COMPOSITIONS WITH ONES AND TWOS

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A great deal of literature has been published on the compositions of integers. In this paper, we attempt to throw some new light by discussing compositions which lead to recurrence relations. Actually, in this article we restrict our attention to compositions using *only* ones and twos. Compositions using 1, 2, and 3, ⋯, or 1 and 3 will lead to more general recurrences, but this will form the subject of later investigations.

Definition 1. Denote by C_n for positive integral n, the number of compositions of n using only 1 and 2.

We make the convention that whenever we refer to the word "composition" in this paper, we mean compositions with 1 and 2 unless specially mentioned.

Examples:

n	Compositions of n	C_n
1	1	1
2	2, 1 + 1	2
3	2 + 1, 1 + 2, 1 + 1 + 1	3
4	2 + 2, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2, 1 + 1 + 1 + 1	5
5	2 + 2 + 1, 2 + 1 + 2, 1 + 2 + 2, 2 + 1 + 1 + 1, 1 + 2 + 1 + 1, 1 + 1 + 2 + 1, 1 + 1 + 1 + 2, 1 + 1 + 1 + 1 + 1	8

The Fibonacci enthusiast will immediately recognize the Fibonacci number pattern in the sequence C_n. So we have

Theorem 1. $$C_n = F_{n+1}, \qquad n = 1, 2, 3, \cdots,$$

where the F_n are the Fibonacci numbers,

$$F_{n+2} = F_{n+1} + F_n, \qquad F_1 = F_2 = 1.$$

Proof 1. It is quite clear from the table that Theorem 1 holds for $n = 1, 2, \cdots, 5$. Let $C_m(1)$ and $C_m(2)$ denote the number of compositions of m that end in 1 or 2, respectively. We then have, trivially,

(1) $$C_{n+1} = C_{n+1}(1) + C_{n+1}(2).$$

Pick a composition of $(n + 1)$, ending in a one. If we remove the one at the end, we get a composition of n. Conversely, to a composition of n by adding a one at the end we get a composition for $(n + 1)$. Therefore,

(2) $$C_{n+1}(1) = C_n.$$

Now consider a composition of $(n + 1)$ ending in a two. If we remove the two at the end, we get a composition for $(n - 1)$. Conversely, we could get a composition for $(n + 1)$ from $(n - 1)$, by adding a two or two ones. The latter case has been counted by (2) and so we have

(3) $$C_{n+1}(2) = C_{n-1}.$$

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Now, (2) and (3) together with (1) establish Theorem 1 by induction.

Proof 2. Consider the generating function

$$C(x) = \frac{x+x^2}{1-(x+x^2)} . \tag{4}$$

Clearly,

$$C(x) = \sum_{n=0}^{\infty} (x+x^2)^{n+1} = \sum_{n=1}^{\infty} (x+x^2)^n .$$

If we now collect the terms with exponent n, we get C_n terms! This gives

$$C(x) = \sum_{n=1}^{\infty} C_n x^n .$$

But we also find from (4) that

$$C(x) = \frac{1}{1-(x+x^2)} - 1 = \sum_{n=1}^{\infty} F_n x^{n-1} - 1 = \sum_{n=1}^{\infty} F_{n+1} x^n = \sum_{n=1} C_n x^n .$$

This proves that $C_n = F_{n+1}$, establishing Theorem 1.

Let $f_1(n)$ and $f_2(n)$ denote the number of ones and the number of twos in the compositions, respectively. Let $p(n)$ denote the number of "+" signs that occur in the compositions of n.

Theorem 2.

$$f_1(n+1) = f_1(n) + f_1(n-1) + F_{n+1}, \qquad f_2(n+1) = f_2(n) + f_2(n-1) + F_n .$$

Proof. Split all the compositions of $(n+1)$ as

$$C_{n+1} = C_{n+1}(1) + C_{n+1}(2).$$

Since $C_{n+1}(2) = C_{n-1}$, we have $f_1(n-1)$ ones since a "2" is not going to affect the counting of ones. We have also by (2) that $C_{n+1}(1) = C_n$, and we have an extra "1" in each composition counted by $C_{n+1}(1)$. So we have counted $f_1(n) + C_n$ ones, proving

$$f_1(n+1) = f_1(n) + f_1(n-1) + F_{n+1} .$$

Now, going back to $C_{n+1}(1)$ and $C_{n+1}(2)$ and using (3) and (2), we can get by similar arguments that

$$f_2(n+1) = f_2(n) + f_2(n-1) + F_n .$$

This proves Theorem 2.

