PALINDROMIC COMPOSITIONS

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In this paper, we discuss palindromic compositions of integers n using members of general sequences of positive integers as summands. A palindromic composition of n is a composition that reads the same forward as backward, as $5 = 1 + 3 + 1$, but not $5 = 3 + 1 + 1$. We derive formulas for the number of palindromic representations of any integer n as well as for the compositions of n. The specialized results lead to generalized Fibonacci sequences, interleaved Fibonacci sequences 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, ⋯, and rising diagonal sums of Pascal's triangle.

1. GENERATING FUNCTIONS

Let

$$\{a_k\}_{k=0}^{\infty}$$

be any increasing sequence of positive integers from which the compositions of a non-negative integer n are made. Then let

$$F(x) = x^{a_0} + x^{a_1} + \cdots + x^{a_k} + \cdots,$$

which will allow us to write generating functions for the number of palindromic compositions P_n as well as the number of compositions C_n made from the sequence

$$\{a_k\}_{k=0}^{\infty}.$$

Theorem 1.1. The number of compositions C_n of a non-negative integer n is given by

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - F(x)}.$$

Proof. Now $C_0 = 1$ and $C_1 = C_2 = \cdots = C_{a_0-1} = 0$ because the numbers $1, 2, 3, \cdots, a_0 - 1$ have no compositions, while the number 0 has a vacuuous composition using no summands from the given sequence. Next,

$$C_n = C_{n-a_0} + C_{n-a_1} + \cdots + C_{n-a_s} + \cdots,$$

where $C_j = 0$ if $j < 0$. Thus,

$$\sum_{n=0}^{\infty} C_n x^n = (x^{a_0} + x^{a_1} + x^{a_2} + \cdots) \sum_{n=0}^{\infty} C_n x^n + 1$$

from which Theorem 1.1 follows immediately.

Theorem 1.2. The number of palindromic compositions P_n of a non-negative integer n is given by

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1 + F(x)}{1 - F(x^2)}$$

or

$$\sum_{n=1}^{\infty} P_n x^n = \frac{F(x)+F(x^2)}{1-F(x^2)} .$$

Proof. First, we can make a palindromic composition by adding an a_k to each side of an existing palindromic composition. Thus

$$P_n = P_{n-2a_0} + P_{n-2a_1} + \cdots + P_{n-2a_s} + \cdots,$$

where $P_j = 0$ if $j < 0$. Thus

$$\sum_{n=0}^{\infty} P_n x^n = x^{2a_0}(P_0 + P_1x + P_2x^2 + \cdots) + x^{2a_1}(P_0 + P_1x + P_2x^2 + \cdots)$$
$$+ x^{2a_2}(P_0 + P_1x + P_2x^2 + \cdots) + \cdots + (x^{a_0} + x^{a_1} + x^{a_2} + \cdots),$$

where the terms $x^{a_0} + x^{a_1} + x^{a_2} + \cdots$ account for the single palindromic compositions not achievable in the first form. Theorem 1.2 is immediate.

We note that the function

$$F(x) = x^{a_0} + x^{a_1} + \cdots + x^{a_s} + \cdots$$

is such that

$$F^i(x) = \sum_{n=0}^{\infty} R(n)x^n ,$$

where $R(n)$ is the i-part composition of n;

$$F^i(x^2) = \sum_{n=0}^{\infty} R^*(n)x^n ,$$

where $R^*(n)$ is the $2i$-part palindromic composition of n; and

$$F(x)F^i(x^2) = \sum_{n=0}^{\infty} R^{**}(n)x^n ,$$

where $R^{**}(n)$ is the $(2i+1)$-part palindromic composition of n.

Next, we find the number of occurrences of a_k in the compositions and in the palindromic compositions of n.

Theorem 1.3. Let A_n be the number of times a_k is used in the compositions of n. Then

$$\sum_{n=0}^{\infty} A_n x^n = \frac{x^{a_k}}{[1-F(x)]^2} .$$

Proof. It is easy to see that

$$A_n = A_{n-a_0} + A_{n-a_1} + \cdots + A_{n-a_k} + C_{n-a_k} + \cdots,$$

where C_j and $A_j = 0$ if $j < 0$.

$$\sum_{n=0}^{\infty} A_n x^n = (x^{a_0} + x^{a_1} + \cdots + x^{a_s} + \cdots) \sum_{n=0}^{\infty} A_n x^n + x^{a_k} \sum_{n=0}^{\infty} C_n x^n$$

from which Theorem 1.3 follows after applying Theorem 1.1.

