SOME IDENTITIES FOR r-FIBONACCI NUMBERS

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Abstract. Let $r \geq 1$ be an integer. The r-generalized Fibonacci sequence $\{G_n\}$ is defined as

$$G_n = \begin{cases} 0, & \text{if } 0 \leq n < r-1; \\ 1, & \text{if } n = r-1; \\ G_{n-1} + G_{n-2} + \cdots + G_{n-r}, & \text{if } n \geq r. \end{cases}$$

We will present several identities and congruences involving r-generalized Fibonacci numbers.

1. Introduction

Definition 1.1. *Let $r \geq 1$ be an integer. The r-generalized Fibonacci sequence $\{G_n\}$ is defined as*

$$G_n = \begin{cases} 0, & \text{if } 0 \leq n < r-1; \\ 1, & \text{if } n = r-1; \\ G_{n-1} + G_{n-2} + \cdots + G_{n-r}, & \text{if } n \geq r. \end{cases}$$

This definition is not new. According to Dickson [2, p. 409], M. d'Ocagne, in a series of papers from 1883 to 1890, considered (with slightly different notation) the sequence $\{U_i\}$ with

$$U_n = c_1 U_{n-1} + c_2 U_{n-2} + \cdots + c_r U_{n-r}$$

and $U_0, \ldots, U_{r-1}$ arbitrary. He also considered the sequence $\{u_n\}$ satisfying the same recurrence, with $u_i = 0$ $(i = 0, \ldots, r-2)$, $u_{r-1} = 1$, and he found a relationship between $\{U_n\}$ and $\{u_n\}$. Evidently r and the c_i's are fixed in these definitions. According to Dickson, for each sequence d'Ocagne "found the sum of any fixed number of consecutive terms and the limit of that sum". In one of the papers involving continued fractions each $c_i = 1$.

More recently (1960), Miles [6] used Definition 1.1 in a paper involving $k \times k$ matrices with k-generalized Fibonacci numbers for elements. Kessler and Schiff [4] stated that the Miles article seems to be the oldest well-known paper on the generalized numbers. Since 1960 many more papers involving the r-generalized Fibonacci numbers have appeared, including several in The Fibonacci Quarterly. See [3] and [4] for example. Kessler and Schiff [4] gave many interesting and relevant references, and they noted that an exhaustive bibliography of these numbers would cover pages.

Though Definition 1.1 is not new, the authors believe that most of the results in this paper are new, or at least not well-known. All of the proofs are original.

In Section 2 we will prove an identity that enables us to find congruences modulo 2^k for the r-generalized Fibonacci sequence. In Section 3 we prove several identities, including a formula for the sum of the squares of the G_n's. In Section 4 we find formulas for G_{2n} and G_{2n+1}, and in Section 5 we prove some miscellaneous results. A table for the r-generalized Fibonacci numbers, for $2 \leq r \leq 8$ can be found in Section 6. Finally, in Section 7 we present possible topics for future research.

2. An Identity and Congruences for $\{G_n\}$

Theorem 2.1. *For $n \geq r+1$,*

$$G_n = 2G_{n-1} - G_{n-r-1}.$$

Proof.

$$\begin{aligned} G_n &= G_{n-1} + G_{n-2} + \cdots + G_{n-r} + 0 \\ &= G_{n-1} + G_{n-2} + \cdots + G_{n-r} \\ &\quad + G_{n-1} - G_{n-2} - \cdots - G_{n-r} - G_{n-r-1} \\ &= 2G_{n-1} - G_{n-r-1}. \end{aligned}$$

□

We note that Gabai [3] stated Theorem 2.1 without proof.

