$-u_0 + v_0\sqrt{D} = u_1 + v_1\sqrt{D}$ belongs to the same class of solutions to $u^2 - Dv^2 = C$ as $u_0 + v_0\sqrt{D}$. Since we are assuming $u_0 < 0$, this contradicts (iv) of Remark A [1]. Hence, even in this case, $v_1 > v_0$. In a similar manner, it is seen that we always have $u_1 > u_0$. Since $u_n > 0$ and $v_n > 0$ for $n \geq 1$, (2) implies that $u_{n+1} > u_n$ and $v_{n+1} > v_n$ for $n \geq 1$.

Theorem 4: If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = -N$, where $N \geq 1$, and if $v \geq ku$, where $k = (y_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = -N$. If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = N$, where $N > 1$, and if $u \geq kv$, where $k = (Dy_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = N$.

Proof: By Theorem 2, $u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n = u_n + v_n\sqrt{D}$, where n is a nonnegative integer and $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = \pm N$. We shall prove $u + v\sqrt{D} = u_0 + v_0\sqrt{D}$. So assume $n \geq 1$. Then we have

$$\begin{aligned} u_n + v_n\sqrt{D} &= (u_{n-1} + v_{n-1}\sqrt{D})(x_1 + y_1\sqrt{D}) \\ &= (x_1 u_{n-1} + Dy_1 v_{n-1}) + (x_1 v_{n-1} + y_1 u_{n-1})\sqrt{D}. \end{aligned}$$

Thus $u_{n-1} = x_1 u_n - Dy_1 v_n$ and $v_{n-1} = -y_1 u_n + x_1 v_n$.

First, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = -N$. We know that

$$v = v_n \geq ku_n = \frac{y_1 u_n}{x_1 - 1}.$$

Hence

$$v_{n-1} = -y_1 u_n + x_1 v_n = (x_1 - 1)v_n - y_1 u_n + v_n \geq v_n.$$

But by the corollary to Lemma 3, $v_{n-1} < v_n$ for $n \geq 1$. Thus $n = 0$ and the proof is complete for the case $u^2 - Dv^2 = -N$.

Now, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = N$. We know that

$$u_n \geq kv_n = \frac{Dy_1 v_n}{x_1 - 1}.$$

(Please turn to page 92)

STRUCTURAL ISSUES FOR HYPERPERFECT NUMBERS

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ABSTRACT

An integer m is said to be n-hyperperfect if $m = 1 + n[\sigma(m) - m - 1]$. These numbers are a natural extension of the perfect numbers, and as such share remarkably similar properties. In this paper we investigate sufficient forms for hyperperfect numbers.

1. INTRODUCTION

Integers having "some type of perfection" have received considerable attention in the past few years. The most well-known cases are: perfect numbers ([1], [12], [13], [14], [15]); multiperfect numbers ([1]); quasiperfect numbers ([2]); almost perfect numbers ([3], [4], [5]); semiperfect numbers ([16], [17]);

and unitary perfect numbers ([11]). The related issue of amicable, unitary amicable, quasiamicable, and sociable numbers ([8], [10], [11], [9], [6], [7]) has also been investigated extensively.

The intent of these variations of the classical definition appears to have been the desire to obtain a set of numbers, of nontrivial cardinality, whose elements have properties resembling those of the perfect case. However, none of the existing definitions generates a rich theory and a solution set having structural character emulating the perfect numbers; either such sets are empty, or their euclidean distance from zero is greater than some very large number, or no particularly unique prime decomposition form for the set elements can be shown to exist.

This is in contrast with the abundance (cardinally speaking) and the crystalized form of the n-hyperperfect numbers (n-HP) first introduced in [18]. These numbers are a natural extension of the perfect case, and, as such, share remarkably similar properties, as described below.

In this paper we investigate sufficient forms for the hyperperfect numbers. The necessity of these forms, though highly corroborated by empirical evidence, remains to be established for many cases.

2. BASIC THEORY

Definition 1:

a. m is n-HP iff $m = 1 + n[\sigma(m) - m - 1]$, m and n positive integers.

b. $M_n = \{m | m \text{ is } n\text{-HP}\}$.

c. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i} \ldots p_j^{\alpha_j} p_{j+1}^k$ be n-HP and be in canonical form

$$(p_1 < p_2 < \cdots < p_j < p_{j+1}).$$

Then $\rho(m) = \{p_1, p_2, \ldots, p_{j-1}, p_j\}$ are the roots of m [if $m = p_1$, $\rho(m) = \emptyset$].

d. $d_1(m) = |\rho(m)| = j$, $d_2(m) = k$.

e. ${}_nM_{h,L} = \{m | m \text{ is } n\text{-HP}, d_1(m) = h, d_2(m) = L\}$.