It follows from Theorem 1.3 that the total use of all a_k is given by all integer counts in the expansion of

$$\frac{F(x)}{[1-F(x)]^2} .$$

Since the number of plus signs occurring is given by the total number of integers used minus the total number of compositions less the one for zero, the number of plus signs has generating function given by

$$\frac{F(x)}{[1-F(x)]^2}-\frac{F(x)}{1-F(x)} = \frac{F^2(x)}{[1-F(x)]^2} .$$

Theorem 1.4. The number of occurrences of a_k in the palindromic compositions of n, denoted by U_n, is given by the generating function

$$\frac{x^{a_k}}{1-F(x^2)}+\frac{2x^{2a_k}(1+F(x))}{[1-F(x^2)]^2} = \sum_{n=0}^{\infty} U_n x^n .$$

Proof. To count the occurrences of a_k in the palindromic compositions of n,

$$U_n = U_{n-2a_0}+U_{n-2a_1}+\cdots+(U_{n-2a_k}+2P_{n-2a_k})+\delta = \begin{cases} 1 \text{ if } n=a_k \\ 0 \text{ if } n \neq a_k \end{cases}$$

the one being for the single palindrome a_k, and U_j and $P_j = 0$ for $j < 0$.

$$\sum_{n=0}^{\infty} U_n x^n = x^{2a_0}(U_0+U_1x+U_2x^2+\cdots)+x^{2a_1}(U_0+U_1x+U_2x^2+\cdots)$$
$$+\cdots+x^{2a_s}(U_0+U_1x+U_2x^2+\cdots)+\cdots+x^{a_k}$$
$$+2x^{2a_k}\sum_{n=0}^{\infty} P_n x^n .$$

Therefore, applying Theorem 1.2 and simplifying yields Theorem 1.4.

As before, from Theorem 1.4 we can write the total number of integers in all palindromic compositions displayed in the form of the generating function

$$\frac{F(x)}{1-F(x^2)}+\frac{2F(x^2)(1+F(x))}{[1-F(x^2)]^2} .$$

Now, in getting all the plus signs counted we need only subtract the generating function for the palindromic compositions of all n except zero. Thus

$$\frac{F(x)}{1-F(x^2)}+\frac{2F(x^2)(1+F(x))}{[1-F(x^2)]^2}-\frac{F(x^2)+F(x)}{1-F(x^2)} = \frac{F(x^2)[1+2F(x)+F(x^2)]}{[1-F(x^2)]^2} .$$

2. APPLICATIONS AND SPECIAL CASES

The results of Section 1 are of particular interest in several special cases.

When the summands are 1 and 2, $F(x) = x + x^2$ gives the result of [1] that the number of compositions of n is F_{n+1}, the $(n+1)^{st}$ Fibonacci number, since by Theorem 1.1,

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-(x+x^2)} = \sum_{n=0}^{\infty} F_{n+1}x^n , \tag{2.1}$$

where we recognize the generating function for the Fibonacci sequence. Theorem 1.2 gives the number of palindromic compositions as

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1+x+x^2}{1-(x^2+x^4)} \tag{2.2}$$

which is the generating function for the interleaved Fibonacci sequence 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, ⋯.

When the summands are 1, 2, and 3, $F(x) = x + x^2 + x^3$ in Theorem 1.1 gives the generating function for the Tribonacci numbers 1, 1, 2, 4, 7, ⋯, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-x-x^2-x^3} = \sum_{n=0}^{\infty} T_{n+1}x^n \tag{2.3}$$

while the number of palindromic compositions from Theorem 1.2 becomes

$$\sum_{n=0}^{\infty} P_n x^n = \frac{1+x+x^2+x^3}{1-x^2-x^4-x^6} \tag{2.4}$$

which generates the interleaved generalized Tribonacci sequence 1, 1, 2, 2, 3, 3, 6, 6, 11, 11, 20, 20,⋯.

When the summands are 1, 2, 3, ⋯, k, then $F(x) = x + x^2 + \cdots + x^k$ in Theorem 1.1 gives the generating function for a sequence of generalized Fibonacci numbers $\{F_n^*\}$ defined by

$$F_{n+k}^* = F_{n+k-1}^* + F_{n+k-2}^* + \cdots + F_n^*, \quad F_1^* = 1, \quad F_n^* = 2^{n-1}, \quad n = 2, 3, 4, \cdots, k,$$

so that $C_n = F_{n+1}^*$.