Theorem 2.2. *For $r \leq n \leq 2r-1$,*

$$G_n = 2^{n-r}.$$

For $0 \leq n \leq r$,

$$G_{2r+n} = 2^{n+r} - (n+2)2^{n-1}.$$

Proof. From Theorem 2.1, we have, for $r \leq n \leq 2r-1$,

$$G_n = 2G_{n-1} = 2^2 G_{n-2} = \cdots = 2^{n-r} G_r = 2^{n-r}.$$

We now note that

$$G_{2r} = 2G_{2r-1} - G_{r-1} = 2(2^{r-1}) - 1 = 2^r - 1.$$

Assume $n > 0$, and assume the theorem is true for G_{2r+n-1}. If $r \geq n$, then $2r-1 \geq r+n-1 \geq r$, so $G_{r+n-1} = 2^{n-1}$ by the first part of Theorem 2.2. We have

$$G_{2r+n} = 2G_{2r+n-1} - G_{r+n-1} = [2^{n+r} - (n+1)2^{n-1}] - 2^{n-1} = 2^{n+r} - (n+2)2^{n-1}.$$

□

In a similar way we can prove the following results.

If $0 \leq n \leq r+1$, then

$$G_{3r+n} = 2^{2r+n} - (r+n+2)2^{r+n-1} + \left[\binom{n+2}{2} - 1\right] 2^{n-2}.$$

If $0 \leq n \leq r+2$, then

$$G_{4r+n} = 2^{3r+n} - (2r+n+2)2^{2r+n-1} + \left[\binom{r+n+2}{2} - 1\right] 2^{r+n-2} - \left[\binom{n+2}{3} - n\right] 2^{n-3}.$$

The general theorem is

Theorem 2.3. *Let $m = kr + n$, with $0 \leq n \leq r+k-2$, and $k \geq 2$. Then*

$$G_m = 2^{m-r} + \sum_{j=1}^{k-1} (-1)^j a_{m,j} 2^{m-(j+1)r-j}$$

where

$$a_{m,j} = a_{m-1,j} + a_{m-r-1,j-1}$$

with $a_{i,0} = 1$ for all i, $a_{i,1} = 0$ for $i < 2r$, and $a_{2r,1} = 2$.

For example, in the Fibonacci case ($r = 2$) the recurrence for the coefficients is

$$a_{m,j} = a_{m-1,j} + a_{m-3,j-1}$$

and we can construct a triangle (like Pascal's) to quickly get

$$a_{10,0} = 1, \quad a_{10,1} = 8, \quad a_{10,2} = 14, \quad a_{10,3} = 2, \text{ so}$$
$$F_{10} = G_{10} = 2^8 - (8)2^5 + (14)2^2 - (2)2^{-1} = 55.$$

Theorem 2.4. *For $r \geq 2$ and $m \geq 2r$,*

$$G_m = 2^{m-r} + \sum_{j=1}^{\lfloor \frac{m+r}{r+1} \rfloor - 1} (-1)^j \left[\binom{m - rj - r + 2}{j} - \binom{m - rj - r}{j - 2} \right] 2^{m-(r+1)j-r}.$$

Before looking at the proof, here are some congruences we get by looking at the last term in the summation. We assume $k > 1$ in the proofs of congruences (2.1)–(2.4), and we can use Theorem 2.2 to verify the cases $k = 0$ and $k = 1$.

Case 1: $m = (r + 1)k$. Then $\lfloor \frac{m+r}{r+1} \rfloor - 1 = k - 1$, and $m - (r + 1) \left(\lfloor \frac{m+r}{r+1} \rfloor - 1 \right) - r = 1$. In the last term $\binom{m-kr+2}{k-1} - \binom{m-kr}{k-3} = \binom{k+2}{3} - \binom{k}{3} = k^2$. The exponent of 2 in the next to last term is $r + 2$. Thus:

$$G_{(r+1)k} \equiv (-1)^{k-1} 2k^2 \pmod{2^{r+2}}. \tag{2.1}$$

Case 2: $m = (r+1)k+t$, $0 < t < r+1$. Then $\lfloor \frac{m+r}{r+1} \rfloor - 1 = k$, and $m-(r+1) \left(\lfloor \frac{m+r}{r+1} \rfloor - 1 \right) - r = t - r$. In the last term $\binom{m-rk-r+2}{k} - \binom{m-rk-r}{k-2} = 0$ if $0 < t < r - 1$. In this case the exponent of 2 in the next to last term is $t + 1$. Thus