Theorem 3. $$f_2(n+1) = f_1(n).$$

Proof. One can verify Theorem 3 for $n = 1, 2, 3$. Now, by Theorem 2, we have

$$f_1(n) = f_1(n-1) + f_1(n-2) + F_n , \tag{5}$$

$$f_2(n+1) = f_2(n) + f_2(n-1) + F_n . \tag{6}$$

Now, Eqs. (5) and (6) establish Theorem 3 by induction.

Theorem 4. The sequence $f_1(n)$ is the Fibonacci convolution sequence.

Proof. By induction and from Theorem 2.

Theorem 5. The sequence $p(n)$ is the convolution sequence of C_n.

Proof. First let us find the generating functions of the sequence $f_1(n)$ and $f_2(n)$. We have by Theorem 3 and Theorem 4 that

$$\sum_{n=1}^{\infty} f_1(n) x^n = \frac{x}{[1-(x+x^2)]^2}$$

and

$$\sum_{n=1}^{\infty} f_2(n)x^n = \frac{x^2}{[1-(x+x^2)]^2} .$$

From the definition of $p(n)$ it trivially follows that

(7) $$p(n) = f_1(n) + f_2(n) - C_n$$

so that we have by (7) that

$$\sum_{n=1}^{\infty} p(n)x^n = \frac{x}{[1-(x+x^2)]^2} + \frac{x^2}{[1-(x+x^2)]^2} - \frac{x+x^2}{1-(x+x^2)} = \frac{(x+x^2)-(x+x^2)[1-(x+x^2)]}{[1-(x+x^2)]^2}$$
$$= \frac{(x+x^2)^2}{[1-(x+x^2)]^2} = [C(x)]^2$$

proving Theorem 5.

We next shift our attention to compositions with special properties. A composition of n is defined to be "palindromic" if written in reverse order it remains unchanged.

Examples: 1 + 2 + 2 + 1 is a palindromic composition of 6 while 1 + 2 + 1 + 2 is not.

Let $\Pi(n)$ denote the number of palindromic compositions of n, $\Pi(n,1)$ the number of those ending with 1, and $\Pi(n,2)$ the number of those ending with 2. Let $\Pi_1(n)$ and $\Pi_2(n)$ denote the number of ones and the number of twos in all the palindromic compositions of n, respectively. Let $\Pi_+(n)$ denote the number of "+" signs in the palindromic compositions of n.

Theorem 6. $$\Pi(n+1) = \Pi(n-1) + \Pi(n-3),$$

and the sequence $\Pi(n)$ is an alternation of Fibonacci sequences

$$1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, \cdots.$$

To be more precise,

$$\Pi(2n+1) = F_n, \qquad \Pi(2n) = F_{n+2}.$$

Proof. We can split

(8) $$\Pi(n+1) = \Pi(n+1, 1) + \Pi(n+1, 2).$$

Since $\Pi(n+1, 1)$ counts the palindromic compositions ending in a 1, by removing the 1's on both sides we get a palindromic composition for $(n-1)$. So we have

(9) $$\Pi(n+1, 1) = \Pi(n-1)$$

and

(10) $$\Pi(n+1, 2) = \Pi(n-3)$$

by similar arguments. Now (9), (10) and (8) together yield Theorem 6. The Π-functions also obey

(11) $$\Pi(n+2) = \Pi(n+1) + (-1)^n \Pi(n).$$

Examples:

n	Palimdromic Compositions of n	$\Pi(n)$
1	1	1
2	2, 1 + 1	2
3	1 + 1 + 1	1
4	2 + 2, 1 + 2 + 1, 1 + 1 + 1 + 1	3
5	2 + 1 + 2, 1 + 1 + 1 + 1 + 1	2
6	2 + 2 + 2, 2 + 1 + 1 + 2, 1 + 2 + 2 + 1, 1 + 1 + 2 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1	5
...	...	...