When the summands are the positive integers, $F(x) = x + x^2 + x^3 + \cdots = x/(1-x)$ in Theorem 1.1 gives the number of compositions of n as 2^{n-1}, $n \geq 1$, since

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x}{1-x}} = \frac{1-x}{1-2x} \tag{2.5}$$

which generates 1, 1, 2, 4, 8, 16, 32, ⋯. Applying Theorem 1.2 to find the number of palindromic compositions gives the generating function for the sequence 1, 1, 2, 2, 4, 4, 8, 8, ⋯, or, $P_n = 2^{[n/2]}$, $n = 0, 1, 2, \cdots$, where $[x]$ is the greatest integer function.

Taking odd summands 1, 3, 5, 7, ⋯, and using $F(x) = x + x^3 + x^5 + x^7 + \cdots = x/(1-x^2)$ in Theorem 1.2 to find the number of palindromic compositions of n again gives the generating function for the interleaved Fibonacci sequence 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, ⋯, while Theorem 1.1 gives the number of compositions of n as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = \sum_{n=0}^{\infty} (F_{n+1} - F_{n-1})x^n \tag{2.6}$$

so that $C_n = F_n$.

If we use the sequence 1, 2, 4, 5, 7, 8, ⋯, the integers omitting all multiples of 3, then

$$F(x) = (x+x^2) + (x^4+x^5) + (x^7+x^8) + \cdots = (x+x^2)/(1-x^3)$$

yields the number of compositions of n as

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x+x^2}{1-x^3}} = \frac{1-x^3}{1-x-x^2-x^3} \tag{2.7}$$

so that, returning to Eq. (2.3), $C_n = T_{n+1} - T_{n-2}$, where T_n is the n^{th} Tribonacci number.

If we take

$$F(x) = x^2 + x^3 + x^4 + x^5 + \cdots = \frac{x^2}{1-x},$$

the number of compositions of n using the sequence of integers greater than 1 is given by

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-\dfrac{x^2}{1-x}} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n-1}x^n \tag{2.8}$$

so that $C_n = F_{n-1}$. Applying Theorem 1.2 we again find the number of palindromic compositions to be the interleaved Fibonacci sequence, but with the subscripts shifted down from before, as 1, 0, 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, ···. (Note: Zero is represented vacuously; one not at all.)

The sequence of multiples of k used for summands leads to

$$F(x) = x^k + x^{2k} + x^{3k} + \cdots = x^k/(1-x^k),$$

which in Theorem 1.1 gives us

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-x^k}{1-2x^k} = 1 + \sum_{m=1}^{\infty} 2^{m-1}x^{km} \tag{2.9}$$

so that the number of compositions of n is 2^{m-1} if $n = km$ or 0 if $n \neq km$ for an integer m.

3. SEQUENCES WHICH CONTAIN REPEATED ONE'S

Compositions formed from sequences which contain repeated one's also lead to certain generalized Fibonacci numbers. We think of labelling the one's in each case so that they can be distinguished. These are weighted compositions.

First, 1, 1, and 2 used as summands gives $F(x) = x + x + x^2$ so that

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} P_{n+1}x^n \tag{3.1}$$

so that $C_n = p_{n+1}$ where p_n is the n^{th} Pell number defined by $p_1 = 1, p_2 = 2, p_{n+2} = 2p_{n+1} + p_n$. Applying Theorem 1.2, we find that we have the generating function for the sequence 1, 2, 3, 4, 7, 10, 17, 24, 41, ···, which is a sequence formed from interleaved generalized Pell sequences, having the same recursion relation as the Pell sequence but different starting values.

In general, if we use the sequence 1, 1, 1, ···, 1, 2 (k one's) as summands, $F(x) = x + x + x + \cdots + x + x^2 = kx + x^2$ in Theorem 1.1 gives

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-kx-x^2} = \sum_{n=0}^{\infty} p^*_{n+1}x^n, \tag{3.2}$$

where

$$p_1^* = 1, \quad p_2^* = k, \quad p^*_{n+2} = kp^*_{n+1} + p^*_n.$$

Thus, the number of compositions of n formed from this sequence is $C_n = p^*_{n+1}$. The number of palindromic compositions is again a sequence formed from two interleaved generalized Pell sequences, having the same recursion relation as p_n^* but different starting values. The starting values for one sequence are 1 and $k + 1$; for the second, k and k^2. Thus, the interleaved sequence begins

$$1,\ k,\ k+1,\ k^2,\ k^2+k+1,\ k^3+k,\ k^3+k^2+2k+1,\ k^4+2k^2,\ \cdots.$$

One other special case using repeated ones is interesting. When the sequence 1, 1, 1, 1, 2 is used as summands,

