RESTRICTED OCCUPANCY OF s KINDS OF CELLS AND GENERALIZED PASCAL TRIANGLES

Spiros D. Dafnis

Department of Mathematics, University of Patras, Patras 26500, Greece

Frosso S. Makri

Department of Mathematics, University of Patras, Patras 26500, Greece

Andreas N. Philippou

Department of Mathematics, University of Patras, Patras 26500, Greece

e-mail: anphilip@math.upatras.gr

(Submitted April 2007 - Final Revision November 2007)

ABSTRACT

There are several well-known formulas counting the number of distinct allocations of n indistinguishable objects into m distinguishable cells, each of which has capacity $k-1$. In the present paper we generalize four of them by relaxing the assumption that each of the m cells has capacity $k-1$ and assuming instead that there are s kinds of cells and each cell of kind i has capacity $k_i - 1$ ($i = 1, \ldots, s$). A generalization of the Pascal triangles of order k is also discussed.

1. INTRODUCTION

Denote by $N_k(m,n)$ the number of distinct allocations of n indistinguishable objects into m distinguishable cells, each of which has capacity $k-1$. It is well-known (see, e.g. Freund [6], Riordan [14, p. 104], and Bondarenko [3, p. 22], that

$$N_k(m,n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-kj+m-1}{m-1}, \tag{1.1}$$

$$N_k(m,n) = \sum_{j=0}^{k-1} N_k(m-1, n-j), \tag{1.2}$$

$$N_k(m,n) = N_k(m,n-1) + N_k(m-1,n) - N_k(m-1,n-k), \tag{1.3}$$

and

$$N_k(m,n) = \sum_{j=0}^{m} \binom{m}{j} N_{k-1}(j, n-j). \tag{1.4}$$

Throughout the paper, for m, n integers, the binomial coefficient $\binom{m}{n}$ is equal to 1, if $m \geq 0$ and $n = 0$ or $m = n$; it is equal to $\prod_{j=1}^{n}(m-j+1)/\prod_{j=1}^{n} j$, if $m > n > 0$, and equals 0, otherwise.

The number $N_k(m,n)$ has been used extensively in reliability and probability studies (see, e.g. Derman, Lieberman and Ross [5], Sen, Agarwal and Bhattacharya [15], Makri and Philippou [7], and Makri, Philippou and Psillakis [9]. Instead of $N_k(m,n)$, some authors (e.g. Bondarenko [3], R. L. Ollerton and A. G. Shannon [11, 12] use the notation $\binom{m}{n}_k$, and name the latter generalized binomial coefficient of order k. For $k = 2$, relations (1.1) and (1.2) reduce to:

$$N_2(m,n) = \binom{m}{n}_2 = \binom{m}{n}, \text{ and } \binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}.$$

As Freund [6] observed, recurrence (1.2), defines a generalized Pascal triangle as an array whose (m,n) entry $(N_k(m,n))$ equals the sum of the k entries above it and to the left $(\sum_{j=0}^{k-1} N_k(m-1,n-j))$. For more on generalized Pascal triangles, or to be more precise Pascal triangles of order k, we refer to Philippou and Georghiou [13], Bollinger [1, 2], and Ollerton and Shannon [10].

In the present paper we generalize relations (1.1)-(1.4) to the case of s kinds of cells. This we do in Section 2. We also discuss, in Section 3, the corresponding generalized Pascal triangles.

2. RESTRICTED OCCUPANCY OF S KINDS OF CELLS

Presently we relax the assumption that each of the m cells has capacity $k-1$ by assuming instead that there are s kinds of cells and each one of kind i has capacity $k_i - 1$ $(i = 1, \ldots, s)$. We first derive the following generalization of (1.1).

Proposition 2.1: *For* $\mathbf{k} = (k_1, \ldots, k_s)$ *and* $\mathbf{m} = (m_1, \ldots, m_s)$, *denote by* $N_{\mathbf{k}}(\mathbf{m}, n)$ *the number of distinct allocations of* n *indistinguishable objects into* $\mathbf{m}$ *distinguishable cells. Assume that each of* m_i *specified cells has capacity* $k_i - 1$ $(i = 1, \ldots, s)$ *and set* $m = m_1 + \ldots + m_s$. *Then,*

$$N_{\mathbf{k}}(\mathbf{m}, n) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\ldots+j_s} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} \binom{m-1+n-k_1 j_1 - \ldots - k_s j_s}{m-1}. \quad (2.1)$$