$$G_{(r+1)k+t} \equiv 0 \pmod{2^{t+1}}, \text{ if } 0 < t < r - 1. \tag{2.2}$$

If $t = r - 1$, then $\binom{m-rk-r+2}{k} - \binom{m-rk-r}{k-2} = (k + 1) - (k - 1) = 2$. Since the exponent of 2 in the next to last term is r, we have

$$G_{(r+1)k+(r-1)} \equiv (-1)^k \pmod{2^r}. \tag{2.3}$$

If $t = r$, then $\binom{m-rk-r+2}{k} - \binom{m-rk-r}{k-2} = \binom{k+2}{k} - \binom{k}{k-2} = 2k + 1$. Thus

$$G_{(r+1)k+r} \equiv (-1)^k (2k + 1) \pmod{2^{r+1}}. \tag{2.4}$$

Proof of Theorem 2.4. We first prove that in Theorem 2.3, the upper limit of the summation, $k - 1$, can be replaced by $\lfloor \frac{m+r}{r+1} \rfloor - 1$.

Case 1: $m = (r + 1)j + t$, with $1 \leq t \leq r$. In this case $m = r(j + 1) + (j - r + t)$, so $k = j + 1$. It is easy to verify that $\lfloor \frac{m+r}{r+1} \rfloor = \lfloor \frac{(r+1)j+r+t}{r+1} \rfloor = j + 1 = k$.

Case 2: $m = (r + 1)j$. In this case, $m = rj + j$, so $k = j$. It is easy to verify that $\lfloor \frac{m+r}{r+1} \rfloor = \lfloor \frac{(r+1)j+r}{r+1} \rfloor = j = k$.

Now we will use Theorem 2.3 to prove Theorem 2.4. Recall $a_{n,0} = 1$ for all n, and $a_{n,1} = 0$ for $n < 2r$. We know that $G_{2r} = 2^r - 1 = 2^r - a_{2r,1} \cdot 2^{-1}$, so $a_{2r,1} = 2$. Using the recurrence

$$a_{m,j} = a_{m-1,j} + a_{m-r-1,j-1}$$

it is easy to see that

$$a_{m,1} = m - 2r + 2 = \binom{m-2r+2}{1} - \binom{m-2r}{-1} \quad (m \geq 2r).$$

Note that the second binomial coefficient is 0.
Now we use induction on j. Suppose we know that for $h = 1, \dots, j$

$$a_{m,h} = \binom{m-rh-r+2}{h} - \binom{m-rh-r}{h-2}.$$

Then

$$\begin{aligned}
a_{m,j+1} &= a_{m-1,j+1} + a_{m-r-1,j} \\
&= a_{m-1,j+1} + \binom{m-r(j+1)-r+1}{j} - \binom{m-r(j+1)-r-1}{j-2} \\
&= a_{m-2,j+1} + \binom{m-r(j+1)-r+1}{j} - \binom{m-r(j+1)-r-1}{j-2} \\
&\quad + \binom{m-r(j+1)-r}{j} - \binom{m-r(j+1)-r-2}{j-2} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&= a_{(j+2)r+j-1,j+1} + \sum_{i=j+1}^{m-r(j+1)-r+1} \left[\binom{i}{j} - \binom{i-2}{j-2} \right].
\end{aligned}$$

Since $a_{(j+2)r+j-1,j+1} = 0$, we have

$$a_{m,j+1} = \sum_{i=j}^{m-r(j+1)-r+1} \left[\binom{i}{j} - \binom{i-2}{j-2} \right] = \binom{m-r(j+1)-r+2}{j+1} - \binom{m-r(j+1)-r}{j-1}.$$

