LATTICE PATHS AND FIBONACCI AND LUCAS NUMBERS

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Several papers have been presented in this quarterly relating lattice paths and Fibonacci numbers: [1], [5], and [6]. In [1] Greenwood remarked about a certain artificialness in his approach. Here we present what we believe is a more natural approach which gives direct derivations of the formulae. We also obtain the Lucas numbers and some generalizations.

1. INTRODUCTION

By a lattice point in the plane is meant a point with integral coordinates. Unless otherwise stated we take these to be non-negative integers. By a path (or lattice path) is meant a minimal path via lattice points taking unit horizontal and unit vertical steps.

It is well known [2, p. 167] that the number of paths from $(0,0)$ to (p,q) is

$$\binom{p+q}{p} . \tag{1}$$

If we associate a plus sign with each horizontal step and a minus sign with each vertical step, there is a one-to-one correspondence between the paths from $(0,0)$ to (p,q) and the arrangements of p pluses and q minuses on a line.

Another well known result [2, p. 127] is that the number of paths from $(0,0)$ to (p,q), $p \geqslant q$, which touch but do not cross the line $y = x$ is

$$\frac{p-q+1}{p+1}\binom{p+q}{q} . \tag{2}$$

In other words (2) gives the number of paths from $(0,0)$ to (p,q) such that at any stage the number of vertical steps never exceeds the number of horizontal steps.

For $p = q$, (2) gives

$$\frac{1}{p+1}\binom{2p}{p} ,$$

the Catalan numbers. These have a number of combinatorial applications [3, p. 192].

Note that if (1) is summed over all $p + q = n$, we get the number of paths from $(0,0)$ to the line $x + y = n$. In this case we get

$$\sum_{p=0}^{n}\binom{n}{p} = 2^n .$$

If each of these paths is reflected in the line $x + y = n$, we have all the symmetric paths from $(0,0)$ to (n,n).

If (2) is summed in the same way, we get the paths from $(0,0)$ to $x + y = n$ which may touch but not cross $y = x$. Reflect each of these to get the symmetric paths from $(0,0)$ to (n,n) which do not cross $y = x$. This is a larger collection than Greenwood's.

2. FIBONACCI NUMBERS

The following problem appears in [4, p. 14]. In how many ways can p pluses and q minuses be placed on a line so that no two minuses are together? In our problem we shall also require an initial plus sign to keep the path from crossing $y = x$.

First, we solve the more general problem of finding the number of arrangements of p pluses and q minuses on a line so that before the first minus and between any two minuses there are at least b pluses, $b \geqslant 0$, $p \geqslant bq$.

Arrange the q minuses and bq of the pluses on a line with b pluses before the first minus and b pluses between each pair of minuses. Distribute the remaining $p - bq$ pluses in the $q + 1$ cells determined by the q minuses. This can be done in

(3) $$\binom{p-(b-1)q}{q}$$

ways [4, p. 92].

For the original problem put $b = 1$ to get

(4) $$\binom{p}{q}.$$

Summed over $p + q = n$, $p \geq q$, (4) gives, with q replaced by k, that there are

(5) $$F_{n+1} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k}$$

paths with the stated conditions from $(0,0)$ to $x + y = n$. These paths begin with a horizontal step and can have no two consecutive vertical steps, so they cannot cross $y = x$. Now reflect each path in $x + y = n$ to get the symmetric paths from $(0,0)$ to (n,n).

Here we have replaced each diagonal step of Greenwood with a horizontal step followed by a vertical step. Thus each of our paths crosses $x + y = n$ on a lattice point. As indicated by Stocks [6, p. 83], this accounts for the fact that we have F_{n+1} such paths where Greenwood gets F_{n+2}. That is, of the $h(n)$ paths of Greenwood only $h(n-1)$ cross $x + y = n$ on a lattice point.

Similarly, (3) summed over $p + q = n$, $p \geq bq$, gives

(6) $$F_{n+1}(b) = \sum_{k=0}^{\left[\frac{n}{b+1}\right]} \binom{n-bk}{k}$$

which has the analogous interpretation with respect to the line $by = x$. These numbers have a Fibonacci character, for it is easy to show that

$$F_{n+1}(b) = \begin{cases} 1, & 0 \leq n \leq b, \\ F_n(b) + F_{n-b}(b), & n \geq b+1 \end{cases}$$

For the enumeration of paths without subpaths [5, p. 143] we note that in Greenwood's terminology these are simply those paths which begin with a diagonal step, and the paths to be deleted are those that begin with a horizontal step. By his proof of the recurrence this is precisely $h(n) - h(n-1) = h(n-2)$. In our terminology the paths without subpaths are those that begin with one horizontal step followed by a vertical step, i.e., paths from (1,1). Thus directly by (5) or a recurrence argument similar to Greenwood's we find that these are F_{n-1} in number.

Analogous results can be gotten for the paths enumerated in (6).

3. LUCAS NUMBERS

Again consider the problem in Riordan [4, p. 14] of arranging p pluses and q minuses on a line with no two minuses together. There are

(7) $$\binom{p+1}{q}$$

such arrangements. These are the paths from $(0,0)$ to (p,q) which do not cross $y = x + 1$. That is, a path may start with a vertical step, but there will be no two in succession.

Now look at the paths as enumerated in (7), but with the added restriction that if the first and last steps are both vertical, we consider them as being consecutive. We thus have two mutually exclusive cases. In the first case, the paths start with a vertical step and must end with a horizontal step. These are the paths from $(1,1)$ to $(p-1,q)$. By (7) there are

(8) $$\binom{p-1}{q-1} = \frac{q}{p}\binom{p}{q}$$

of these. In the second case, the paths start with a horizontal step and end with either. These are the paths from $(1,0)$ to (p,q). By (7), there are

$$\binom{p}{q} \tag{9}$$

of these. Note that the second case is really only (4).

Add (8) and (9) to get the solution

$$\frac{p+q}{p}\binom{p}{q} . \tag{10}$$

Summed over $p+q=n$, with $k=q$, (10) gives

$$L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k}\binom{n-k}{k} . \tag{11}$$

Reflect each of these paths in $x+y=n$ to get the symmetric paths from $(0,0)$ to (n,n).

Again it is easy to see that the number of paths without subpaths is L_{n-2}.

In analogy with (3), these results can also be generalized for arbitrary $b \geqslant 0$. In fact, (7) becomes

$$\binom{p-(b-1)(q-1)+1}{q} ,$$

(10) becomes

$$\frac{p+q}{p+q-bq}\binom{p+q-bq}{q} ,$$

and (11) becomes

$$L_n(b) = \sum_{k=0}^{\left[\frac{n}{b+1}\right]} \frac{n}{n-bk}\binom{n-bk}{k} .$$

Again parallel results follow with respect to the line $y=x+b$.

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