Proof: Let $g(t)$ be the generating function of $N_{\mathbf{k}}(\mathbf{m}, n)$. Then,

$$g(t)=\sum_{n=0}^{\infty}N_{\mathbf{k}}(\mathbf{m},n)t^n=\prod_{i=1}^{s}(1+t+t^2+\ldots+t^{k_i-1})^{m_i}$$

$$=\left[\prod_{i=1}^{s}(1-t^{k_i})^{m_i}\right](1-t)^{-m},\quad m=\sum_{i=1}^{s}m_i$$

$$=\left[\prod_{i=1}^{s}\sum_{j_i=0}^{m_i}(-1)^{j_i}\binom{m_i}{j_i}t^{k_ij_i}\right]\sum_{j=0}^{\infty}\binom{m-1+j}{m-1}t^j,$$

by the binomial theorem,

$$=\sum_{n=0}^{\infty}\sum\left[\prod_{i=1}^{s}(-1)^{j_i}\binom{m_i}{j_i}\right]\binom{m-1+j}{m-1}t^n,$$

where the inner summation is over all nonnegative integers $j, j_1, j_2, \ldots, j_s$, satisfying the conditions $j_i \leq m_i \quad (i=1,\ldots,s)$ and $j+\sum_{i=1}^{s}k_ij_i=n$. Therefore,

$$N_{\mathbf{k}}(\mathbf{m},n)=\sum\left[\prod_{i=1}^{s}(-1)^{j_i}\binom{m_i}{j_i}\right]\binom{m-1+j}{m-1},$$

from which the proposition follows. □

For s = 1, Proposition 1.1 reduces to relation (1.1). For s = 2, it reduces to

$$N_{k_1,k_2}(m_1,m_2,n)=\sum_{j_1=0}^{m_1}\sum_{j_2=0}^{m_2}(-1)^{j_1+j_2}\binom{m_1}{j_1}\binom{m_2}{j_2}\binom{m-1+n-k_1j_1-k_2j_2}{m-1},\qquad(2.2)$$

a result derived and employed by Makri, Philippou and Psillakis [8] (2007a) to study Polya, inverse Polya and circular Polya distributions of order k for l-overlapping success runs. We proceed now to generalize recurrences (1.2) - (1.4).

Proposition 2.2: Let $N_{\mathbf{k}}(\mathbf{m},n)$ be as in Proposition 2.1. Then,

$$N_{\mathbf{k}}(\mathbf{m},n)=\sum_{j_1=0}^{k_1-1}N_{\mathbf{k}}(m_1-1,m_2,\ldots,m_s,n-j_1),\qquad(2.3)$$

$$N_{\mathbf{k}}(\mathbf{m},n)=\sum_{j_i=0}^{k_i-1}N_{\mathbf{k}}(m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_s,n-j_i),\quad i=2,\dots,s-1,\quad (2.4)$$

and

$$N_{\mathbf{k}}(\mathbf{m},n)=\sum_{j_s=0}^{k_s-1}N_{\mathbf{k}}(m_1,m_2,\dots,m_{s-1},m_s-1,n-j_s). \quad (2.5)$$

Proof: It suffices to show (2.3). We first note that by employing (2.1) and the Pascal triangle identity $\binom{m_1}{j_1}=\binom{m_1-1}{j_1-1}+\binom{m_1-1}{j_1}$, we get

$$N_{\mathbf{k}}(\mathbf{m},n)=S_1+S_2 \quad (2.6)$$

with

$$S_1=\sum_{j_1=1}^{m_1}\sum_{j_2=0}^{m_2}\dots\sum_{j_s=0}^{m_s}(-1)^{j_1+j_2+\dots+j_s}\binom{m_1-1}{j_1-1}\binom{m_2}{j_2}\dots\binom{m_s}{j_s}\binom{m-1+n-\sum_{i=1}^{s}k_ij_i}{m-1}$$

$$=\sum_{j_1'=0}^{m_1-1}\sum_{j_2=0}^{m_2}\dots\sum_{j_s=0}^{m_s}(-1)^{j_1'+j_2+\dots+j_s+1}\binom{m_1-1}{j_1'}\binom{m_2}{j_2}\dots\binom{m_s}{j_s}$$