Here we have used the identity

$$\sum_{i=j}^{n} \binom{i}{j} = \binom{n+1}{j+1}.$$

This completes the induction argument. $\square$

Notice that the induction argument works in the case $j = 1$, since in that case we have

$$a_{m,2} = \sum_{i=2}^{m-3r+1} \binom{i}{1} = \sum_{i=1}^{m-3r+1} \binom{i}{1} - 1 = \binom{m-3r+2}{2} - \binom{m-3r}{0}.$$

We have used the fact that $a_{m,j} = 0$ for $m < (j+1)r + (j-1)$. For example $a_{m,1} = 0$ if $m < 2r$; $a_{m,2} = 0$ if $m < 3r + 1$, etc. This follows from the recurrence for $a_{m,j}$. Note that we have used the convention $\binom{a}{b} = 0$ if $a < b$ or if either a or b is negative.

When $r = 2$ in Theorem 2.4, we have the following formula for the Fibonacci numbers.

Theorem 2.5. *For $m \geq 2$,*

$$F_m = 2^{m-2} + \sum_{j=1}^{\lfloor \frac{m+2}{3} \rfloor - 1} (-1)^j \left[\binom{m-2j}{j} - \binom{m-2j-2}{j-2} \right] 2^{m-3j-2}.$$

Corresponding to (2.1), (2.3), (2.4), the congruences for the Fibonacci numbers (for $k \geq 0$) are:

$$F_{3k} \equiv (-1)^{k-1} 2k^2 \pmod{16}, \tag{2.5}$$
$$F_{3k+1} \equiv (-1)^k \pmod{4}, \tag{2.6}$$
$$F_{3k+2} \equiv (-1)^k (2k+1) \pmod{8}. \tag{2.7}$$

If we look at the last two terms of the summation in Theorem 2.5, we see that

$$F_{3k+1} \equiv (-1)^k + (-1)^k (10k)(k+1)(2k+1) \pmod{32}.$$

Thus, if $k \equiv 0$ or $3 \pmod 4$,

$$F_{3k+1} \equiv (-1)^k \pmod{8}.$$

If $k \equiv 1$ or $2 \pmod 4$,

$$F_{3k+1} \equiv (-1)^k 5 \pmod{8}.$$

The A array in Theorem 2.3 and Theorem 2.4 is interesting. For $r = 2$, the Fibonacci case, the A array is

$m\backslash j$	0	1	2	3	4	5
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	1	2	0	0	0	0
5	1	3	0	0	0	0
6	1	4	0	0	0	0
7	1	5	2	0	0	0
8	1	6	5	0	0	0
9	1	7	9	0	0	0
10	1	8	14	2	0	0
11	1	9	20	7	0	0
12	1	10	27	16	0	0
13	1	11	35	30	2	0
14	1	12	44	50	9	0
15	1	13	54	77	25	0

$a_{m,j}$ for $r = 2$ (Fibonacci case).

For $r = 3$, the Tribonacci case, the Tribonacci A array is similar to the Fibonacci A array; there are just more 0's at the top of each column.

Condensing the A array in the Fibonacci case to a B array by defining $b_{m,j} = a_{m+3j,j}$ we have the following B array.

$m\backslash j$	0	1	2	3	4	5
1	1	2	2	2	2	2
2	1	3	5	7	9	11
3	1	4	9	16	25	36
4	1	5	14	30	55	91
5	1	6	20	50	105	196
6	1	7	27	77	182	378
7	1	8	35	112	294	672
8	1	9	44	156	450	1122
9	1	10	54	210	660	1782
10	1	11	65	275	935	2717
11	1	12	77	352	1287	4004
12	1	13	90	442	1729	5733
13	1	14	104	546	2275	8008
14	1	15	119	665	2940	10948
15	1	16	135	800	3740	14688

B array.

The first column of the B array is 1 and after the first entry, the first row is 2. Entries in the middle of the B array can be determined by adding the elements in the B array immediately to its left and directly above the entry, i.e.,

$$b_{m,j} = b_{m,j-1} + b_{m-1,j}.$$

A closed form formula for $b_{m,j}$ is

$$\frac{m(m+1)\cdots(m+j-2)(m+2j-1)}{j!}.$$

Here, we assume the $j = 1$ column is

$$\frac{m+1}{1!}.$$

The entries in the $j = 2$ column of B are A000096 in Sloane's OEIS [7]. The $j = 3$ column of B is A005581 in Sloane's OEIS [7] and the $j = 4$ column of B is A005582 in Sloane's OEIS [7].