We now define enumerating polynomials on the above compositions. For a certain n, $\phi_n(x)$ contains the term "ax^b" if there are "a" compositions with "b" + signs. The sequence of polynomials $\phi_n(x)$ is:

$$1,\ x+1,\ x^2,\ x^3+2x^2+x,\ x^2(x^2+1),\ x^5+x^4+2x^3+x^2,\ \cdots$$

obeying the recurrence

$$\phi_{n+2}(x) = x^2[\phi_n(x)+\phi_{n-2}(x)] \tag{12}$$

and this is quite obvious, for

$$\phi_{n+2}(x) = \Pi(n+2). \tag{13}$$

Theorem 7.

$$\Pi_+(n+2) = \Pi_+(n)+\Pi_+(n-2)+2\Pi(n+2), \tag{14}$$

$$\Pi_1(n+2) = \Pi_1(n)+\Pi_1(n-2)+2\Pi(n), \tag{15}$$

$$\Pi_2(n+2) = \Pi_2(n)+\Pi_2(n-2)+2\Pi(n-2). \tag{16}$$

Proof. First we prove (14). From the definition of $\phi_n(x)$ it is evident that

$$\left.\frac{d\phi_n(x)}{dx}\right|_{at\,x=1} = \Pi_+(n).$$

By (12) we have

$$\frac{d\phi_{n+2}(x)}{dx} = x^2\left[\frac{d\phi_n(x)}{dx}+\frac{d\phi_{n-2}(x)}{dx}\right]+2x[\phi_n(x)+\phi_{n-2}(x)].$$

Now, using (13) and Theorem 6 we get

$$\Pi_+(n+2) = \Pi_+(n)+\Pi_+(n-2)+2\Pi(n+2).$$

We prove (15) and (16) combinatorially. Split the compositions of $(n+2)$ as

$$\Pi(n+2) = \Pi(n+2,1)+\Pi(n+2,2).$$

We know

$$\Pi(n+2,1) = \Pi(n), \quad \text{and} \quad \Pi(n+2,2) = \Pi(n-2).$$

Now, in the compositions counted by $\Pi(n+2,2)$, the extra 2 does not affect the counting of 1's. Therefore, we have counted $\Pi_1(n-2)$ "ones." The compositions counted by $\Pi(n+2,1)$ contain two extra ones, compared to those counted by $\Pi(n)$, and so we count $\Pi_1(n)+2\Pi(n)$ ones. This proves

$$\Pi_1(n+2) = \Pi_1(n)+\Pi_1(n-2)+2\Pi(n).$$

By the same arguments we find the compositions counted by $\Pi(n+2,1)$ contains the same number of twos as those counted by $\Pi(n)$ and so we have counted $\Pi_2(n)$ twos. But the compositions counted by $\Pi(n+2,2)$ contain two extra 2's compared to those counted by $\Pi(n-2)$ giving $\Pi_2(n-2)+2\Pi(n-2)$. Putting these together,

$$\Pi_2(n+2) = \Pi_2(n)+\Pi_2(n-2)+2\Pi(n-2).$$

Theorem 8.

$$\Pi_+(n) = \Pi_+(n-1)+(-1)^{n-2}\Pi(n-2)+\Pi(n), \tag{17}$$

$$\Pi_1(n+1) = \Pi_1(n)+(-1)^{n-1}\Pi_1(n-1)+\Pi(n), \tag{18}$$

$$\Pi_2(n+2) = \Pi_2(n+1)+(-1)^n\Pi_2(n)+(-1)^n\Pi(n). \tag{19}$$

Proof. We know by Theorem 7 that the following hold:

$$\Pi_+(n+2)-\Pi_+(n)-\Pi_+(n-2) = 2\Pi(n+2)$$

$$\Pi_+(n+1)-\Pi_+(n-1)-\Pi_+(n-3) = 2\Pi(n+1)$$

$$\Pi_+(n)-\Pi_+(n-2)-\Pi_+(n-4) = 2\Pi(n).$$

We also know that the Π-functions satisfy

$$\Pi_+(n+2) = \Pi_+(n+1)+(-1)^n\Pi(n).$$

If we put these together we get

$$\Pi_+(n+2)-\Pi_+(n)-\Pi_+(n-2) = \Pi_+(n+1)-\Pi_+(n-1)-\Pi_+(n-3)+(-1)^n[\Pi_+(n)-\Pi_+(n-2)-\Pi_+(n-4)].$$

Assume that for a fixed n, (14) holds for n and $(n-2)$. This means that we get from the above the following:

$$\Pi_+(n+2) - \Pi_+(n-1) - (-1)^{n-2}\Pi_+(n+2) - \Pi(n) - \Pi_+(n-3) - (-1)^{n-4}\Pi_+(n-4) - \Pi(n-2)$$
$$= \Pi_+(n+1) - \Pi_+(n-1) - \Pi_+(n-3) + (-1)^n[\Pi_+(n) - \Pi_+(n-2) - \Pi_+(n-4)]$$

which simplifies to

$$\Pi_+(n+2) = \Pi_+(n+1) + (-1)^n\Pi_+(n) + \Pi(n+2)$$

establishing (17) for $(n+2)$. Now one can verify (17) for $n = 0,1,2,\cdots,5$, and so (17) holds by induction.