$$\times\binom{m-1-k_1+n-k_1j_1'-\sum_{i=2}^{s}k_ij_i}{m-1}$$

on setting $j_1'=j_1-1$, and

$$S_2=\sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2}\dots\sum_{j_s=0}^{m_s}(-1)^{j_1+j_2+\dots+j_s}\binom{m_1-1}{j_1}\binom{m_2}{j_2}\dots\binom{m_s}{j_s}\binom{m-1+n-\sum_{i=1}^{s}k_ij_i}{m-1}$$

$$=\sum_{j_1=0}^{m_1-1}\sum_{j_2=0}^{m_2}\dots\sum_{j_s=0}^{m_s}(-1)^{j_1+j_2+\dots+j_s}\binom{m_1-1}{j_1}\binom{m_2}{j_2}\dots\binom{m_s}{j_s}$$

$$\times\left[\binom{m-1-k_1+n-\sum_{i=1}^{s}k_ij_i}{m-1}+\sum_{j=0}^{k_1-1}\binom{(m-1)-1+n-j-\sum_{i=1}^{s}k_ij_i}{(m-1)-1}\right].$$

The last equality follows by means of the"vertical" recurrence relation (Charalambides [4, p. 129]

$$\binom{x}{k} = \binom{x-r-1}{k} + \sum_{j=0}^{r} \binom{x-j-1}{k-1},$$

which holds true for any real number x and any nonnegative integer k. By interchanging the order of summation we obtain that

$$S_2 = \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1}\binom{m_2}{j_2}\cdots\binom{m_s}{j_s}\binom{m-1-k_1+n-\sum_{i=1}^{s} k_i j_i}{m-1}$$

$$+ \sum_{j=0}^{k_1-1} \left[\sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+j_2+\ldots+j_s} \binom{m_1-1}{j_1}\binom{m_2}{j_2}\cdots\binom{m_s}{j_s} \right.$$

$$\left. \times \binom{(m-1)-1+n-j-\sum_{i=1}^{s} k_i j_i}{(m-1)-1} \right]$$

$$= -S_1 + \sum_{j=0}^{k_1-1} N_{\mathbf{k}}(m_1-1, m_2, \ldots, m_s, n-j).$$

Substituting S_2 in (2.6) the proposition follows. □

For $s = 1$ Proposition 2.2 reduces to (1.2). For $s = 2$, it reduces to

$$N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_1=0}^{k_1-1} N_{k_1,k_2}(m_1-1, m_2, n-j_1), \tag{2.7}$$

and

$$N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_2=0}^{k_2-1} N_{k_1,k_2}(m_1, m_2-1, n-j_2). \tag{2.8}$$

Furthermore, by usage of (2.3)-(2.5) we get

$$N_{\mathbf{k}}(\mathbf{m}, n) = \sum_{j_1=0}^{k_1-1} \cdots \sum_{j_s=0}^{k_s-1} N_{\mathbf{k}}(m_1-1, \ldots, m_s-1, n-j_1-\ldots-j_s), \tag{2.9}$$

which, for $s = 2$, reduces to

$$N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_1=0}^{k_1-1} \sum_{j_2=0}^{k_2-1} N_{k_1,k_2}(m_1 - 1, m_2 - 1, n - j_1 - j_2).$$

Proposition 2.3: *Let $N_{\mathbf{k}}(\mathbf{m}, n)$ be as in Proposition 2.1. Then,*

$$N_{\mathbf{k}}(\mathbf{m}, n) = N_{\mathbf{k}}(\mathbf{m}, n-1) + N_{\mathbf{k}}(m_1-1, m_2, \dots, m_s, n) - N_{\mathbf{k}}(m_1-1, m_2, \dots, m_s, n-k_1), \quad (2.11)$$

$$\begin{aligned} N_{\mathbf{k}}(\mathbf{m}, n) = N_{\mathbf{k}}(\mathbf{m}, n-1) &+ N_{\mathbf{k}}(m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_s, n) \\ &- N_{\mathbf{k}}(m_1, \dots, m_{i-1}, m_i - 1, m_{i+1}, \dots, m_s, n - k_i), \quad i = 2, \dots, s-1, \end{aligned} \quad (2.12)$$

and

$$N_{\mathbf{k}}(\mathbf{m}, n) = N_{\mathbf{k}}(\mathbf{m}, n-1) + N_{\mathbf{k}}(m_1, m_2, \dots, m_s - 1, n) - N_{\mathbf{k}}(m_1, m_2, \dots, m_{s-1}, m_s - 1, n - k_s). \quad (2.13)$$