3. The Sum of the Squares and Other Identities

Theorem 3.1. *For $n \geq 2r - 1$,*

$$G_n = 2^{r-1} G_{n-r+1} - \sum_{k=1}^{r-1} 2^{k-1} G_{n-r-k}.$$

Proof. We iterate $r-1$ times the recurrence in Theorem 2.1:

$$\begin{aligned}
G_n &= 2G_{n-1} - G_{n-r-1} = 2(2G_{n-2} - G_{n-r-2}) - G_{n-r-1} \\
&= 2^2 G_{n-2} - 2G_{n-r-2} - G_{n-r-1} \\
&= 2^2(2G_{n-3} - G_{n-r-3}) - 2G_{n-r-2} - G_{n-r-1} \\
&= 2^3 G_{n-3} - 2^2 G_{n-r-3} - 2G_{n-r-2} - G_{n-r-1} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&= 2^{r-1} G_{n-r+1} - \sum_{k=1}^{r-1} 2^{k-1} G_{n-r-k}.
\end{aligned}$$

□

Theorem 3.2. *For $r \geq 2$ and $n \geq 2r-1$,*

$$G_n = 2^{r-1} G_{n-r} + \sum_{k=1}^{r-1} \left(\sum_{i=k}^{r-1} 2^{i-1} \right) G_{n-r-k}.$$

For the Fibonacci sequence, this identity is

$$F_n = 2F_{n-2} + F_{n-3}.$$

Listing this identity for $r = 2, 3, 4, 5,$ and 6 we have the resulting formulas.

$$\begin{aligned}
r = 2: &\quad G_n = 2G_{n-2} + G_{n-3} \\
r = 3: &\quad G_n = 4G_{n-3} + 3G_{n-4} + 2G_{n-5} \\
r = 4: &\quad G_n = 8G_{n-4} + 7G_{n-5} + 6G_{n-6} + 4G_{n-7} \\
r = 5: &\quad G_n = 16G_{n-5} + 15G_{n-6} + 14G_{n-7} + 12G_{n-8} + 8G_{n-9} \\
r = 6: &\quad G_n = 32G_{n-6} + 31G_{n-7} + 30G_{n-8} + 28G_{n-9} + 24G_{n-10} + 16G_{n-11}.
\end{aligned}$$

Proof of Theorem 3.2. From Theorem 3.1, we have

$$G_n = 2^{r-1} G_{n-r+1} - \sum_{k=1}^{r-1} 2^{k-1} G_{n-r-k}.$$

Thus, by Definition 1.1 applied to G_{n-r+1}, we have (note that the 1's cancel in the first equation):

$$\begin{aligned}
G_n &= 2^{r-1} G_{n-r} + \sum_{k=1}^{r-1} (2^{r-1} - 1) G_{n-r-k} - \sum_{k=1}^{r-1} (2^{k-1} - 1) G_{n-r-k} \\
&= 2^{r-1} G_{n-r} + \sum_{k=1}^{r-1} \left(\sum_{i=1}^{r-1} 2^{i-1} \right) G_{n-r-k} - \sum_{k=1}^{r-1} \left(\sum_{i=1}^{k-1} 2^{i-1} \right) G_{n-r-k} \\
&= 2^{r-1} G_{n-r} + \sum_{k=1}^{r-1} \left(\sum_{i=k}^{r-1} 2^{i-1} \right) G_{n-r-k}.
\end{aligned}$$

□

Theorem 3.3. *For $r \geq 2$, $n \geq 0$,*

$$\sum_{k=0}^{n} G_k^2 + \sum_{i=2}^{r-1}\sum_{k=0}^{n-i} G_k G_{k+i} = G_n G_{n+1}.$$

The special case of this identity for the Fibonacci sequence was discovered by Lucas in 1876. He discovered that for $n \geq 0$,

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.$$

Its proof can be found in [5, pp. 77–78].