Note that for $n = 1$ the perfect numbers are recaptured. Clearly one has

$$M_n = \left[\bigcup_L {}_nM_{0,L}\right] \cup \left[{}_nM_{1,1}\right] \cup \left[\bigcup_{h=2}^{\infty} {}_nM_{h,1}\right] \cup \left[\bigcup_{h=1}^{\infty} \bigcup_{L=2}^{\infty} {}_nM_{h,L}\right].$$

Definition 2:

a. If $m \in \bigcup_L {}_nM_{0,L}$ we say that m is a *Sublinear HP*.

b. If $m \in {}_nM_{1,1}$ we say that m is a *Linear HP*.

c. If $m \in \bigcup_{h=2}^{\infty} {}_nM_{h,1}$ we say that m is a *Superlinear HP*.

d. If $m \in \bigcup_{h=1}^{\infty} \bigcup_{L=2}^{\infty} {}_nM_{h,L}$ we say that m is a *Nonlinear HP*.

It has already been shown [18] that

Proposition 1: There are no Sublinear n-HPs.

Table 1 below shows the n-HP numbers less than 1,500,000. In each case, m is a Linear HP. We thus give an exhaustive theory for Linear HPs. Superlinear and Nonlinear results will be presented elsewhere; however, it appears that the only n-HP are *Linear* n-HP. In fact, several nonlinear forms have been shown to be impossible.

Table 1. n-HP up to 1,500,000, $n \geq 2$

n	m	Prime Decomposition for m	n	m	Prime Decomposition for m
2	21	3×7	2	176,661	$3^5 \times 727$
6	301	7×43	31	214,273	$47^2 \times 97$
3	325	$5^2 \times 13$	168	250,321	193×1297
12	697	17×41	108	275,833	133×2441
18	1,333	31×43	66	296,341	67×4423
18	1,909	23×83	35	306,181	$53^2 \times 109$
12	2,041	13×157	252	389,593	317×1229
2	2,133	$3^3 \times 79$	18	486,877	79×6163
30	3,901	47×83	132	495,529	137×3617
11	10,693	$17^2 \times 37$	342	524,413	499×1087
6	16,513	$7^2 \times 337$	366	808,861	463×1747
2	19,521	$3^4 \times 241$	390	1,005,421	479×2099
60	24,601	73×337	168	1,005,649	173×5813
48	26,977	53×509	348	1,055,833	401×2633
19	51,301	$29^2 \times 61$	282	1,063,141	307×3463
132	96,361	173×557	498	1,232,053	691×1783
132	130,153	157×829	540	1,284,121	829×1549
10	159,841	$11^2 \times 1321$	546	1,403,221	787×1783
192	163,201	293×557	59	1,433,701	$89^2 \times 181$

3. LINEAR THEORY

The following basic theorem of Linear n-HP gives a sufficient form for a hyperperfect number.

Theorem 1: m is a Linear n-HP if and only if

$$p_2 = \frac{np_1^{\alpha_1+1} - (n-1)p_1 - 1}{p_1^{\alpha_1+1} - (n+1)p_1^{\alpha_1} + n}.$$

Proof: $(\rightarrow)$ m is a Linear n-HP, if $m = p_1^{\alpha_1}p_2$; then

$$\sigma(m) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}(1 + p_2).$$

But m n-HP implies that $(n+1)m = (1-n) + n\sigma(m)$. Substituting for $\sigma(m)$ and solving for p_2, we obtain the desired result. Note that p_2 must be a prime.

$$(\leftarrow) \text{ if } m = p_1^{\alpha_1}\frac{np_1^{\alpha_1+1} - (n-1)p_1 - 1}{p_1^{\alpha_1+1} - (n+1)p_1^{\alpha_1} + n}$$

where the second term is prime, then

$$\sigma(m) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\left[1 + \frac{np_1^{\alpha_1+1} - (n - 1)p_1 - 1}{p_1^{\alpha_1\ 1} - (n + 1)p_1^{\alpha_1} + n}\right],$$

from which one sees that the condition for a *Linear n-HP* is satisfied. Q.E.D.

We say that n is convolutionary if $n + 1$ is prime p_1.