Proof: It suffices to show (2.11). By Proposition 2.1 and the Pascal triangle identity

$$N_{\mathbf{k}}(\mathbf{m}, n) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\dots+j_s} \binom{m_1}{j_1}\binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i}{m - 1}$$

$$+ \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\dots+j_s} \binom{m_1}{j_1}\binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 + n - 1 - \sum_{i=1}^{s} k_i j_i}{m - 1 - 1}$$

$$= N_{\mathbf{k}}(\mathbf{m}, n-1)$$

$$+ \sum_{j_1=0}^{m_1-1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\dots+j_s} \binom{m_1 - 1}{j_1}\binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 - 1 + n - \sum_{i=1}^{s} k_i j_i}{m - 1 - 1}$$

$$+ \sum_{j_1=1}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\dots+j_s} \binom{m_1 - 1}{j_1 - 1}\binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 - 1 + n - \sum_{i=1}^{s} k_i j_i}{m - 1 - 1}$$

$$= N_{\mathbf{k}}(\mathbf{m}, n-1) + N_{\mathbf{k}}(m_1 - 1, m_2, \dots, m_s, n)$$

$$+ \sum_{j_1=1}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_s=0}^{m_s} (-1)^{j_1+\dots+j_s} \binom{m_1 - 1}{j_1 - 1}\binom{m_2}{j_2} \cdots \binom{m_s}{j_s} \binom{m - 1 - 1 + n - \sum_{i=1}^{s} k_i j_i}{m - 1 - 1}.$$

The result follows by setting $j_1 - 1 = j_1^{'}$ in the sum of the last equality. □

For $s = 1$ Proposition 2.3 reduces to (1.3). For $s = 2$, it reduces to

$$N_{k_1,k_2}(m_1, m_2, n) = N_{k_1,k_2}(m_1, m_2, n-1) + N_{k_1,k_2}(m_1-1, m_2, n) - N_{k_1,k_2}(m_1-1, m_2, n-k_1), \tag{2.14}$$

and

$$N_{k_1,k_2}(m_1, m_2, n) = N_{k_1,k_2}(m_1, m_2, n-1) + N_{k_1,k_2}(m_1, m_2-1, n) - N_{k_1,k_2}(m_1, m_2-1, n-k_2). \tag{2.15}$$

Proposition 2.4: *Let $N_{\mathbf{k}}(\mathbf{m}, n)$ be as in Proposition 2.1. Then,*

$$N_{\mathbf{k}}(\mathbf{m}, n) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_s=0}^{m_s} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} N_{\mathbf{k-1}}(j_1, \ldots, j_s, n - j_1 - \cdots - j_s). \tag{2.16}$$

Proof: We consider the proof of (2.16) as a classical occupancy problem. Let A be the set of allocations of n indistinguishable objects into m distinguishable cells such that each of m_i specified cells may be occupied by at most $k_i - 1$ objects (cells of the ith kind), $i = 1, \ldots, s$ ($m = m_1 + \cdots + m_s$).

For $i = 1, \ldots, s$, let $A_{j_i}^{(i)}$ be the subset of these allocations in which j_i cells, $j_i = 0, 1, \ldots, m_i$, of the ith kind are occupied (and consequently the remaining $m_i - j_i$ cells of the ith kind remain empty). For given $j_1, \ldots, j_s$ and any specified selection of j_1 cells out of m_1 of the 1st kind, ..., j_s cells out of m_s of the sth kind, one object is placed in each of these $j_1 + \cdots + j_s$ specified cells. Next, note that the number of allocations of the remaining $n - (j_1 + \cdots + j_s)$ objects into the $j_1 + \cdots + j_s$ cells, under the restrictions of the capacities of the cells, equals

$$N_{\mathbf{k-1}}(j_1, \ldots, j_s, n - (j_1 + \cdots + j_s))$$

by Proposition 2.1. Further, the $j_1, \ldots, j_s$ cells can be chosen in

$$\binom{m_1}{j_1} \cdots \binom{m_s}{j_s}, \quad j_i = 0, 1, \ldots, m_i, \quad i = 1, 2, \ldots, s$$

ways. So, according to the multiplicative principle, the number of the elements of the set $A_{j_1}^{(1)} \bigcap \cdots \bigcap A_{j_s}^{(s)}$ equals

$$\binom{m_1}{j_1} \cdots \binom{m_s}{j_s} N_{\mathbf{k-1}}(j_1, \ldots, j_s, n - (j_1 + \cdots + j_s)).$$