Proof. For $j = n, n-1, \ldots, r-1$, we start with

$$G_j = G_{j+1} - G_{j-1} - G_{j-2} - \cdots - G_{j-r+1}$$

and multiply by G_j to get

$$\begin{aligned}
G_n^2 &= G_nG_{n+1} - G_nG_{n-1} - \cdots - G_nG_{n-i} - \cdots - G_nG_{n-r+1},\\
G_{n-1}^2 &= G_{n-1}G_n - G_{n-1}G_{n-2} - \cdots - G_{n-1}G_{n-1-i} - \cdots - G_{n-1}G_{n-r},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
G_{r-1}^2 &= G_{r-1}G_r - G_{r-1}G_{r-2} - \cdots - G_{r-1}G_{r-1-i} - \cdots - G_{r-1}G_0.
\end{aligned}$$

When we add, we get $\sum_{k=0}^{n} G_k^2$ on the left side, and all of the nonzero terms in columns 1 and 2 on the right side cancel except for G_nG_{n+1}. The $i+1$ column is $-\sum_{k=0}^{n-i} G_kG_{k+i}$ and since $2 \leq i \leq r-1$,

$$\sum_{k=0}^{n} G_k^2 + \sum_{i=2}^{r-1}\sum_{k=0}^{n-i} G_k G_{k+i} = G_n G_{n+1}.$$

□

We note that Gabai [3] proved a result equivalent to Theorem 3.3.

4. Formulas for G_{2n} and G_{2n+1}

Theorem 4.1. *For $r \geq 2$, $n > 0$, $m > 0$,*

$$G_{n+m} = G_nG_m + G_nG_{m-1} + G_{n-1}G_m + \sum_{i=1}^{r-2} G_{n+i}A_i,$$

where

$$A_i = \begin{cases} G_{m+r-i-1} - G_{m+r-i-2} - \cdots - G_{m+1} & (i < r-2);\\ G_{m+1} & (i = r-2). \end{cases}$$

Before giving the proof, here are some applications. If we let $m = n$, we have:

$$\begin{aligned}
r = 2:&\quad F_{2n} = F_n^2 + 2F_{n-1}F_n\\
r = 3:&\quad G_{2n} = G_n^2 + G_{n+1}^2 + 2G_{n-1}G_n\\
r = 4:&\quad G_{2n} = G_n^2 - G_{n+1}^2 + 2G_{n-1}G_n + 2G_{n+1}G_{n+2}\\
r = 5:&\quad G_{2n} = G_n^2 - G_{n+1}^2 + G_{n+2}^2 + 2G_{n-1}G_n - 2G_{n+1}G_{n+2} + 2G_{n+1}G_{n+3}.
\end{aligned}$$

If we let $m = n+1$, we have (after a little manipulation)

$$\begin{aligned}
r = 2: &\quad F_{2n+1} = F_n^2 + F_{n+1}^2 \\
r = 3: &\quad G_{2n+1} = G_n^2 + G_{n+1}^2 + 2G_{n-1}G_{n+1} + 2G_nG_{n+1} \\
r = 4: &\quad G_{2n+1} = G_n^2 + G_{n+1}^2 + G_{n+2}^2 + 2G_{n-1}G_{n+1} + 2G_nG_{n+1} \\
r = 5: &\quad G_{2n+1} = G_n^2 + G_{n+1}^2 - G_{n+2}^2 + 2G_{n-1}G_{n+1} + 2G_nG_{n+1} + 2G_{n+2}G_{n+3}.
\end{aligned}$$