Corollary 1: If n is convolutionary, a sufficient form for $m = p_1^{\alpha_1+1}p_2$ to be Linear n-HP is that for some α_1, $p_2 = (n + 1)^{\alpha_1} - n$ is a prime. In this case,

$$m = (n + 1)^{\alpha_1+1}[(n + 1)^{\alpha_1} - n].$$

Corollary 2: If $m = p_1p_2$ is a Linear n-HP, then

$$p_2 = \frac{np_1^2 - (n - 1)p_1 - 1}{p_1^2 - (n + 1)p_1 + n}.$$

We would expect these n-HPs to be the most abundant, since they have the simplest structure. This appears to be so, as indicated by Table 1.

Corollary 3: If $m = p_1p_2$ is a Linear n-HP with $p_1 = n + 1$, then $p_2 = n^2 + n + 1$, so that

$$m = (n + 1)(n^2 + n + 1).$$

In view of these corollaries, the following issues are of capital importance for cardinality considerations of Linear n-HP.

Definition 3:

a. We say that $(n + 1)^{\alpha} - n$, $\alpha = 1, 2, 3, \ldots$, is a Legitimate Mersenne sequence rooted on n (n-LMS), if $n + 1$ is a prime.

b. Given an n-LMS, we say that $(n + 1)^{\alpha} - n$ is an nth-order Mersenne prime (n-MP), if $(n + 1)^{\alpha} - n$ is prime.

A 1-LMS is the well-known sequence $2^{\alpha} - 1$.

Question 1. Does there exist an n-MP for each n?

Question 2. Do there exist infinitely many n-MP for each n?

Question 3. Are there infinitely many primes of the form $n^2 + n + 1$, where $n + 1$ is prime?

Extensive computer searches (not documented here) seem to indicate that the answer to these questions is affirmative.

Theorem 2: If m is a Linear n-HP, then $n + 1 \leq p_1 \leq 2n - 1$ if $n > 1$ and $p_1 \leq 2$ if $n = 1$.

Proof: It can be shown that if m is n-HP and $j|m$, then $j > n$. Thus, for a Linear n-HP, $p_1 > n$; equivalently, $p_1 \geq n + 1$. Now, since

$$p_2 = p_1\frac{np_1^{\alpha_1} - (n - 1) - 1/p_1}{(p_1 - n - 1)p_1^{\alpha_1} + n},$$

we let

$$p_1 = n + 1 + \mu;$$

then,

$$p_2 = p_1\frac{np_1^{\alpha_1} - (n - 1) - 1/p_1}{\mu p_1^{\alpha_1} + n}.$$

(Note: $\mu = 0$ implies $p_1 = n + 1 < 2n$.) Since we want the second factor of this expression larger than 1, we must have $\mu < n$ or $n < p_1 \le 2n$, from which we get $n + 1 \le p_1 \le 2n - 1$ if $n > 1$, for primality, and $1 < p_1 \le 2$ if $n = 1$. Q.E.D.

Observe that the upper bound is necessary for a Linear n-HP, *but not for a general n-HP.*

Corollary 4: m is a Linear 1-HP iff it is of the form $m = 2^{t-1}(2^t - 1)$.

Proof: From Theorem 2, $p_1 = 2$. Now we can apply Corollary 1 to obtain the necessary part of this result. The sufficiency part follows from the definition.

4. BOUNDS FOR LINEAR n-HP

We now establish important bounds for Linear n-HP.

Proposition 2: Let m be Linear n-HP. Consider $p_2 = F(\alpha)$. The p_2 is monotonically increasing on α.

Proof: Omitted.

Proposition 3:

$$\lim_{\alpha\to\infty} p_2 = \begin{cases} \dfrac{np_1}{p_1 - n - 1} & p_1 \ne n + 1 \\ \infty & p_1 = n + 1 \end{cases}$$

This follows directly from Theorem 1 and Corollary 1. Using Proposition 2, we obtain

Proposition 4:

$$\frac{np_1^2 - (n-1)p_1 - 1}{p_1^2 - (n+1)p_1 + n} \le p_2 \le \frac{np_1}{p_1 - n - 1} \qquad p_1 \ne n + 1$$

$$n^2 + n + 1 \le p_2 < \infty \qquad p_1 = n + 1$$

Using these propositions, we have essentially proved the following important theorem.

Theorem 3: Given n, $n+1 \le p_1 \le 2n - 1$, if n is not convolutionary, then there can be at most finitely many n-HP of the form $m = p_1^{\alpha_1}p_2$.

Table 2 and Table 3 show the allowable values for p_1, given n, along with the bounds for p_2. We can now obtain results similar to those of Corollary 4.