Now, to prove (18), we observe from Theorem 7 that

$$\Pi_1(n+2) - \Pi_1(n) - \Pi_1(n-2) = 2\Pi(n)$$
$$\Pi_1(n+1) - \Pi_1(n-1) - \Pi_1(n-3) = 2\Pi(n-1)$$
$$\Pi_1(n) - \Pi_1(n-2) - \Pi_1(n-4) = 2\Pi(n-2).$$

If we again use (11) we find

$$\Pi_1(n+2) - \Pi_1(n) - \Pi_1(n-2) = \Pi_1(n+1) - \Pi_1(n-1) - \Pi_1(n-3) + (-1)^n[\Pi_1(n) - \Pi_1(n-2) - \Pi_1(n-4)].$$

Now, if we assume that for a fixed n, Eq. (18) holds for $(n-2)$ and n, then we have

$$\Pi_1(n+2) - \Pi_1(n-1) - (-1)^{n-2}\Pi_1(n-2) - \Pi_1(n-1) - \Pi_1(n-3) - (-1)^n\Pi_1(n-4) - \Pi(n-3)$$
$$= \Pi_1(n+1) - \Pi_1(n-1) - \Pi_1(n-3) + (-1)^n[\Pi_1(n) - \Pi_1(n-2) - \Pi_1(n-4)]$$

which simplifies to

$$\Pi_1(n+2) = \Pi_1(n+1) + (-1)^n\Pi_1(n) + \Pi(n+1)$$

establishing (18) for $(n+2)$. Again one can verify (18) for $n = 1, 2, 3, 4, 5$, and so (18) holds by induction for all positive integral n.

We prove (19) with the aid of (17) and (18). From the definitions of Π_1, Π_2, and Π_+ we get

$$\Pi_2(n) = \Pi_+(n) + \Pi(n) - \Pi_1(n).$$

If (19) were to hold, we must have

$$\Pi_+(n+2) + \Pi(n+2) - \Pi_1(n+2) = \Pi_+(n+1) + \Pi(n+1) - \Pi_1(n+1)$$
$$+ (-1)^n[\Pi_+(n) + \Pi(n) - \Pi_1(n)] + (-1)^n\Pi(n).$$

Since (17) and (18) holds, we have

$$\Pi_+(n+1) + (-1)^n\Pi_+(n) + \Pi(n+2) + \Pi(n+2) - \Pi_1(n+1) - (-1)^n\Pi_1(n) - \Pi(n+1)$$
$$= \Pi_+(n+1) + \Pi(n+1) - \Pi_1(n+1) + (-1)^n[\Pi_+(n) + \Pi(n) - \Pi_1(n)] + (-1)^n\Pi(n)$$

which reduces to

$$2\Pi(n+2) = 2\Pi(n+1) + 2(-1)^n\Pi(n),$$

which we know is true. This establishes (19) and so Theorem 8. Note that we could have proved (19) in the same way as we did (17) and (18).

Definitions. If in a composition of n, a 2 follows a 1, we say it is a "rise," and if a 1 follows a 2, it is a "fall." Two 1's or two 2's contribute a "straight."

Let $R(n)$, $F(n)$, and $S(n)$ denote the number of rises, falls, and straights, respectively, in the compositions of n. It is easy to establish that

(20) $$R(n) = F(n)$$

and

$$p(n) = R(n) + F(n) + S(n).$$

Theorem 9.

$$R(n+2) = R(n+1) + R(n) + F_n$$

and $R(n)$ is the Fibonacci convolution sequence displaced.