Proof. We will use Zhou's "Theory of Constructing Identities" (TCI) [8].
Let $F_r(x) = x^r - x^{r-1} - \cdots - x - 1$. Then clearly

$$F_r(x) \sum_{i=0}^{m-1} G_{n+i} x^{m-1-i} \equiv 0 \pmod{F_r(x)}.$$

That is, modulo $F_r(x)$:

$$\begin{aligned}
0 &\equiv (x^r - x^{r-1} - \cdots - x - 1)(G_n x^{m-1} + G_{n+1}x^{m-2} + \cdots + G_{n+m-r}x^{r-1} + \cdots + G_{n+m-1}) \\
&\equiv G_n x^{m+r-1} + (G_{n+1} - G_n)x^{m+r-2} + (G_{n+2} - G_{n+1} - G_n)x^{m+r-3} + \cdots \\
&\quad + (G_{n+r-2} - G_{n+r-3} - \cdots - G_n)x^{m+1} + (G_{n+r-1} - G_{n+r-2} - \cdots - G_n)x^m \\
&\quad + 0 \cdot x^{m-1} + 0 \cdot x^{m-2} + \cdots + 0 \cdot x^r - (G_{n+m-1} + G_{n+m-2} + \cdots \\
&\quad + G_{n+m-r})x^{r-1} - \cdots - G_{n+m-1}.
\end{aligned}$$

By TCI, we can replace x^k by G_k and congruence is changed to equality. Thus

$$\begin{aligned}
G_nG_{m+r-1} + (G_{n+1} - G_n)G_{m+r-2} + (G_{n+2} - G_{n+1} - G_n)G_{m+r-3} + \cdots \\
+ (G_{n+r-2} - G_{n+r-3} - \cdots - G_n)G_{m+1} + G_{n-1}G_m - G_{n+m} = 0.
\end{aligned} \tag{4.1}$$

Notice we have used the identities

$$\begin{aligned}
&G_{n+r-1} - G_{n+r-2} - \cdots - G_n = G_{n-1} \quad \text{(in the } x^m \text{ term)}, \\
&-(G_{n+m-1} + G_{n+m-2} + \cdots + G_{n+m-r}) = -G_{n+m} \quad \text{(in the } x^{r-1} \text{ term)}, \\
&G_{r-1} = 1; \quad G_i = 0 \quad \text{for} \quad i < r-1.
\end{aligned}$$

To simplify further, use

$$G_{m+r-1} = G_{m+r-2} + G_{m+r-3} + \cdots + G_{m+1} + G_m + G_{m-1}$$

in the first term of (4.1) and notice that all the other G_n terms are $-(G_{m+r-2} + G_{m+r-3} + \cdots + G_{m+1})G_n$. Thus the G_n terms are reduced to

$$G_nG_m + G_nG_{m-1}.$$

Now for $i = 1, \ldots, r-3$, we get all the G_{n+i} terms together:

$$(G_{m+r-i-1} - G_{m+r-i+2} - \cdots - G_{m+1})G_{n+i}$$

and we note that the only term with G_{n+r-2} is $G_{m+1}G_{n+r-2}$. This completes the proof. □

5. Miscellaneous Results

(a) "Lucas numbers".

Define $K_n = 0$ for $0 \le n \le r-3$, $K_{r-2} = a$, $K_{r-1} = b$ and $K_n = \sum_{i=n-r}^{n-1} K_i$ if $n \ge r$. Let $K(x)$ be the generating function $K(x) = \sum_{i=0}^{\infty} K_i x^i$. It is easy to see that

$$K(x) = \frac{ax^{r-2} + (b-a)x^{r-1}}{1 - x - x^2 - \cdots - x^r}.$$

Note that if $a = 0$ and $b = 1$, then $K_n = G_n$, and we use the notation $K(x) = G(x)$. From the generating functions, it is easy to see that

$$K_n = aG_{n+1} + (b-a)G_n.$$

In particular, if $a = 2$, $b = 1$, we get the "Lucas" sequence $\{L_n\}$:

$$L(x) = \frac{2x^{r-2} - x^{r-1}}{1 - x - x^2 - \cdots - x^r} = \sum_{i=0}^{\infty} L_i x^i$$

which gives

$$L_n = 2G_{n+1} - G_n.$$

Since $\frac{x}{2-x} L(x) = G(x)$, we have

$$G_n = \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^{n-i} L_i.$$

(b) Relationship between $G_n^{(r)}$ and $G_n^{(r+1)}$.