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1-4x-x^2} = \sum_{n=0}^{\infty} \frac{F_{3(n+1)}}{2}\, x^n \tag{3.3}$$

using the known generating function [2], where L_k is the k^{th} Lucas number, (3.4) $$\frac{F_k x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn} x^n .$$

Actually, as a bonus, this gives us two simple results; F_{3k} is always divisible by 2, since C_n is an integer, and, from the recursion relation $C_{n+2} = 4C_{n+1} + C_n$, we have

$$F_{3(n+2)} = 4F_{3(n+1)} + F_{3n} .$$

But, we can go further. Equation (3.4) combined with Theorem 1.1 for odd k gives us

(3.5) $$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - L_k x - x^2} = \sum_{n=0}^{\infty} (F_{k(n+1)}/F_k) x^n, \quad k \text{ odd},$$

so that

$$C_n = F_{k(n+1)}/F_k$$

when L_k repeated ones and a 2 are used for the sequence from which the compositions of n are made, k odd. Since C_n is an integer, we prove in yet another way that F_k divides F_{kn} [3], as well as write the formula

(3.6) $$F_{k(n+2)} = L_k F_{k(n+1)} + F_{kn}, \quad k \text{ odd},$$

4. APPLICATIONS TO RISING DIAGONAL SUMS IN PASCAL'S TRIANGLE

The generalized Fibonacci numbers of Harris and Styles [4], [5] are the numbers $u(n;p,q)$ which are found by taking the sum of elements appearing along diagonals of Pascal's triangle written in left-justified form. The number $u(n;p,q)$ is the sum of the elements found by beginning with the left-most element in the n^{th} row and taking steps of p units up and q units right throughout the array. We recall that

(4.1) $$\frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;p,q) x^n .$$

Note that $p = q = 1$ yields the Fibonacci numbers, or, $F_{n+1} = u(n;1,1)$. Now, Eq. (4.1) combined with Theorem 1.1 gives us the number of compositions of n from the sequence $\{1, p+1\}$ as

(4.2) $$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - x - x^{p+1}} = \sum_{n=0}^{\infty} u(n;p,1) x^n$$

so that $C_n = u(n;p,1)$, the sequence of diagonal sums found in Pascal's triangle by taking steps of p units up and 1 unit right throughout the array. Note again that $p = 1$ gives us the Fibonacci sequence.

Suppose that the compositions are made from the sequence of integers greater than or equal to $p + 1$. Then

$$F(x) = x^{p+1} + x^{p+2} + x^{p+3} + \cdots = x^{p+1}/(1-x),$$

so that Theorem 1.1 gives

(4.3) $$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - \dfrac{x^{p+1}}{1-x}} = \frac{1-x}{1 - x - x^{p+1}} = \sum_{n=0}^{\infty} [u(n;p,1) - u(n-1;p,1)] x^n$$

and the number of compositions of n becomes

$$C_n = u(n;p,1) - u(n-1;p,1) .$$

Again the special case $p = 1$ yields Fibonacci numbers, with $C_n = F_{n-1}$.

Now, if the compositions are made from the sequence $1, p+2, 2p+3, \cdots$ or the sequence formed by taking every $(p+1)^{st}$ integer,

$$F(x) = x + x^{p+2} + x^{2p+3} + x^{3p+4} + \cdots = x/(1 - x^{p+1})$$

in Theorem 1.1 gives

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1}{1 - \dfrac{x}{1 - x^{p+1}}} = \frac{1 - x^{p+1}}{1 - x - x^{p+1}} \qquad (4.4)$$

so that

$$C_n = u(n;p,1) - u(n-p-1;p,1).$$

Again, $p = 1$ yields Fibonacci numbers, being the case of the sequence of odd integers, where $C_n = F_n$, as in (2.6).

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A NOTE ON TOPOLOGIES ON FINITE SETS

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In an article [1] by D. Stephen, it was shown that an upper bound for the number of elements in a non-discrete topology on a finite set with n elements is $3(2^{n-2})$ and moreover, that this upper bound is attainable. The following example and theorem furnish a much easier proof of these results.

Example. Let b, c be distinct elements of a finite set X with $n (n \geq 2)$ elements. Define

$$\Gamma = \{ A \subset X \mid b \in A \text{ or } c \notin A \}.$$

Now Γ is a topology on X and since there are 2^{n-1} subsets of X containing b and 2^{n-2} subsets of X which do not intersect $\{ b, c \}$ we have

$$2^{n-1} + 2^{n-2} = 3(2^{n-2})$$

elements in Γ.

Theorem. If Σ is a non-discrete topology on a finite set X, then Σ is contained in a topology of the type defined in the example.

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