Proof. Partition the compositions of $(n+2)$ as

$$C_{n+2} = C_{n+2}(1) + C_{n+2}(2).$$

We know

$$C_{n+2}(1) = C_{n+1}, \quad \text{and} \quad C_{n+2}(2) = C_n .$$

The 1 at the end of the compositions counted by $C_{n+2}(1)$ will not affect the counting of rises counted in the compositions included in C_{n+1}. But the 2 at the end of the compositions counted by $C_{n+2}(2)$ will contribute an extra rise if and only if the compositions counted by C_n end in a 1. This is true for $C_n(1) = F_n$ compositions. This proves

(21) $$R(n+2) = R(n+1) + R(n) + F_n .$$

The form of the recurrence in (21) and induction establishes the second part of Theorem 9.

Theorem 10. $$S(n+1) = S(n) + S(n-1) + L_{n-1},$$

where $L_n = F_{n+1} + F_{n-1}$ are Lucas numbers. Further,

(22) $$S(n) = R(n+1) + R(n-1).$$

Proof. Partition as before

$$C_{n+1} = C_{n+1}(1) + C_{n+1}(2).$$

We know that $C_{n+1}(1) = C_n$. The extra 1 at the end, in the compositions counted by $C_{n+1}(1)$ will give an extra "straight" if the corresponding composition counted by C_n ends in 1. So we have $C_n(1) = F_n$ extra "straights."

Now,

$$C_{n+1}(2) = C_{n-1} = F_n,$$

and so the 2 at the end of the compositions counted by $C_{n+1}(2)$ will contribute an extra "straight," if the corresponding compositions counted by C_{n-1} end in 2. This happens for $C_{n-1}(2) = F_{n-2}$ compositions, and so we have

(23) $$S(n+1) = S(n) + F_n + S(n-1) + F_{n-2} = S(n) + S(n-1) + L_{n-1}.$$

We can establish the second part of Theorem 10 by induction on (22). Let

$$S(n) = R(n+1) + R(n-1)$$

for $n = 1, 2, 3, \cdots, m$. We know by (23) that

$$S(m+1) = S(m) + S(m-1) + L_{m-1}$$

which can be split up as

$$S(m+1) = R(m+1) + R(m-1) + R(m) + R(m-2) + F_m + F_{m-2}.$$

This can be grouped as

$$\begin{aligned} S(m+1) &= R(m+1) + R(m) + F_m + R(m-1) + R(m-2) + F_{m-2} \\ &= R(m+2) + R(m) \end{aligned}$$

by Theorem 9, establishing (22) for $n = m+1$.

This proves the theorem.

Theorem 11. The sequence $S(n)$ is a convolution of the Fibonacci and Lucas sequences.

Proof. One could say that Theorem 11 follows by observing the form of (23). We, however, use generating functions to prove Theorem 11.

By Theorem 9 we know the "R" to be the displaced Fibonacci convolution sequence. So

$$\sum_{n=1}^{\infty} S(n)x^n = \sum_{n=1}^{\infty} [R(n+1) + R(n-1)]x^n$$

$$= \frac{x^2}{[1-(x+x^2)]^2} + \frac{x^4}{[1-(x+x^2)]^2} = \frac{x(x+x^3)}{[1-(x+x^2)]^2} = \frac{x}{1-(x+x^2)} \cdot \frac{x+x^3}{1-(x+x^2)}$$

which says that the $S(n)$ is the convolution of the Fibonacci and Lucas sequences shown below:

Lucas (with extra 1): 1, 1, 3, 4, 7, 11, 18, 29, ⋯
Fibonacci: 1, 1, 2, 3, 5, 8, 13, 21, ⋯

This completes the proof.

We can actually state a stronger form of Theorem 10. If $S_1(n)$ and $S_2(n)$ are defined to be the number of "straights" counted as 1 + 1 and 2 + 2, respectively, in the compositions of n, then it is obvious that

$$S(n) = S_1(n) + S_2(n).$$

We also know

$$S(n) = R(n+1) + R(n).$$

It is indeed remarkable that

Theorem 12. $$R(n+1) = S_1(n) \quad \text{and} \quad R(n) = S_2(n).$$

Tables:

n	C_n	$f_1(n)$	$f_2(n)$	$p(n)$	$R(n)$	$S(n)$	$\Pi(n)$	$\Pi_1(n)$	$\Pi_2(n)$	$\Pi_+(n)$
1	1	1	0	0	0	0	1	1	0	0
2	2	2	1	1	0	1	2	2	1	1
3	3	5	2	4	1	2	1	3	0	2
4	5	10	5	10	2	6	3	6	3	6
5	8	20	10	22	5	12	2	6	2	6
6	13	38	20	63	10	25	5	14	8	17
.	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

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