For fixed r, write $G(x) = G_r(x)$ and $G_n = G_n^{(r)}$; likewise $L(x) = L_r(x)$ and $L_n = L_n^{(r)}$. Since

$$\frac{G_r(x)}{G_{r+1}(x)} = \frac{1}{x}\left[1 - \frac{x^{r+1}}{1 - x - x^2 - \cdots - x^r}\right],$$

we have

$$xG_r(x) = G_{r+1}(x) - x^2 G_r(x) G_{r+1}(x),$$

which gives

$$G_{n+1}^{(r+1)} = G_n^{(r)} + \sum_{i=0}^{n-1} G_i^{(r)} G_{n-1-i}^{(r+1)}.$$

For example, when $r = 2$

$$G_{n+1}^{(3)} = F_n + \sum_{i=0}^{n-1} F_i G_{n-1-i}^{(3)}.$$

Similarly we have

$$L_{r+1}(x) = xL_r(x) + x^2 L_{r+1}(x) G_r(x)$$

so

$$L_{n+1}^{(r+1)} = L_n^{(r)} + \sum_{i=0}^{n-1} L_i^{(r+1)} G_{n-1-i}^{(r)}.$$

The case $r = 1$ gives the well-known formulas

$$F_{n+1} = 1 + \sum_{i=0}^{n-1} F_i, \qquad L_{n+1} = 1 + \sum_{i=0}^{n-1} L_i.$$

(c) Another example using Zhou's TCI.

If we use Zhou's Theory of Constructing Identities on $G_n = 2G_{n-1} - G_{n-1-r}$, we have

$$(x^{r+1} - 2x^r + 1)(G_n x^{m-1} + G_{n+1}x^{m-2} + \cdots + G_{n-m-1}) \equiv 0 \pmod{x^{r+1} - 2x^r + 1}.$$

Simplifying as we did before, we get

$$G_{n+m} = G_{n+m-r} + G_n G_{m+r} - \sum_{i=0}^{r-1} G_{n-1-i}G_{m+i},$$

which could be written

$$G_{n+m+1} - G_{n+m} = G_n G_{m+r} - \sum_{i=0}^{r-1} G_{n-1-i}G_{m+i},$$

or

$$G_{2n} = G_{2n-r} + G_n G_{n+r} - \sum_{i=0}^{r-1} G_{n-1-i}G_{n+i}.$$

6. Table of r-Generalized Fibonacci Numbers

The first few terms of the r-generalized Fibonacci sequence for $2 \le r \le 8$ are given in the following table.

r-generalized Fibonacci Sequences

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136
4	0	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872
5	0	0	0	0	1	1	2	4	8	16	31	61	120	236	464	912	1793
6	0	0	0	0	0	1	1	2	4	8	16	32	63	125	248	492	976
7	0	0	0	0	0	0	1	1	2	4	8	16	32	64	127	253	504
8	0	0	0	0	0	0	0	1	1	2	4	8	16	32	64	128	255

The r-generalized Fibonacci sequences for $r = 2, 3, 4, 5, 6, 7, 8$ can be found in Sloane [7] as sequences A000045, A000073, A000078, A001591, A001592, A122189, and A079262, respectively.

7. Topics for Future Study

Many of the theorems, like Theorem 3.3, can undoubtedly be proved using combinatorial arguments in the manner of Benjamin and Quinn [1]. In fact, G_{n+r-1} (for $n \geq 0$) counts the number of tilings of an n-board with tiles of length at most r. It would be interesting to see different approaches to our theorems.

The r-generalized Lucas sequence could obviously be examined more thoroughly, and more relationships to the r-generalized Fibonacci numbers can probably be found.

The more general recurrence

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \cdots + c_r G_{n-r},$$

where the c_i's are constants, can certainly be studied in more detail.

We leave all of these topics for future research.

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