Corollary 5: If m is Linear 2-HP, then it can only be of the form

$$3^{t-1}(3^t - 2).$$

Corollary 6: *a*) If m is Linear 3-HP, then it must be of the form

$$5^{t-1}\frac{3 \cdot 5^t - 11}{5^{t-1} + 3}$$

b) There is exactly one Linear 3-HP (see Tables 2 and 3).

Corollary 7: *a*) There are no Linear 4-HP rooted on 7 (see Tables 2 and 3).
b) There are Linear 4-HP rooted on 5. For example,

$$m = 5^4(5^4 - 4) = 5^4(3121).$$

Corollary 8: There are no Linear 5-HP.

Corollary 9: There are no Linear 7-HP.

Table 2. Allowable Values of p_1 and Bounds on p_2

n	Allowable Roots			
2	3(7, ∞)			
3	5(8, 15)			
4	5(21, ∞)	7(9, 14)		
5	7(18, 35)			
6	7(43, ∞)	11(13, 17)		
7	11(19, 26)	13(15, 19)		
8	11(29, 44)	13(21, 26)		
9	11(50, 99)	13(29, 39)	17(19, 22)	
10	11(111, ∞)	13(43, 65)	17(24, 29)	19(21, 24)

Table 3. p_2 as a Function of p_1, n, and α

α	$p_1 = 5$		$p_1 = 7$	
↓	$n = 3$	$n = 4$	$n = 4$	$n = 5$
1	8	21	9.66	18
2	13	121	13.23	31.22
3	14.56	621	13.88	34.41
4	14.91	3121	13.98	34.91
5	14.98	15621	13.99	34.98
6	14.99	78121	13.999	34.99
7	14.999	390621	13.9999	34.999
8	14.999	1953121	13.99999	34.9999

α	$p_1 = 11$			$p_1 = 13$	
↓	$n = 7$	$n = 8$	$n = 9$	$n = 7$	$n = 8$
1	19.50	29.66	50	15.33	21
2	25	42.28	91.46	17.95	25.56
3	25.60	43.83	98.26	18.18	25.96
4	25.66	43.98	98.93	18.19	25.997
5	25.666	43.99	98.99	18.199	25.9997
6	25.6666	43.999	98.999	18.1999	25.99998
7	25.66666	43.9999	98.9999	18.19999	
8		43.99999	98.99999		

Further bounds are derived below. We have already given one such bound:

$$p_2 \le \frac{np_1}{p_1 - n - 1} \quad \text{for } n \text{ nonconvolutionary}$$

$$p_1 = n + 2 \qquad p_2 \le n^2 + 2n$$

$$p_1 = n + 3 \qquad p_2 \le \frac{n(n + 3)}{2}$$

$$p_1 = n + 4 \qquad p_2 \le \frac{n(n + 4)}{3}$$

Therefore,

Proposition 5: Let n be nonconvolutionary. If m is Linear n-HP, then

$$p_2 \leq n^2 + 2n.$$

More generally,

Proposition 6: Let $d_n = p - n$, where p is the first prime larger than n. Then

$$\frac{n(n + d_n)^2 - (n - 1)(n + d) - 1}{(n + d_n)^2 \leq (n + 1)(n + d_n) + n} \leq p_2 \leq \frac{n(n + d_n)}{d_n - 1},$$

which is valid for $d_n \geq 1$.

Proposition 7: Let m be Linear n-HP, n nonconvolutionary. Then $p_2 \geq 2n + 1$.

Proof: (From the previous general bound on p_2, we see that this statement is also true for convolutionary n.) The proof involves looking at the expression for p_2, given that $p_1 = n + i$, $2 \leq i \leq n - 1$. Suppose $p_1 = n + 2$. Since m is Linear n-HP, we have

$$p_2 \geq \frac{np_1^2 - (n - 1)p_1 - 1}{p_1^2 - (n + 1)p_1 + 1}.$$

But $p_1 = n + 2$, so that

$$p_2 \geq \frac{n(n + 2)^2 - (n - 1)(n + 2) - 1}{(n + 2)^2 - (n + 1)(n + 2) + n} = \frac{(n + 1)^3}{2(n + 1)} = \frac{(n + 1)^2}{2} = \frac{n^2 + 2n + 1}{2}.$$

However, $n^2 > 2n$ $(n > 2)$, so that for this case $p_2 \geq 2n$ or $p_2 \geq 2n + 1$. Similar arguments hold for $p = n+3, n+4, \ldots$. We show the case $p = 2n - 1$. We have

$$p_2 \geq \frac{n(2n - 1)^2 - (n - 1)(2n - 1) - 1}{(2n - 1)^2 - (n + 1)(2n - 1) + n} = \frac{2n^3 - 3n^2 + 2n - 1}{n^2 - 2n + 1}$$

$$= 2n + 1 + \frac{2n - 2}{n^2 - 2n + 1}.$$

(Note that $n \neq 1$.) Therefore, again, $p_2 \geq 2n + 1$. Q.E.D.

