, then the expression

$$d_n = 2^{-n/2}\, i^n H_n(-i/\sqrt{2}), \tag{5.14}$$

gives a recurrent sequence of positive integers.

The pure recurrence relation for Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x),$$

together with (5.14) gives

$$d_{n+1} = d_n + nd_{n-1},\ n \geq 1,\ d_0 = 1,\ d_1 = 1. \tag{5.15}$$

Sequence (5.15) has been studied by P. Rubio, *Dragados y Construcciones* (Madrid, Spain), although (5.14) was, to my knowledge, discovered by me (see [10]).

The first few d_n's are

$$d_0 = 1,\ d_1 = 1,\ d_2 = 2,\ d_3 = 4,\ d_4 = 10,\ d_5 = 26, \text{ etc.}$$

Known relations for Hermite polynomials provide interesting expansions with the d_n's as coefficients. For instance, the generating relation [9]

$$(1 - 4t^2)^{-\frac{1}{2}}\exp\left[y^2 - \frac{(y - 2xt)^2}{1 - 4t^2}\right] = \sum_{k=0}^{\infty} \frac{H_k(x)H_k(y)t^k}{k!}$$

gives, on changing both x and y to $-i/\sqrt{2}$, and t to $-t/2$, the interesting relation

$$(1 - t^2)^{-\frac{1}{2}}\exp\left(\frac{t}{1 - t}\right) = \sum_{k=0}^{\infty} \frac{d_k^2 t^k}{k!}. \tag{5.16}$$

Changing $\frac{t}{1 - t}$ to x, we find

$$e^x = (2x + 1)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{d_k^2}{k!} \frac{x^k}{(x + 1)^{k+1}},\ x > -\tfrac{1}{2}. \tag{5.17}$$

Series (5.9), (5.13), and (5.17) are offered only as mathematical curiosities. None of them converges faster than Euler's exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Series (5.9) and (5.17), in particular, converge very slowly.

These recurrent sequences of positive integers associated with classical polynomials seem not to have been studied in the existing literature, in spite of the fact that they may well be used to advantage in numerical work.

6. Continued Fraction Expansions for Fibonacci and Lucas Numbers

We will close this paper by showing how to expand Fibonacci and Lucas numbers in nontrivial finite continued fractions. This result is rather surprising inasmuch as Fibonacci and Lucas numbers are integers.

The expression

$$S = \alpha_0 + \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3\alpha_4 + \cdots, \tag{6.1}$$

is easily seen to be equivalent to the infinite continued fraction

$$S = \alpha_0 + \frac{\alpha_1}{1} - \frac{\alpha_2}{(1 + \alpha_2)} - \frac{\alpha_3}{(1 + \alpha_3)} - \cdots . \tag{6.2}$$

If we let

$$\alpha_0 = \beta(z), \tag{6.3}$$

where $\beta(z)$ is an arbitrary function of z,

$$\alpha_1 = \beta(z)\frac{a_1 a_2 \ldots a_p}{b_1 b_2 \ldots b_q} z, \tag{6.4}$$

$$\alpha_k = \frac{(a_1 + k - 1)(a_2 + k - 1) \ldots (a_p + k - 1)}{(b_1 + k - 1)(b_2 + k - 1) \ldots (b_q + k - 1)} \frac{z}{k}, \; k > 1, \tag{6.5}$$

where none of the b's is zero or a negative integer, then (6.1) becomes

$$\beta(z)\left[1 + \frac{a_1 a_2 \ldots a_p}{b_1 b_2 \ldots b_q} \frac{z}{1!} + \frac{(a_1)_2 (a_2)_2 \ldots (a_p)_2}{(b_1)_2 (b_2)_2 \ldots (b_q)_2} \frac{z^2}{2!} + \ldots\right]$$

$$= \beta(z)\,{}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\; z\right]. \tag{6.6}$$

Use of (6.2) with the values (6.3), (6.4), and (6.5) gives a continued fraction expansion for the generalized hypergeometric function ${}_pF_q(z)$ times an arbitrary function of z, $\beta(z)$. The continued fraction expansion converges, of course, whenever the infinite series defining the hypergeometric function converges. The continued fraction and the series converge and diverge together.

One of the known expressions for Jacobi polynomials is

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n (1 + \beta)_n}{n!}\;{}_2F_1\left[\begin{matrix} -n, 1 + \alpha + \beta + n; \\ 1 + \beta; \end{matrix}\; \frac{1 + z}{2}\right]. \tag{6.7}$$

In terms of Jacobi polynomials, Chebyshev polynomials are given by

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x), \tag{6.8}$$

and

$$U_n(x) = \frac{(n + 1)!}{(\frac{3}{2})_n} P_n^{(\frac{1}{2}, \frac{1}{2})}(x). \tag{6.9}$$

Simple substitutions, and use of (2.3) through (2.6), gives continued fraction expansions for Fibonacci and Lucas numbers.

Let us illustrate this by finding a continued fraction expansion for L_4. One has

$$L_{2n} = 2\,{}_2F_1\left[\begin{matrix} -2n, 2n; \\ \frac{1}{2}; \end{matrix}\; \frac{2 + \sqrt{5}}{4}\right],$$

from which one gets, for $n = 2$,

$$\beta(z) = 2,$$
$$\alpha_0 = 2,$$
$$\alpha_1 = -16(2 + \sqrt{5}),$$
$$\alpha_2 = -\frac{5}{4}(2 + \sqrt{5}),$$
$$\alpha_3 = -\frac{2}{5}(2 + \sqrt{5}),$$
$$\alpha_4 = -\frac{1}{8}(2 + \sqrt{5}),$$

$\alpha_k = 0,\ k > 4.$

From these, it follows that

$$L_4 = 2 - \frac{16(2+\sqrt{5})}{1} \; \frac{5(2+\sqrt{5})}{+\ -6-5\sqrt{5}} \; \frac{8(2+\sqrt{5})}{+\ 1-2\sqrt{5}} \; \frac{5(2+\sqrt{5})}{+\ 6-\sqrt{5}} = 7,$$

as the reader can verify easily.

Putting the Jacobi polynomial into its several equivalent forms [9] gives different, but equivalent, continued fractions for Fibonacci and Lucas numbers.

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