Thus, summing for all values of $j_i = 0, 1, \ldots, m_i$, $i = 1, \ldots, s$, according to the addition principle, we deduce (2.16). □

For $s = 1$ Proposition 2.4 reduces to (1.4). For $s = 2$, it reduces to

$$N_{k_1,k_2}(m_1, m_2, n) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \binom{m_1}{j_1} \binom{m_2}{j_2} N_{k_1-1,k_2-1}(j_1, j_2, n - j_1 - j_2). \tag{2.17}$$

3. GENERALIZED PASCAL TRIANGLES OF ORDER k

In this section, we note that the s recurrences (2.3)-(2.5) define a generalized Pascal triangle (hyper cube), which we call Pascal triangle of order $\mathbf{k}$ and denote by $T_{\mathbf{k}}(\mathbf{m}, n)$, as the hyper cube whose $(\mathbf{m}, n)$ entry $N_{\mathbf{k}}(\mathbf{m}, n)$ equals any one of the k_i sums $(i = 1, \dots, s)$ appearing on the right-hand side of (2.3)-(2.5). For example, recurrence (2.3) gives the $(\mathbf{m}, n)$ entry $N_{\mathbf{k}}(\mathbf{m}, n)$ of $T_{\mathbf{k}}(\mathbf{m}, n)$ as the sum of the k_1 entries $N_{\mathbf{k}}(m_1 - 1, m_2, \dots, m_s, n - j), j = 0, 1, \dots, k_1 - 1$. For $s = 2$, the (m_1, m_2, n) entry of the Pascal triangle (cube) of order (k_1, k_2) equals the sum of the k_1 entries $N_{k_1,k_2}(m_1 - 1, m_2, n - j), j = 0, 1, \dots, k_1 - 1$. It is also equal to the sum of the k_2 entries $N_{k_1,k_2}(m_1, m_2 - 1, n - j), j = 0, 1, \dots, k_2 - 1$.

Geometrically, we could use recurrence (2.7) to construct a cube with entries N_{k_1,k_2} (m_1, m_2, n). Consider a cube such that, on its upper (horizontal) side (P_u), a generalized Pascal triangle of order k_1, $T_{k_1}(m_1, n)$ is created Freund [6], e.g., its first row $m_1 = 0$ consists of a 1 and no other entries and each other entry is obtained as the sum of the entry immediately above and the $k_1 - 1$ entries to its left.

Next, on the left vertical side of the cube (P_v), perpendicular to the upper side, a generalized Pascal triangle of order k_2, $T_{k_2}(m_2, n)$ is created (see the following figure, which provides an illustration for $k_1 = 3$, $k_2 = 4$).

Note that the (m_1, n) entry of $T_{k_1}(m_1, n)$ is simultaneously the $(m_1, 0, n)$ entry of the cube, and the (m_2, n) entry of $T_{k_2}(m_2, n)$ is simultaneously the $(0, m_2, n)$ entry of the cube.

For a given value of $m_2 = m$ we consider a plane parallel to the upper side of the cube which intersects the left vertical side of the cube at the row $m_2 = m$ of $T_{k_2}(m_2, n)$. On this new plane an array is constructed with its first row ($m_1 = 0$) being the $m_2 = m$ row of $T_{k_2}(m_2, n)$ and each other entry is obtained as the sum of the entry immediately above and the $k_1 - 1$ entries to its left. $N_{k_1,k_2}(m_1, m_2, n)$, which represents the number of distinct allocations of n indistinguishable objects into m_1 distinguishable cells each of which has capacity $k_1 - 1$ and m_2 distinguishable cells each of which has capacity $k_2 - 1$, is the (m_1, n) entry of this array. A similar procedure could be followed using recurrence (2.8).

To make it more clear, we note that in order to calculate $N_{k_1,k_2}(u, v, n)$ we first construct $T_{k_2}(m_2, n)$ until its line $m_2 = v$. In the sequel we construct an array $(a_{m_1,n})$ with its first row $(m_1 = 0)$ being the $m_2 = v$ row of $T_{k_2}(m_2, n)$ and each other entry of the array is obtained as the sum of the entry above and $k_1 - 1$ entries to the left of the one immediately above.