Proposition 8: If $m = p_1^{\alpha_1} p_2$ is a Linear n-HP, n nonconvolutionary, then

$$\alpha_1 \leq \frac{\log\left[\dfrac{n^2 p_1}{p_1 - n - 1} + (n - 1)p_1 + 1\right]}{\log p_1}$$

Proof: We have shown that p_2 tends monotonically to $e = (np_1)/(p_1 - n - 1)$ as $\alpha \to \infty$. Let e' be the greatest integer smaller than e. Setting

$$\frac{np_1^{\alpha_1+1} - (n - 1)p_1 - 1}{p_1^{\alpha_1}(p_1 - n - 1) + n} = e'$$

and solving for α_1, we obtain

$$\alpha_1 = \frac{\log\left[\dfrac{ne' + (n-1)p_1 + 1}{np_1 - e'(p_1 - n - 1)}\right]}{\log p_1}.$$

However,

$$\frac{ne' + (n-1)p_1 + 1}{np_1 - e'(p_1 - n - 1)} \le n\,\frac{np_1}{p_1 - n - 1} + (n-1)p_1 + 1,$$

and, in fact, the equality holds in many cases. The result follows. Q.E.D.

The following statement summarizes the bounds for a linear n-HP:

1. $n + 1 \le p_1 \le 2n - 1$;
2. $\begin{cases} \text{if } p_1 = n + 1, \text{ then } n^2 + n + 1 \le p_2 < \infty, \\ \text{if } p_1 > n + 1, \text{ then } 2n + 1 \le p_2 \le n^2 + 2n; \end{cases}$
3. if $p_1 > n + 1$, then $\alpha_1 < \dfrac{\log\left[\dfrac{n^2 p_1}{p_1 - n - 1} + (n-1)p_1 + 1\right]}{\log p_1}$.

Notwithstanding the fact that no Superlinear and Nonlinear n-HP have been observed, we can still derive sufficient forms for these numbers (if they exist). It may be shown that

Proposition 9: $m = p_1^{\alpha_1} p_2^{\alpha} \ldots p_{j-1}^{\alpha_{j-1}} p_j$ is a Superlinear n-HP if and only if

$$p_j = \frac{n\Pi(p_1^{\alpha_i + 1} - 1) + (1 - n)\Pi(p_i - 1)}{(n + 1)\Pi(p_i - 1)\Pi p_i^{\alpha_i} - n\Pi(p^{\alpha_i + 1} - 1)}.$$

3. CONCLUSION

Theorem 1 and Proposition 9 guarantee that, if an integer has a specific prime decomposition, then it is n-HP. However, no n-HP with these forms was observed in the search up to 1,500,000. One reason for such an unavailability could be the fact that the search was limited. The last term required by these theorems is a fraction or even involves a radical; hence, to ask that this expression turn out to be an integer and, moreover, a prime, is a strong demand. Possibly, very rare combinations of primes could generate the required conditions. It has been shown that indeed some forms are impossible.

The other explanation is that there are only Linear n-HP, and thus Theorem 1 is *necessary* and *sufficient* for a number to be n-HP, just as in the regular perfect number case. Such a statement would have a critical impact on the generalized perfect number problem. In fact, in view of the corollaries presented above, there would be no n-HP for various values of n.

Computer time (PDP 11/70) for Table 1 was over ten hours.

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ON RECIPROCAL SERIES RELATED TO FIBONACCI NUMBERS WITH SUBSCRIPTS IN ARITHMETIC PROGRESSION

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1. INTRODUCTION

Recently, interest has been shown in summing infinite series of reciprocals of Fibonacci numbers [1], [2], and [3]. As V. E. Hoggatt, Jr., and Marjorie Bicknell state [2]: "It is not easy, in general, to derive the sum of a series whose terms are reciprocals of Fibonacci numbers such that the subscripts are terms of geometric progressions." It seems even more difficult if the subscripts are in arithmetic progression. To take a very simple example, to my knowledge the series

$$\sum_1^\infty \frac{1}{F_n} \tag{1.1}$$

has not been evaluated in closed form, although Brother U. Alfred has derived formulas connecting it with other highly convergent series [4].

In this note, we develop formulas for closely related series of the form

$$\sum_0^\infty \frac{1}{F_{an+b} + c} \tag{1.2}$$