As an example, we give the calculation of $N_{3,4}(m_1, 6, n)$. First, we construct $T_4(m_2, n)$.

$m_2 \backslash n$	0	1	2	3	4	5	6
0	1						
1	1	1	1	1			
2	1	2	3	4	3	2	1
3	1	3	6	10	12	12	10
4	1	4	10	20	31	40	44
5	1	5	15	35	65	101	135
6	**1**	**6**	**21**	**56**	**120**	**216**	**336**

Then we construct $T_{3,4}(m_1, 6, n) = T_3(m_1, n)$ with $N_3(0, n) = N_4(6, n)$,

$m_1 \backslash n$	0	1	2	3	4	5	6
0	**1**	**6**	**21**	**56**	**120**	**216**	**336**
1	1	7	28	83	197	392	672
2	1	8	36	118	308	672	1261
3	1	9	45	162	462	1098	2241
4	1	10	55	216	669	1722	2865
5	1	11	66	281	940	2607	3750
6	1	12	78	358	1287	3828	4971

from which $N_{3,4}(m_1, 6, n)$ are readily available. For example,

$$N_{3,4}(2, 6, 5) = N_3(2, 5) = 672,$$

$$N_{3,4}(5, 6, 3) = N_3(5, 3) = 281,$$

$$N_{3,4}(6, 6, 4) = N_3(6, 4) = 1287.$$

REFERENCES

[1] R. C. Bollinger. "A Note on Pascal-T Triangles, Multinomial Coefficients and Pascal Pyramids." *The Fibonacci Quarterly* **24** (1986): 140–144.

[2] R. C. Bollinger. "Extended Pascal Triangles." *Mathematics Magazine* **66.2** (1993): 87–94.

[3] B. A. Bondarenko. *Generalized Pascal Triangles and Pyramids, Their Fractals, Graphs, and Applications*, The Fibonacci Association, (1993).

[4] C. A. Charalambides. *Enumerative Combinatorics*, Chapman and Hall/CRC, Boca Raton, FL, (2002).

[5] C. Derman, G. Lieberman, S. Ross. "On the Consecutive-k-of n:F System." *IEEE Trans. Reliability* **31** (1982): 57–63.

[6] J. E. Freund. "Restricted Occupancy Theory. A Generalization of Pascal's Triangle." *American Mathematical Monthly* **63** (1956): 20–27.

[7] F. S. Makri and A. N. Philippou. "On Binomial and Circular Binomial Distributions of Order k for l-overlapping Success Runs of Length k." *Statistical Papers* **46.3** (2005): 411–432.

[8] F. S. Makri, A. N. Philippou, and Z. M. Psillakis. "Polya, Inverse Polya and Circular Polya Distributions of Order k for l-overlapping Success Runs." *Communications in Statistics-Theory and Methods* **36** (2007a): 657–668.

[9] F. S. Makri, A. N. Philippou, and Z. M. Psillakis. "Shortest and Longest Length of Success Runs in Binary Sequences." *Journal of Statistical Planning and Inference* **137** (2007b): 2226–2239.

[10] R. L. Ollerton and A. G. Shannon. "Some Properties of Generalized Pascal Squares and Triangles." *The Fibonacci Quarterly* **36** (1998): 140–144.

[11] R. L. Ollerton and A. G. Shannon. "Extensions of Generalized Binomial Coefficients." *In Applications of Fibonacci Numbers*, Volume 9. Edited by F. T. Howard, Dordrecht: Kluwer Academic Publishers, 2004, 187–199.

[12] R. L. Ollerton and A. G. Shannon. "Further Properties of Generalized Binomial Coefficient k-extensions." *The Fibonacci Quarterly* **43** (2005): 124–129.

[13] G. N. Philippou and C. Georghiou. "Fibonacci-type Polynomials and Pascal Triangles of Order k." *In Fibonacci Numbers and Their Applications*. Edited by A. N. Philippou, G. E. Bergum, and A. F. Horadam, Dordrecht: D. Reidel Publishing Company, 1986, 229–233.

[14] J. Riordan. *An Introduction to Combinatorial Analysis*. Wiley, New York, (1964).

[15] K. Sen, M. Agarwal and S. Bhattacharya. "On Circular Distributions of Order k Based on Polya-Eggenberger Sampling Scheme." *The Journal of Mathematical Sciences* **2** (2003): 34–54.

AMS Classification Numbers: 05A10, 11B65

✠ ✠ ✠